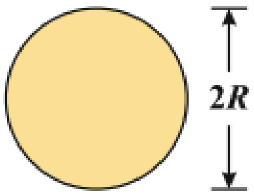
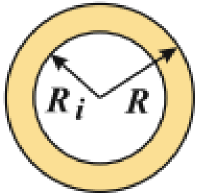
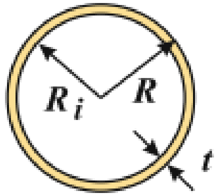
# Chapter 5. Torsion Members
# الفصل الخامس عناصر الفتل

Shear Stress:

$$\tau(r) = \frac{Tr}{J},$$

$$\tau_{max} = \frac{TR}{J}$$

| Cross-section | Cross-section | Polar Moment of Inertia, $J$ |
|---|---|---|
| |  | $\frac{\pi R^4}{2} = \frac{\pi D^4}{32}$ |
| **Thick-walled (Hollow) Shaft** $R = D/2$, $R_i = D_i/2$ |  | $\frac{\pi(R^4 - R_i^4)}{2} = \frac{\pi(D^4 - D_i^4)}{32}$ |
| **Thin-walled Shaft *** $t << R$ <br> $t = R - R_i = (D - D_i)/2$ <br> $R_{ave} = (R+R_i)/2$; $D_{ave} = (D+D_i)/2$ |  | $2\pi R_{ave}^3 t = \frac{\pi D_{ave}^3 t}{4}$ |

Shear Strain:

$$\gamma(r) = \frac{\tau(r)}{G} = \frac{Tr}{JG}$$

Angle of twist:

$$\theta = \frac{TL}{JG}$$

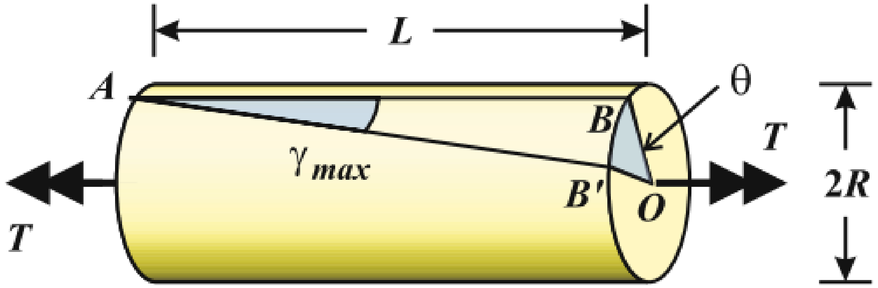
Torsional Stiffness

$$K_T = \frac{T}{\theta} = \frac{JG}{L}$$

## 2. Torsion Members – Force Method

The following examples demonstrate the *force method* applied to torsion members. The torque everywhere in the system is first determined, followed by the shear stress and the angle of twist.

### *Example 5.2 Solid Shaft under Torsion*




**Given:** A solid shaft, $L = 1.0$ m long, with a diameter $D = 2R = 5.0$ cm., is subjected to a torque $T = 460$ Nm. (*Fig.*). The shaft is made of steel with shear modulus $G = 80$ GPa and yield strength $S_y = 350$ MPa.

**Required:** Determine (a) the maximum shear stress, (b) the angle of twist of the shaft, and (c) the factor of safety FS against yielding.

**Solution:** *Step 1.* The polar moment of inertia is: $J = \pi D^4/32 \quad = 61.4 \times 10^{-8}\ m^4$

*Step 2.* The maximum shear stress is: $\tau_{max} = TR/J \quad = 0.187 \times 10^8\ Nm = 18.7\ MPa$

*Step 3. The angle of twist is:* $\theta = TL/JG \quad = 0.112 \times 10^{-2}\ Rad = 0.45^{\circ}$

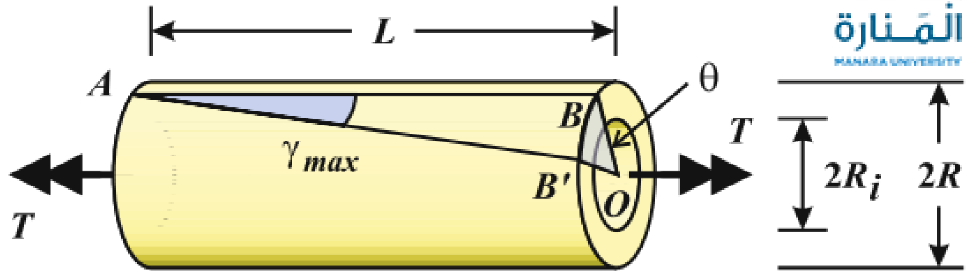
*Step 4.* $\tau_y$ is not given, but for *ductile materials* it can be approximated from the yield strength $S_y$

$\tau_y = S_y/\sqrt{3} = 202\ MPa \quad FS = \tau_y/\tau_{max} \approx 11$

Design of shafts is often limited by the angle of twist.

## *Example 5.3 Hollow Shaft under Torsion*

**Given:** A hollow shaft, $L = 1.0$ m long, with outer diameter $D = 5.0$ cm., and inner diameter $Di = 2.5$ cm., is subjected to a torque $T = 460$ N-m. (*Fig.*). The shaft is made of steel with shear modulus $G = 80$ GPa and yield strength $S_y = 350$ MPa.




**Required:** Determine (a) the shear stresses at the outer and inner surfaces, (b) the angle of twist of the shaft, and (c) the factor of safety FS against yielding.

**Solution:** *Step 1.* The polar moment of inertia is: $J = \pi(D^4 - D_i^4)/32$

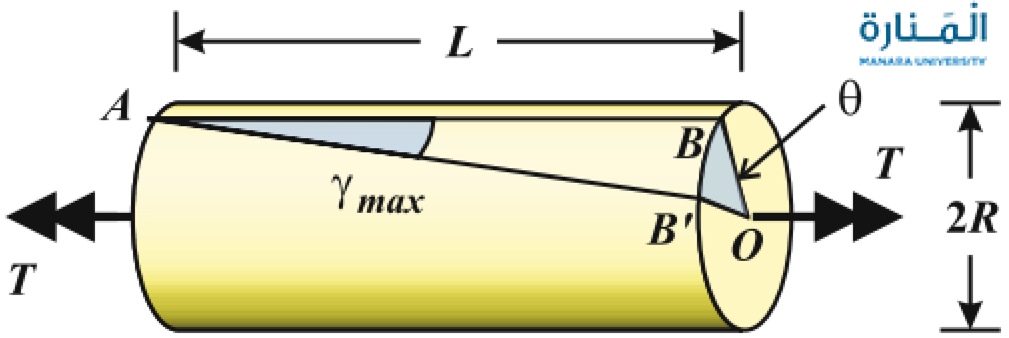
*Step 2.* the shear stresses at the outer and inner surfaces $\tau(r) = Tr/J$,

*Step 3. The angle of twist is:* $\theta = TL/JG$

*Step 4.* the factor of safety FS against yielding $FS = \tau_y/\tau_{max}$

Note that there is not much difference in the responses of the current hollow shaft and the solid shaft of *Example 5.2*. Both shafts have the same outer diameter, but the hollow shaft provides a weight savings of 25% with a modest increase in stress of ?? %.

## *Example 5.4* Pre-torqued Suspension Torsion Bar




**Given:** In high-performance cars, the steering and handling characteristics are improved by introducing a prestressed circular torsion bar. The torsion bar is pre-torqued by rotating one end of the bar with respect to the other end. One end of the torqued bar is then attached to the frame, and the other end to the wheel suspension. Such a solid shaft, with diameter $D = 30$ mm and length $L = 1.3$ m, is made of steel ($G = 77$ GPa). The pre-torque causes a relative angular displacement between the ends of the bar of $\theta = 3.0°$ (*Fig.*).

**Required:** Determine (a) the pre-torque required, and (b) the maximum shear stress.

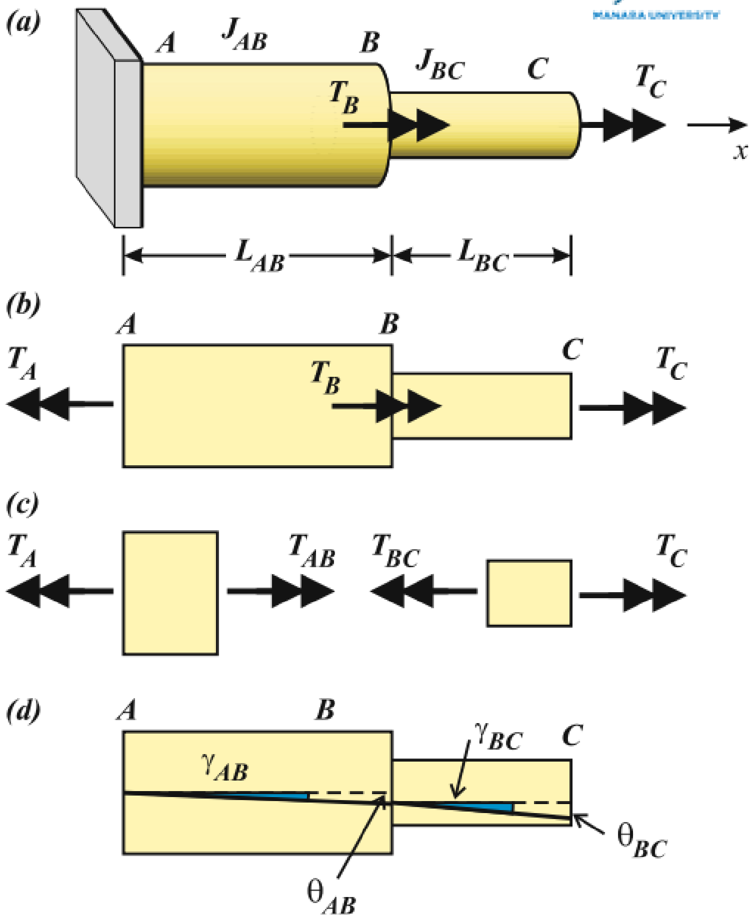
**Solution:** *Step 1.* The polar moment of inertia is: $J = \pi D^4/32$

*Step 2.* Solving for the torque, and noting that 3.0° = ???? rad: $T = JG\theta/L$

*Step 3.* The maximum shear stress is: $\tau_{max} = TR/J$

## *Example 5.5* Stepped Shaft in Torsion

**Given:** The solid stepped shaft, *ABC*, is fixed at *A*, and is subjected to torques $T_B = 880$ N·m and $T_C = 275$ N·m (*Fig.*). Segment *AB* has length $L_{AB} = 1.5$ m and diameter $D_{AB} = 50$ mm. Segment *BC* has length $L_{BC} = 1.0$ m and diameter $D_{BC} = 30$ mm. The material is steel with a shear modulus of $G = 77$ GPa.




**Required:** Determine (a) the maximum shear stress in *AB*, (b) the maximum shear stress in *BC*, and (c) the total angle of twist of shaft *ABC*, $\theta_{AC}$.

**Solution:** *Step 1.* The polar moment of inertia of each cross-section is:

$$J_{AB} = \pi D_{AB}^4/32 \qquad J_{BC} = \pi D_{BC}^4/32$$

*Step 2.* From a FBD of the shaft (*Fig.b*), the reaction at the wall is:

$$T_A = T_B + T_C$$

*Step 3.* The maximum shear stress and angle of twist of segment *AB* are:

$$\tau_{AB,max} = T_A R_{AB}/J_{AB} \qquad \theta_{AB} = T_A L_{AB}/J_{AB} G$$

*Step 4. The maximum shear stress and angle of twist of segment BC are:*

$$\tau_{BC,max} = T_C R_{BC}/J_{BC} \qquad \theta_{BC} = T_C L_{BC}/J_{BC} G$$

*Step 5.* Since the internal torques twist both segments in the same direction, the total angle of twist is (*Fig.d*):

$$\theta_{AC} = \theta_{AB} + \theta_{BC}$$

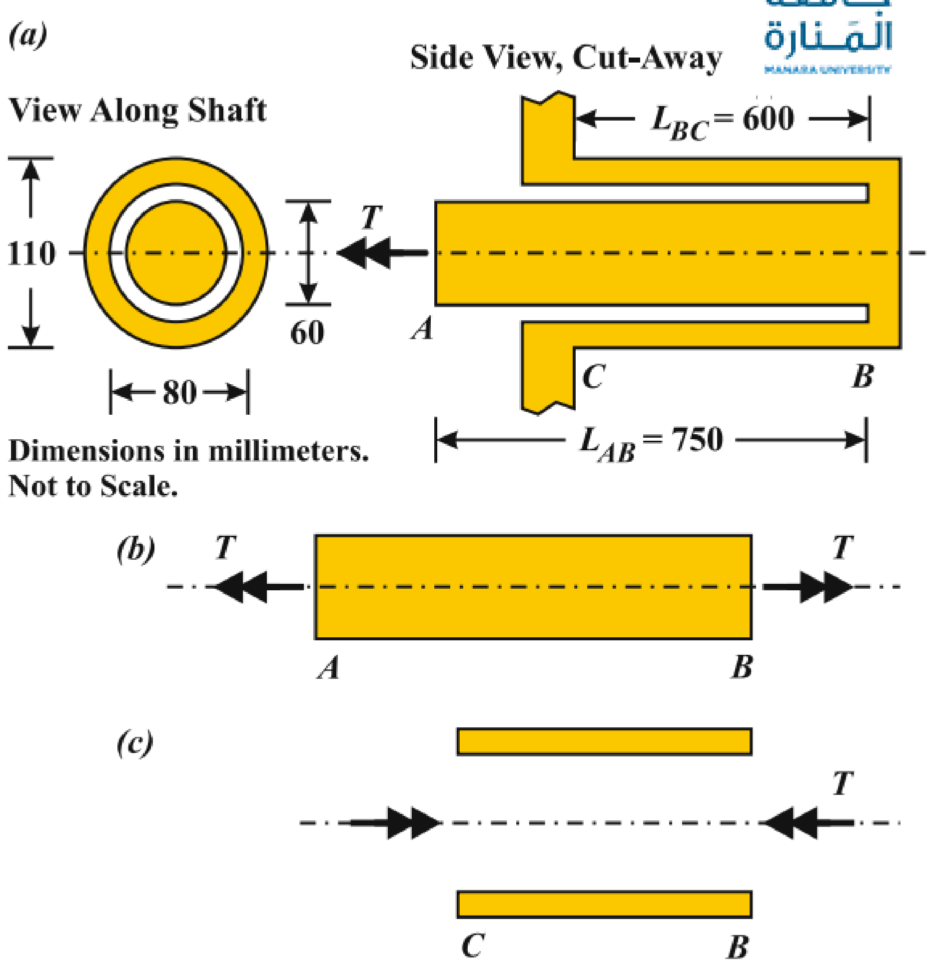
## *Example 5.6* Torsional Vibration Damper

**Given:** A damper used to reduce the torsional vibration in a machine consists of a solid shaft *AB*, set inside hollow shaft *BC* (*Fig.*). For *AB*: $D_{AB}$ = 60 mm, $L_{AB}$ = 750 mm. For *BC*: $D_o$ = 110 mm, $D_i$ = 80 mm, $L_{BC}$ = 600 mm.

The shafts are joined at cross-section *B*. The vibrating machine is at point *A* and the assembly is anchored at fixed base point *C*. The damper is made of steel *G* = 77 GPa.

The damper is designed to occupy minimal space by doubling back upon itself.

**Required:** Determine (a) the maximum shear stress in each segment for an applied torque *T* = 8.60 kN·m, (b) the rotations of cross-sections *A* and *B* with respect to fixed cross-section *C*, and (c) the torsional stiffness $K_T = T/\theta$ of the damper system.



## *Example 5.7* Drill Bit

**Given:** A drill bit is stuck in a piece of wood, but the user attempts to continue to operate it. Assume that the bit is a simple cylinder of radius *R* and embedded length *L*. The drill applies torque *T*. The surrounding wood applies a uniform interfacial shear stress $\tau_i$ on the surface of the cylinder, reducing the internal torque $T(x)$ carried in the bit linearly from the drill-end ($x = L$) to the tip of the bit ($x = 0$). No torque is transferred across the free end (tip) of the bit.

**Required:** Determine the angle of twist of the embedded length of the drill bit. Measure *x* from the tip (free end) of the bit.

**Solution:** *Step 1. Equilibrium* of the embedded bit (*Fig.b*). Assuming that the bit attempts to rotate clockwise as it cuts, the value of the torque *T* at the drill end is: $T = \tau_i(2\pi RL)R = 2\pi\tau_i R^2 L$

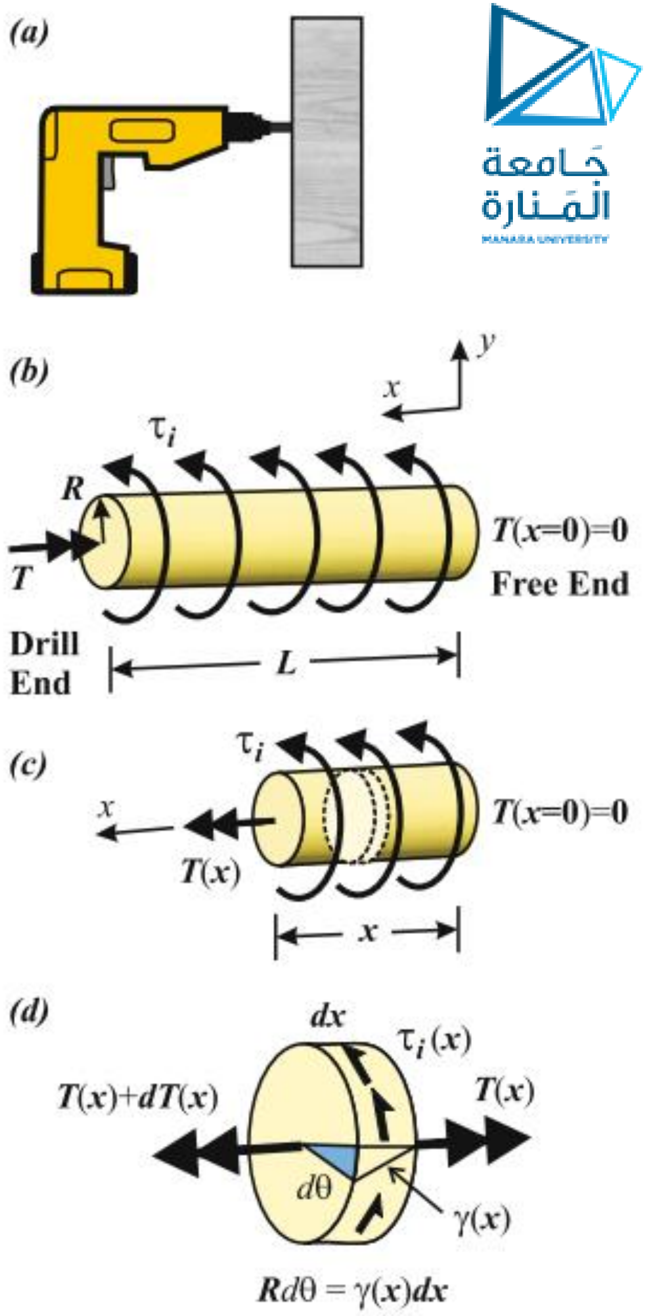
Equilibrium on a FBD from the tip of the bit to the cross-section at *x* (*Fig.c*) gives the internal torque $T(x)$:

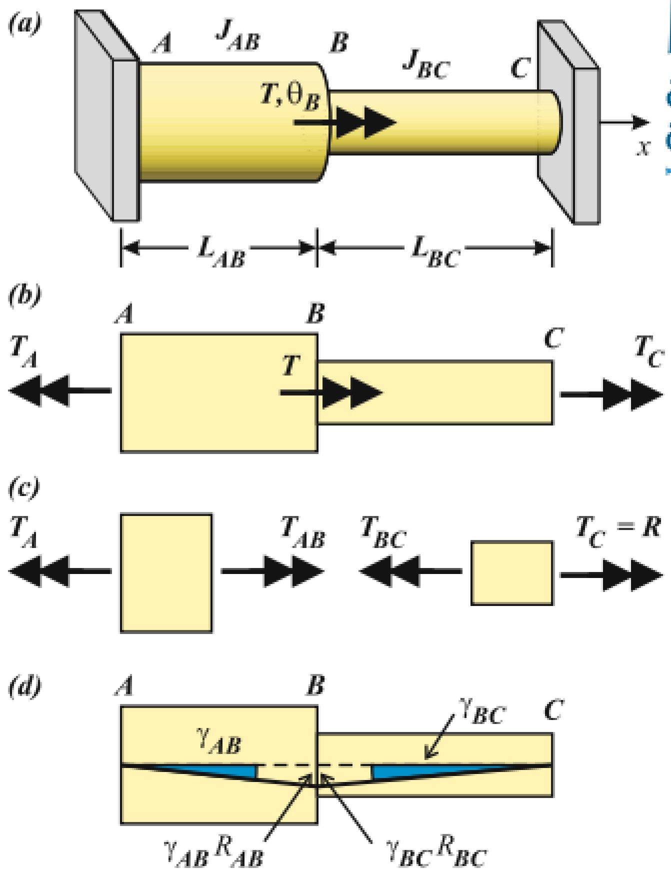
$$T(x) = -2\pi\tau_i R^2 x = -Tx/L$$

*Step 2. Torque–twist.* Consider a length *dx* so small that the change in $T(x)$ over *dx* is negligible. Over *dx* (*Fig.d*), the angle of twist is: $d\theta(x) = T(x)dx/JG = -2\pi\tau_i R^2 x dx/JG = -Txdx/JGL$

*Step 3. Compatibility* (*rotation–twist*). The angle of twist from the tip ($x = 0$) to any cross-section at *x* is found by integrating $d\theta(x)$ from 0 to *x*:

$$\theta(x) = \int_0^x \frac{T(x)dx}{JG} = \frac{-Tx^2}{2JGL} \qquad \Rightarrow \theta = \theta(L) = \frac{-TL}{2JG}$$






## *Example 5.8* Stepped-Shaft with Fixed Ends – Statically Indeterminate

**Given:** The stepped shaft *ABC* shown in (*Fig.*) is fixed at ends *A* and *C*. Torque *T* is applied at the juncture point *B*. The shear modulus of both segments of the shaft is *G*.

**Required:** Using the force method, determine expressions for the torques carried in each segment of the shaft, $T_{AB}$ and $T_{BC}$. Take the system to remain elastic

**Solution:**

*Step 1. Equilibrium* of the entire system provides only one equation – the torque about the *x*-axis (*Fig.b*):

$$\sum T_x = 0: T - T_A + T_C = 0$$

*Step 2.* The *twist–torque* elastic relationship gives the angles of twist for *AB* and *BC*:

$$\theta_{AB} = \frac{T_A L_{AB}}{J_{AB} G} \qquad \theta_{BC} = \frac{T_C L_{BC}}{J_{BC} G}$$

*Step 3. Compatibility* (*rotation–twist* relationship) relates the angle of twist of each member ($\theta_{AB}$ and $\theta_{BC}$): $\theta_{AB} + \theta_{BC} = 0$

*Step 4.* Solving for torques $T_{AB}$ and $T_{BC}$: