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MATHEMATICAL ANALAYSIS 2

Lecture 1

Prepared by

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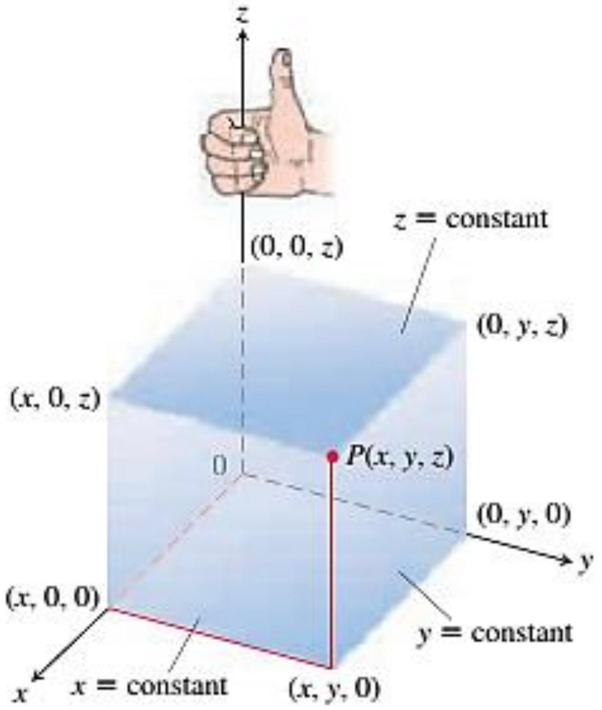
Vectors and the Geometry of Space

- Three-Dimensional Coordinate Systems
- Vectors
- The Dot Product
- The Cross Product
- Lines and Planes in Space
- Cylinders and Quadric Surfaces

Three-Dimensional Coordinate Systems



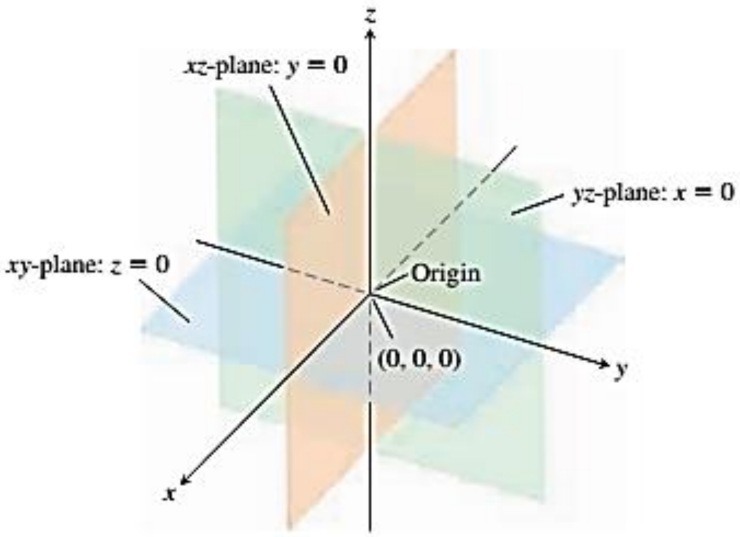


FIGURE 12.2 The planes $x = 0$, $y = 0$, and $z = 0$ divide space into eight octants.

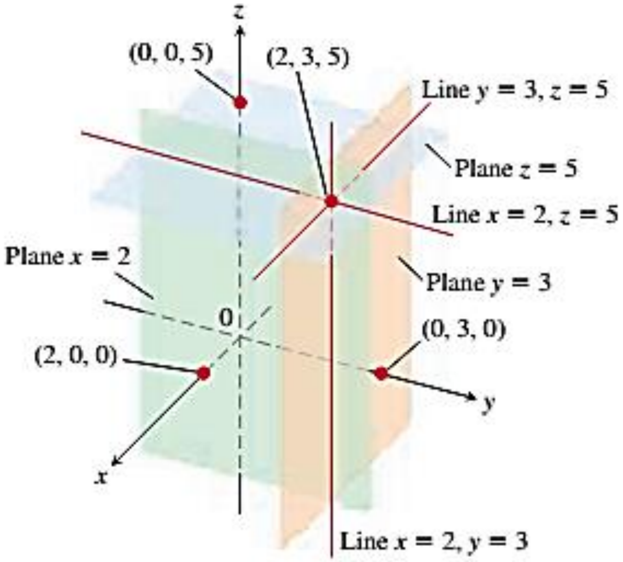


FIGURE 12.3 The planes $x = 2$, $y = 3$, and $z = 5$ determine three lines through the point $(2, 3, 5)$.

EXAMPLE 1 We interpret these equations and inequalities geometrically.

(a) $z \geq 0$ — The half-space consisting of the points on and above the xy-plane.

(b) $x = -3$ — The plane perpendicular to the x-axis at $x = -3$. This plane lies parallel to the yz-plane and 3 units behind it.

(c) $z = 0, x \leq 0, y \geq 0$ — The second quadrant of the xy-plane.

(d) $x \geq 0, y \geq 0, z \geq 0$ — The first octant.

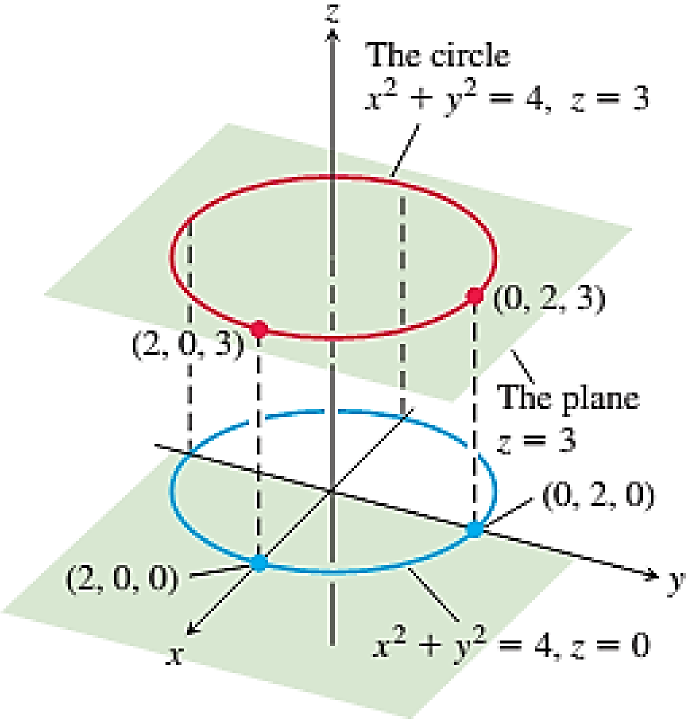
(e) $-1 \leq y \leq 1$ — The slab between the planes $y = -1$ and $y = 1$ (planes included).

(f) $y = -2, z = 2$ — The line in which the planes $y = -2$ and $z = 2$ intersect. Alternatively, the line through the point $(0, -2, 2)$ parallel to the x-axis.

Three-Dimensional Coordinate Systems

EXAMPLE 2 What points (x, y, z) satisfy the equations

$$x^2 + y^2 = 4 \qquad \text{and} \qquad z = 3?$$



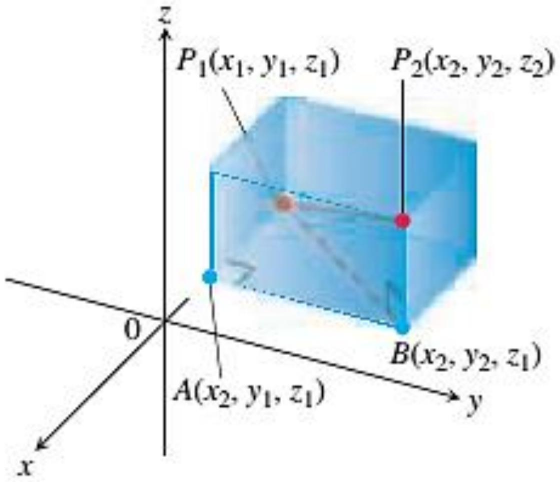
Distance and Spheres in Space

The Distance Between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

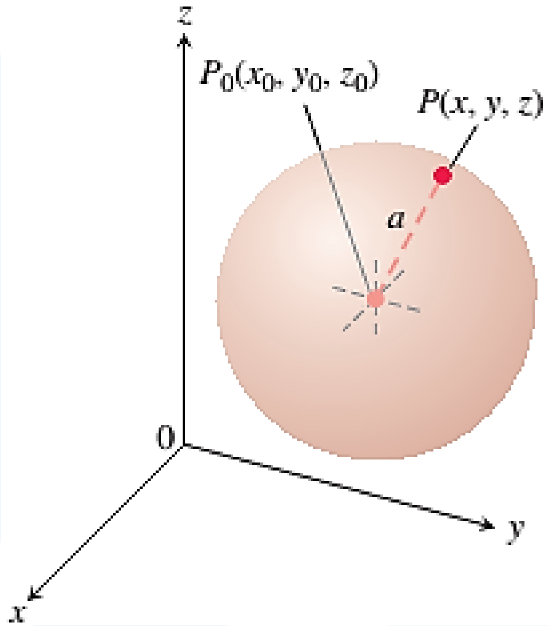
EXAMPLE 3 The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$\begin{aligned} |P_1P_2| &= \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2} \\ &= \sqrt{16 + 4 + 25} \\ &= \sqrt{45} \approx 6.708. \end{aligned}$$



The Standard Equation for the Sphere of Radius a and Center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$



Distance and Spheres in Space

EXAMPLE 4 Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

$$\left(x+\frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4} \Rightarrow (-3/2, 0, 2), r = \sqrt{21}/2$$

EXAMPLE 5 Here are some geometric interpretations of inequalities and equations involving spheres.

(a) $x^2 + y^2 + z^2 < 4$ — The interior of the sphere $x^2 + y^2 + z^2 = 4$.

(b) $x^2 + y^2 + z^2 \le 4$ — The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$. Alternatively, the sphere $x^2 + y^2 + z^2 = 4$ together with its interior.

(c) $x^2 + y^2 + z^2 > 4$ — The exterior of the sphere $x^2 + y^2 + z^2 = 4$.

(d) $x^2 + y^2 + z^2 = 4, z \le 0$ — The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy-plane (the plane $z = 0$).

EXERCISES

Find the perimeter of the triangle with vertices $A(-1, 2, 1)$, $B(1, -1, 3)$, and $C(3, 4, 5)$.

$\sqrt{17}+\sqrt{33}+6.$

Find an equation for the set of all points equidistant from the planes $y = 3$ and $y = -1$.

$y = 1$

Find the point on the sphere $x^2 + (y - 3)^2 + (z + 5)^2 = 4$ nearest

a. the xy-plane.

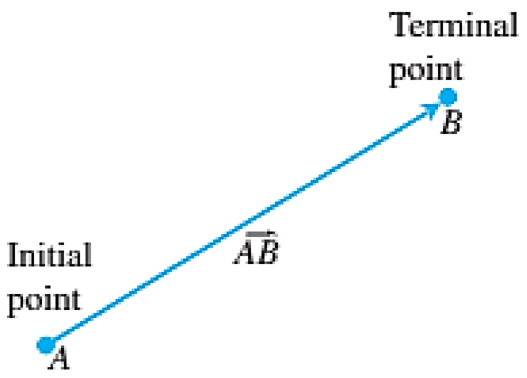
$(0, 3, -3).$

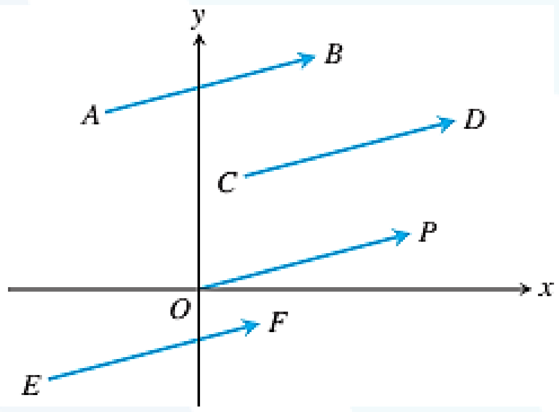
b. the point $(0, 7, -5)$.

$(0, 5, -5).$

Vectors

DEFINITIONS The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.





$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}.$

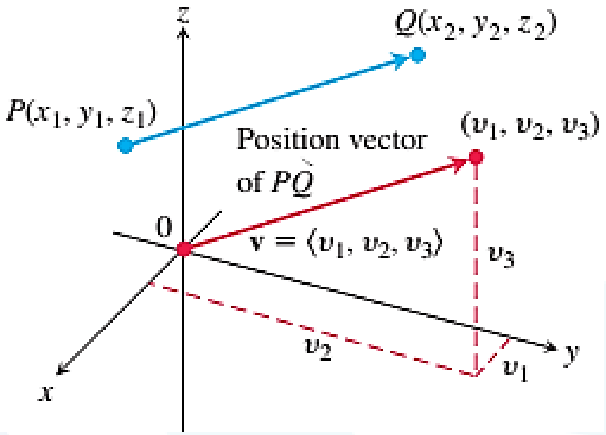
Vectors

DEFINITION If $\mathbf{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

If $\mathbf{v}$ is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$



$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(see Figure 12.10).

Vectors

EXAMPLE 1 Find the **(a)** component form and **(b)** length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

The component form of $\overrightarrow{PQ}$ is

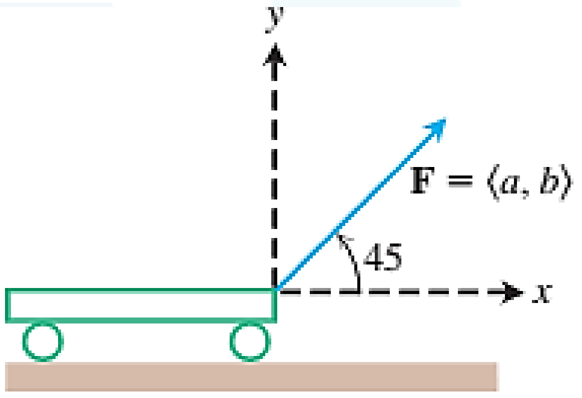
$$\mathbf{v} = \langle -2, -2, 1 \rangle.$$

The length or magnitude of $\mathbf{v} = \overrightarrow{PQ}$ is

$$|\mathbf{v}| = \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3.$$

EXAMPLE 2 A small cart is being pulled along a smooth horizontal floor with a 20-lb force $\mathbf{F}$ making a 45° angle to the floor (Figure 12.11). What is the *effective* force moving the cart forward?

$$a = |\mathbf{F}| \cos 45° = (20)\left(\frac{\sqrt{2}}{2}\right) \approx 14.14 \text{ lb.}$$

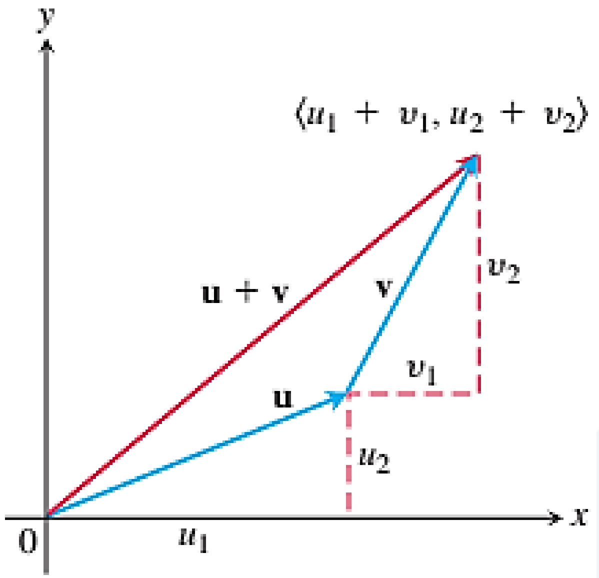


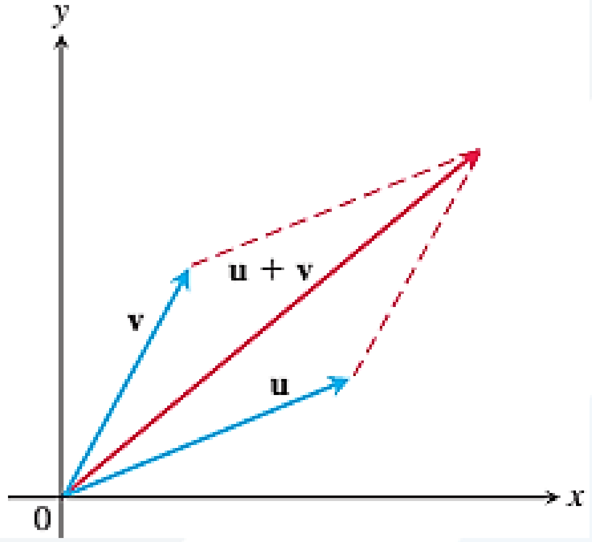
Vector Algebra Operations

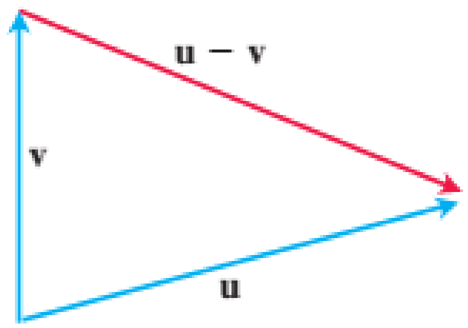
DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

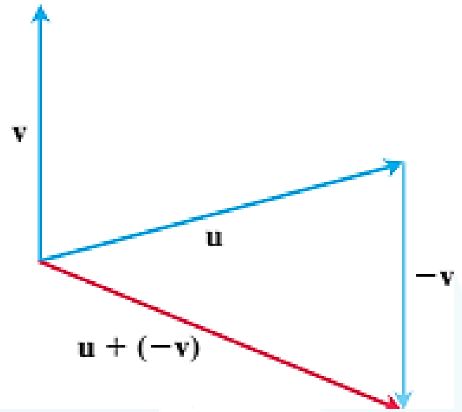
Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$









Vector Algebra Operations

EXAMPLE 3 Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left|\frac{1}{2}\mathbf{u}\right|$.

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left|\frac{1}{2}\mathbf{u}\right| = \left|\left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle\right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$

Vector Algebra Operations

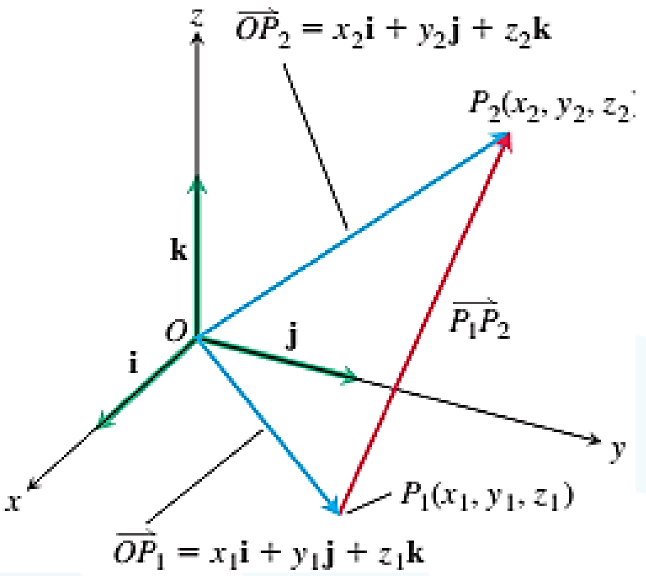
Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}.$$



If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector called the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

EXAMPLE 4 Find a unit vector $\mathbf{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

$$\overrightarrow{P_1P_2} = (3 - 1)\mathbf{i} + (2 - 0)\mathbf{j} + (0 - 1)\mathbf{k} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\mathbf{u} = \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{2\mathbf{i} + 2\mathbf{j} - \mathbf{k}}{3} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}.$$

Unit Vectors

EXAMPLE 5 If $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$ is a velocity vector, express $\mathbf{v}$ as a product of its speed times its direction of motion.

Speed is the magnitude (length) of $\mathbf{v}$:

$$|\mathbf{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9 + 16} = 5.$$

The unit vector $\mathbf{v}/|\mathbf{v}|$ is the direction of $\mathbf{v}$:

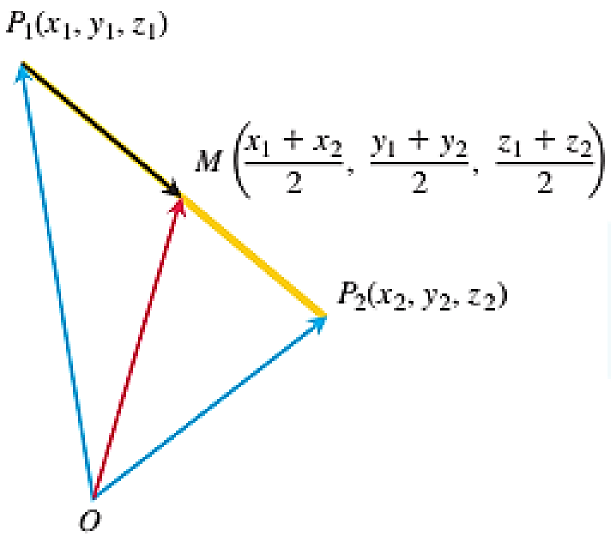
$$\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3\mathbf{i} - 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

$$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} = \underset{\text{Length (speed)}}{5}\underbrace{\left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right)}_{\text{Direction of motion}}.$$

Midpoint of a Line Segment

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

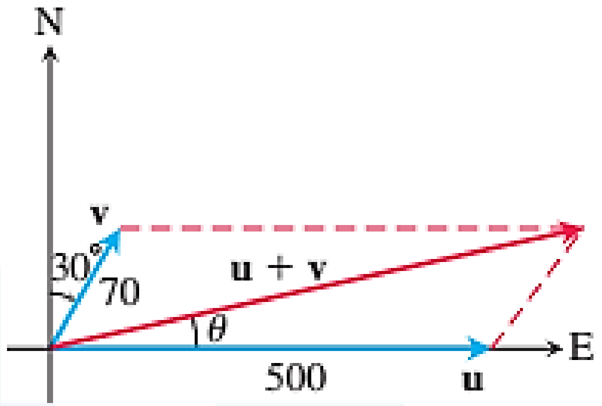


EXAMPLE 7 The midpoint of the segment joining $P_1(3, -2, 0)$ and $P_2(7, 4, 4)$ is

$$\left(\frac{3 + 7}{2}, \frac{-2 + 4}{2}, \frac{0 + 4}{2}\right) = (5, 1, 2).$$

Applications

EXAMPLE 8 A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?



If $\mathbf{u}$ is the velocity of the airplane alone

$\mathbf{v}$ is the velocity of the tailwind,

$$\mathbf{u} = \langle 500, 0 \rangle \qquad \text{and} \qquad \mathbf{v} = \langle 70\cos 60°, 70\sin 60° \rangle = \langle 35, 35\sqrt{3} \rangle$$

$$\mathbf{u} + \mathbf{v} = \langle 535, 35\sqrt{3} \rangle = 535\mathbf{i} + 35\sqrt{3}\,\mathbf{j}$$

$$|\mathbf{u} + \mathbf{v}| = \sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4 \qquad \theta = \tan^{-1}\frac{35\sqrt{3}}{535} \approx 6.5°.$$

The new ground speed of the airplane is about 538.4 mph, and its new direction is about 6.5° north of east. ■

Applications

EXAMPLE 9 A 75-N weight is suspended by two wires, as shown in Figure 12.18a. Find the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in both wires.

$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°,\ |\mathbf{F}_1|\sin 55° \rangle \quad \text{and} \quad \mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°,\ |\mathbf{F}_2|\sin 40° \rangle.$$

$$\mathbf{F}_1 + \mathbf{F}_2 = \langle 0, 75 \rangle \quad \Rightarrow \quad \begin{aligned} -|\mathbf{F}_1|\cos 55° + |\mathbf{F}_2|\cos 40° &= 0 \\ |\mathbf{F}_1|\sin 55° + |\mathbf{F}_2|\sin 40° &= 75 \end{aligned}$$

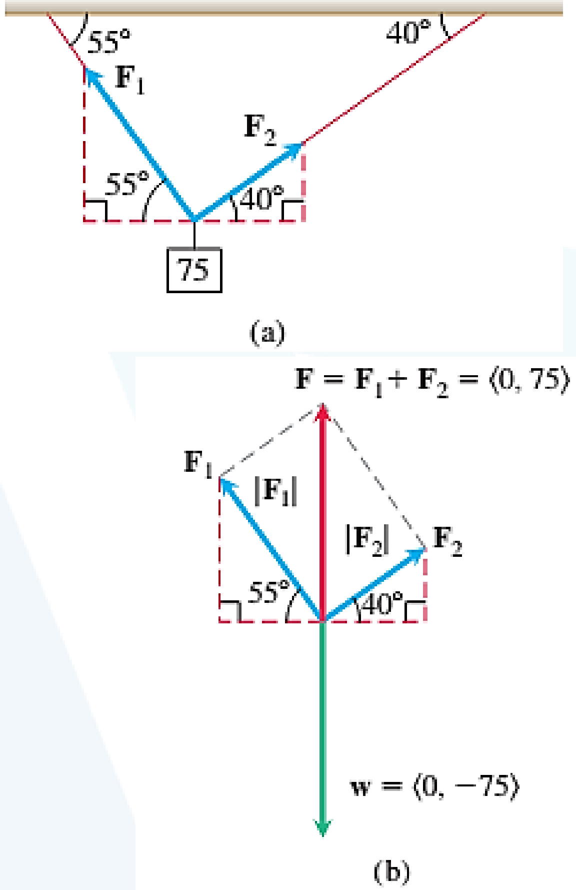
Cramer's Rule $\Rightarrow$

$$|\mathbf{F}_1| \approx 57.67 \text{ N}$$

$$|\mathbf{F}_2| \approx 43.18 \text{ N.}$$

$$\mathbf{F}_1 = \langle -|\mathbf{F}_1|\cos 55°,\ |\mathbf{F}_1|\sin 55° \rangle \approx \langle -33.08, 47.24 \rangle$$

$$\mathbf{F}_2 = \langle |\mathbf{F}_2|\cos 40°,\ |\mathbf{F}_2|\sin 40° \rangle \approx \langle 33.08, 27.76 \rangle.$$

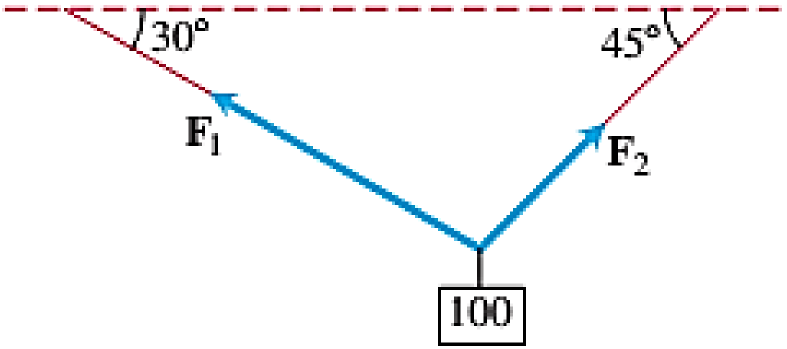


EXERCISES

- **Velocity** An airplane is flying in the direction 25° west of north at 800 km/h. Find the component form of the velocity of the airplane, assuming that the positive x-axis represents due east and the positive y-axis represents due north.

$\langle -338.095, 725.046 \rangle$

- Consider a 100-N weight suspended by two wires as shown in the accompanying figure. Find the magnitudes and components of the force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$.



$|\mathbf{F}_1| = \frac{200}{1+\sqrt{3}} \approx 73.205$ N

$|\mathbf{F}_2| = \frac{100\sqrt{6}}{1+\sqrt{3}} \approx 89.658$ N

$\mathbf{F}_1 \approx \langle -63.397, 36.603 \rangle$ and $\mathbf{F}_2 = \approx \langle 63.397, 63.397 \rangle$

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The Dot Product

DEFINITION The **dot product** $\mathbf{u} \cdot \mathbf{v}$ (“**u** dot **v**”) of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

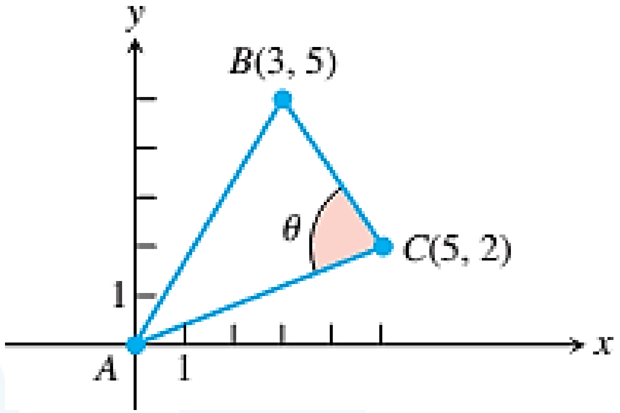
Dot Product and Angles

The angle between two nonzero vectors **u** and **v** is $\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$.

The dot product of two vectors **u** and **v** is given by $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$.

The Dot Product

EXAMPLE 3 Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$, and $C = (5, 2)$ (Figure 12.22).



$$\overrightarrow{CA} = \langle -5, -2 \rangle \quad \text{and} \quad \overrightarrow{CB} = \langle -2, 3 \rangle.$$

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-5)(-2) + (-2)(3) = 4 \qquad |\overrightarrow{CA}| = \sqrt{(-5)^2 + (-2)^2} = \sqrt{29}$$

$$|\overrightarrow{CB}| = \sqrt{(-2)^2 + (3)^2} = \sqrt{13}$$

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}\right) = \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right) \approx 78.1^\circ \text{ or } 1.36 \text{ radians.}$$

The Dot Product

Orthogonal Vectors

DEFINITION Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

Properties of the Dot Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

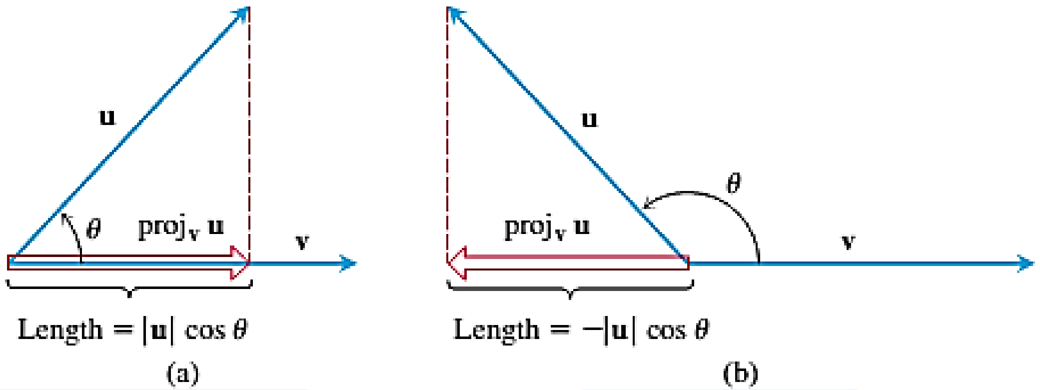
The Dot Product

The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|}. \qquad (1)$$

The scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$|\mathbf{u}| \cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}. \qquad (2)$$



The Dot Product

EXAMPLE 5 Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.$$

We find the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ from Equation (2):

$$|\mathbf{u}| \cos\theta = \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = -\frac{4}{3}.$$

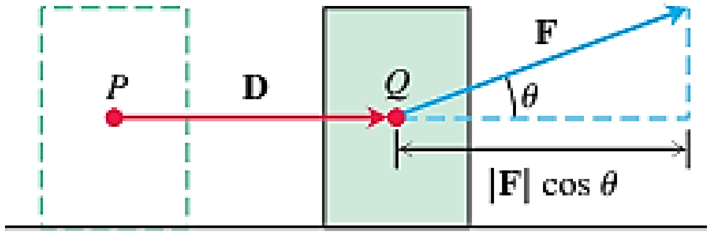
The Dot Product

Work

$$\text{Work} = \begin{pmatrix}\text{scalar component of } \mathbf{F} \\ \text{in the direction of } \mathbf{D}\end{pmatrix}(\text{length of } \mathbf{D})$$

$$= (|\mathbf{F}|\cos\theta)|\mathbf{D}|$$

$$= \mathbf{F}\cdot\mathbf{D}.$$



DEFINITION The **work** done by a constant force $\mathbf{F}$ acting through a displacement $\mathbf{D} = \overrightarrow{PQ}$ is

$$W = \mathbf{F}\cdot\mathbf{D}.$$

The Dot Product

Work

EXAMPLE 8 If $|\mathbf{F}| = 40$ N (newtons), $|\mathbf{D}| = 3$ m, and $\theta = 60°$, the work done by $\mathbf{F}$ in acting from P to Q is

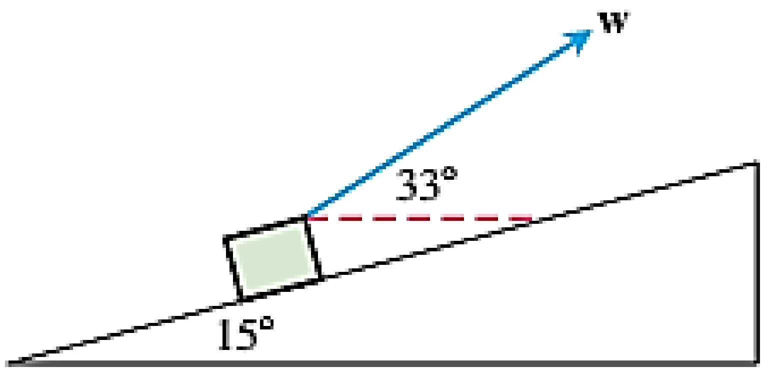
$$\begin{aligned}
\text{Work} &= \mathbf{F} \cdot \mathbf{D} && \text{Definition} \\
&= |\mathbf{F}||\mathbf{D}| \cos\theta \\
&= (40)(3)\cos 60° && \text{Given values} \\
&= (120)(1/2) = 60 \text{ J (joules).}
\end{aligned}$$

EXERCISES

Projectile motion A gun with muzzle velocity of 1200 ft/sec is fired at an angle of 8° above the horizontal. Find the horizontal and vertical components of the velocity.

1188 ft/s 167 ft/s

Inclined plane Suppose that a box is being towed up an inclined plane as shown in the figure. Find the force **w** needed to make the component of the force parallel to the inclined plane equal to 2.5 lb.



$\langle 2.205, 1.432 \rangle$

EXERCISES

Work along a line Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) in moving an object along the line from the origin to the point (1, 1) (distance in meters).

5 J

The Cross Product

DEFINITION The **cross product** $\mathbf{u} \times \mathbf{v}$ ("$\mathbf{u}$ cross $\mathbf{v}$") is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta)\,\mathbf{n}.$$

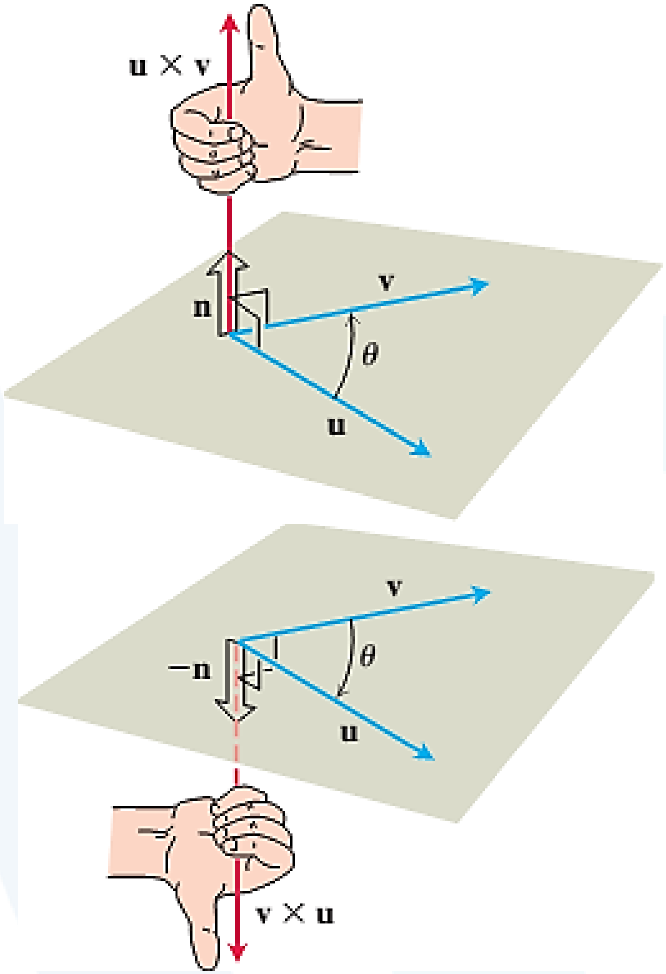
Parallel Vectors

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$



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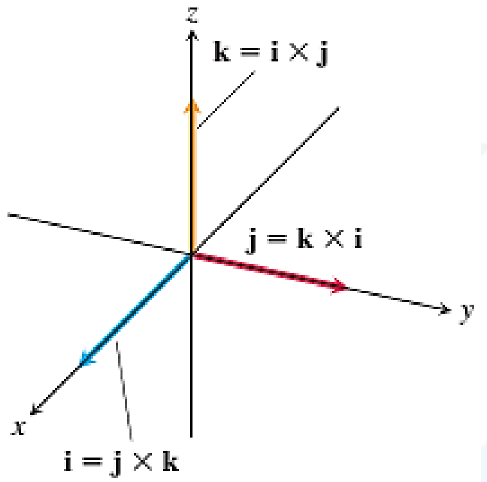
The Cross Product

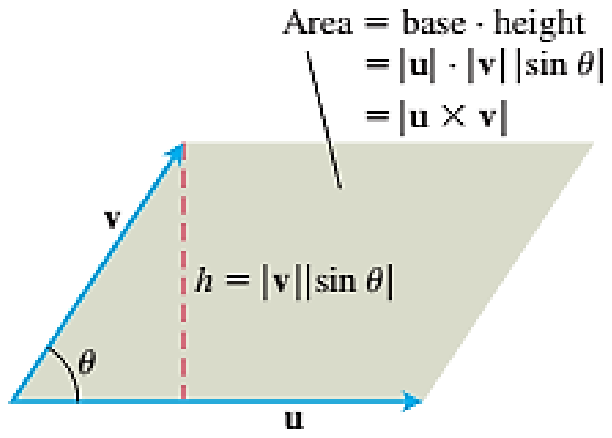
$$\mathbf{i} \times \mathbf{j} = -(\mathbf{j} \times \mathbf{i}) = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = -(\mathbf{k} \times \mathbf{j}) = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = -(\mathbf{i} \times \mathbf{k}) = \mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$



$|\mathbf{u} \times \mathbf{v}|$ Is the Area of a Parallelogram

$$\boxed{|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\,|\sin\theta|\,|\mathbf{n}| = |\mathbf{u}||\mathbf{v}|\sin\theta.}$$



The Cross Product

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

EXAMPLE 1 Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k} = -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

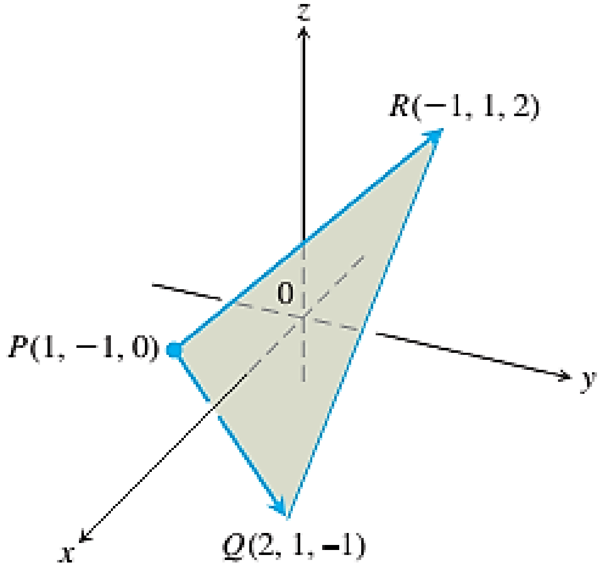
$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

The Cross Product

EXAMPLE 2 Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.32).

$$\overrightarrow{PQ} = (2-1)\mathbf{i} + (1+1)\mathbf{j} + (-1-0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$
$$\overrightarrow{PR} = (-1-1)\mathbf{i} + (1+1)\mathbf{j} + (2-0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix}\mathbf{k}$$
$$= 6\mathbf{i} + 6\mathbf{k}.$$



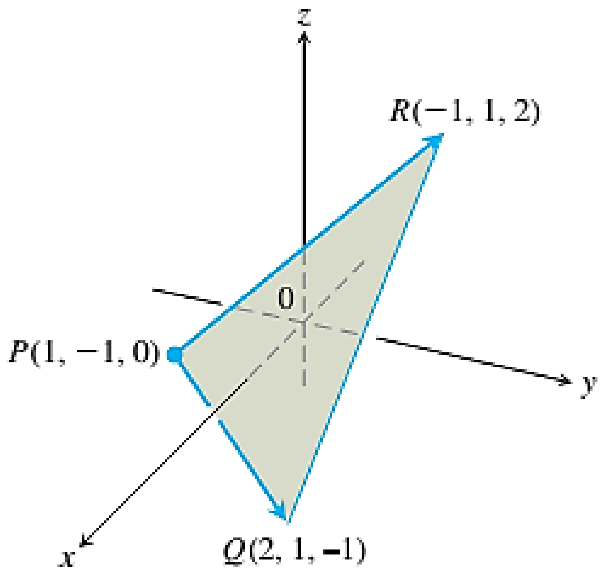
The Cross Product

EXAMPLE 3 Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ (Figure 12.32).

The area of the parallelogram determined by P, Q, and R is

$$\begin{aligned} |\overrightarrow{PQ} \times \overrightarrow{PR}| &= |6\mathbf{i} + 6\mathbf{k}| \\ &= \sqrt{(6)^2 + (6)^2} = \sqrt{2 \cdot 36} = 6\sqrt{2}. \end{aligned}$$

The triangle's area is half of this, or $3\sqrt{2}$.



EXAMPLE 4 Find a unit vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

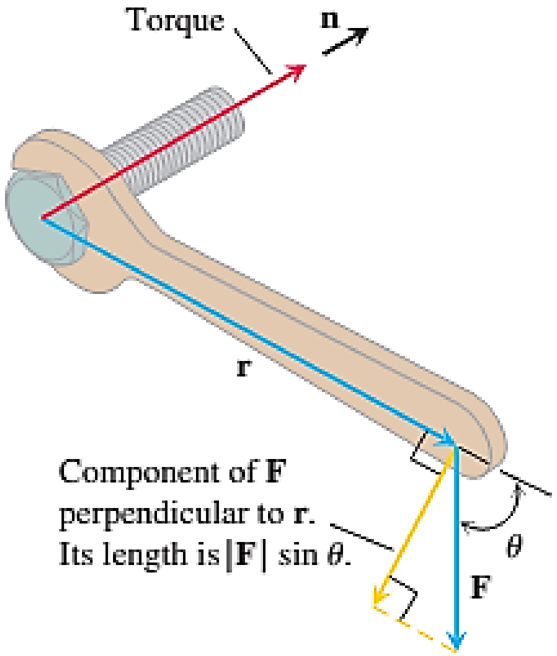
$$\mathbf{n} = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6\mathbf{i} + 6\mathbf{k}}{6\sqrt{2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}.$$

The Cross Product

Torque

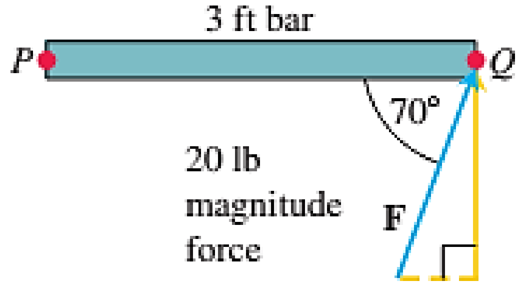
Magnitude of torque vector $= |\mathbf{r}||\mathbf{F}|\sin\theta,$

Torque vector $= \mathbf{r} \times \mathbf{F} = (|\mathbf{r}||\mathbf{F}|\sin\theta)\,\mathbf{n}.$



EXAMPLE 5 The magnitude of the torque generated by force **F** at the pivot point P in Figure 12.34 is

$$|\overrightarrow{PQ} \times \mathbf{F}| = |\overrightarrow{PQ}||\mathbf{F}|\sin 70^\circ \approx (3)(20)(0.94) \approx 56.4 \text{ ft-lb}.$$

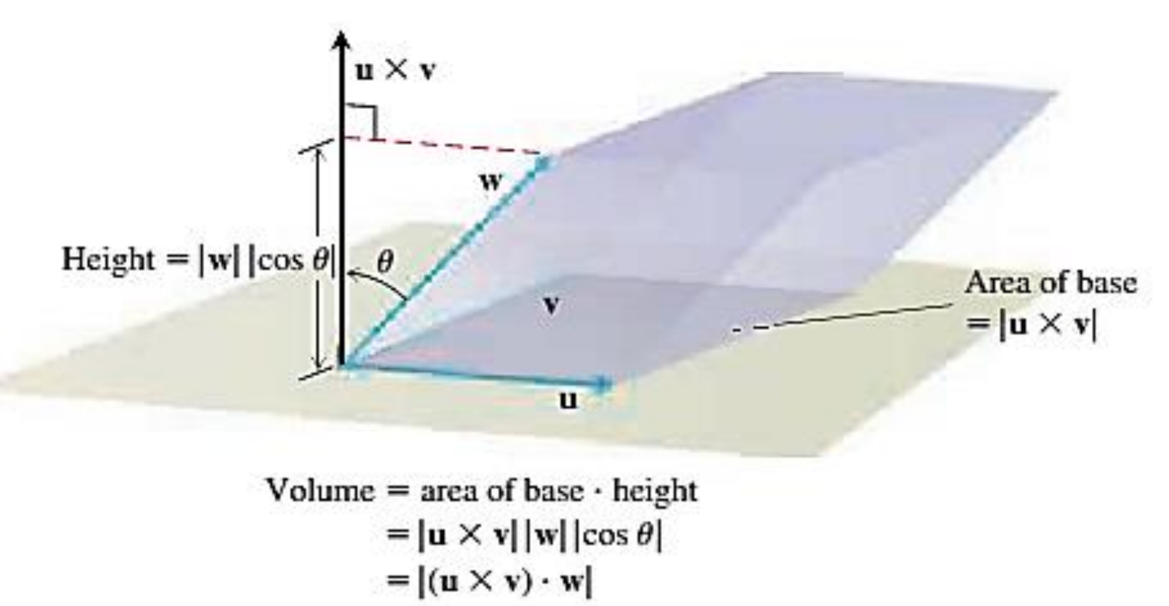


The Cross Product

Triple Scalar or Box Product

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



The Cross Product

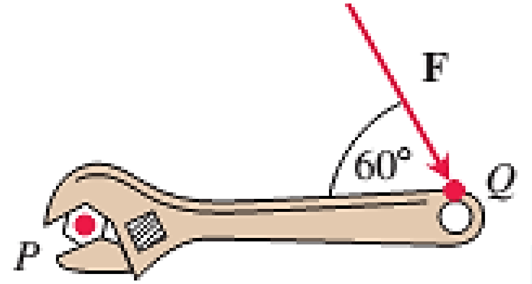
EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - (2)\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} + (-1)\begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -23$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.

EXERCISES

find the magnitude of the torque exerted by **F** on the bolt at P if $|\overline{PQ}| = 8$ in. and $|\mathbf{F}| = 30$ lb. Answer in foot-pounds.



$10\sqrt{3}$ ft · lb

Find the volume of a parallelepiped if four of its eight vertices are $A(0, 0, 0)$, $B(1, 2, 0)$, $C(0, -3, 2)$, and $D(3, -4, 5)$.

5

Determine whether the given points are coplanar.

$A(0, 1, 2)$, $B(-1, 1, 0)$, $C(2, 0, -1)$, $D(1, -1, 1)$

Lines and Planes in Space

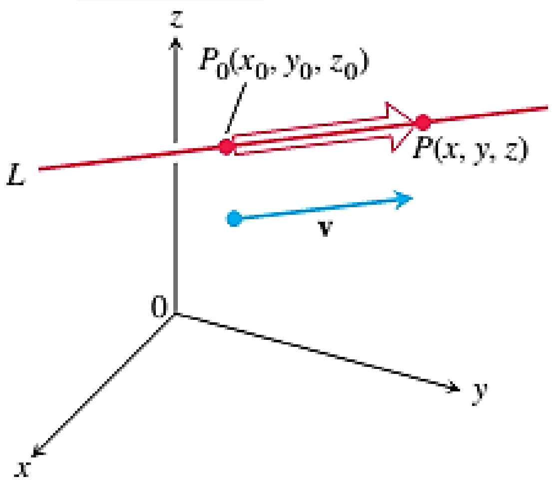
Lines and Line Segments in Space

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to v is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.



Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$

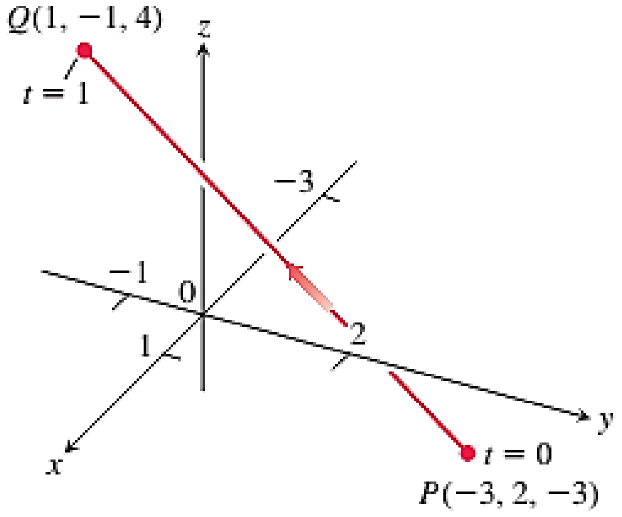
Lines and Planes in Space

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

$$\overrightarrow{PQ} = (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} = 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}$$

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.38).



$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \le t \le 1.$$

Lines and Planes in Space

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

$$= \underbrace{\mathbf{r}_0}_{\text{Initial position}} + \underbrace{t}_{\text{Time}}\underbrace{|\mathbf{v}|}_{\text{Speed}}\underbrace{\frac{\mathbf{v}}{|\mathbf{v}|}}_{\text{Direction}}.$$

EXAMPLE 4 A helicopter is to fly directly from a helipad at the origin in the direction of the point (1, 1, 1) at a speed of 60 ft/sec. What is the position of the helicopter after 10 sec?

Then the unit vector $\mathbf{u} = \frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}$

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\text{speed})\mathbf{u} = \mathbf{0} + t(60)\left(\frac{1}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{1}{\sqrt{3}}\mathbf{k}\right) = 20\sqrt{3}t(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

$$\mathbf{r}(10) = 200\sqrt{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$= \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3}\rangle$$

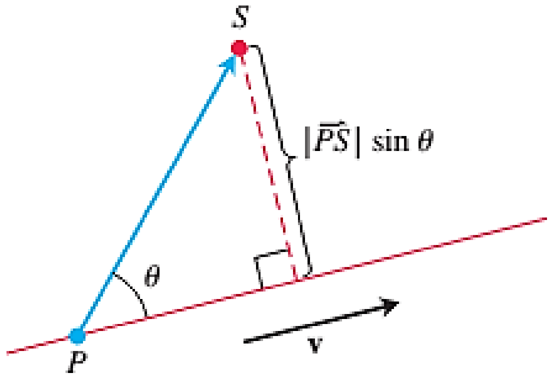
$\Rightarrow$ $|r(10)| = 600\text{ft}$

Lines and Planes in Space

The Distance from a Point to a Line in Space

Distance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \qquad (5)$$



EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

$P(1, 3, 0)$ $\qquad$ $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$$\overrightarrow{PS} = (1-1)\mathbf{i} + (1-3)\mathbf{j} + (5-0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k} \qquad \overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Lines and Planes in Space

An Equation for a Plane in Space

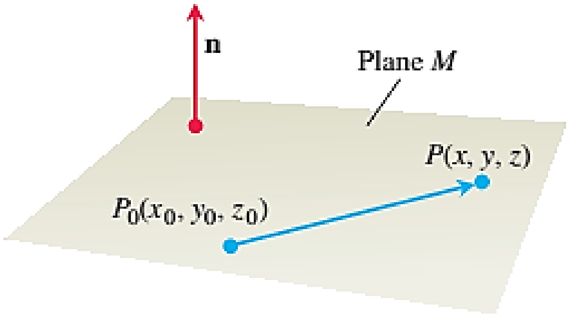
Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$



EXAMPLE 6 Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\mathbf{n} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0.$$

$$5x + 2y - z = -22$$

An Equation for a Plane in Space

EXAMPLE 7 Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$, and $C(0, 3, 0)$.

The normal

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$$

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

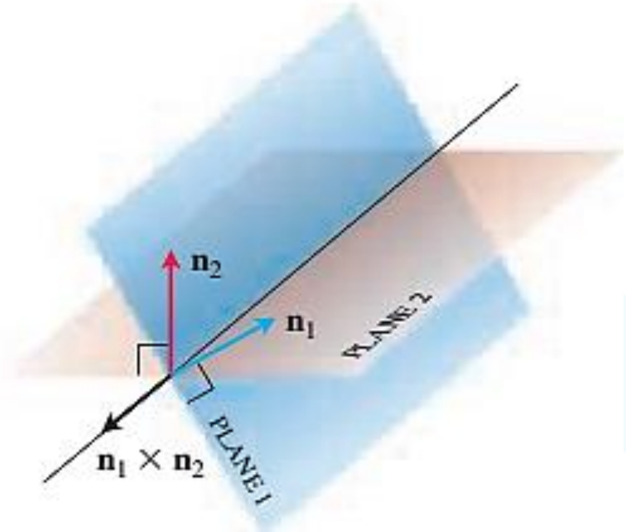
$$3x + 2y + 6z = 6.$$

Lines and Planes in Space

Lines of Intersection

EXAMPLE 8 Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\mathbf{i} + 2\mathbf{j} + 15\mathbf{k}$$



EXAMPLE 9 Find parametric equations for the line in which the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ intersect.

Finding point common to the two planes

$$3x - 6y - 2z = 15$$
$$2x + y - 2z = 5$$

$z = 0$ $\Rightarrow$ $(3, -1, 0)$

$$x = 3 + 14t, \qquad y = -1 + 2t, \qquad z = 15t.$$

Lines and Planes in Space

Lines of Intersection

EXAMPLE 10 Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

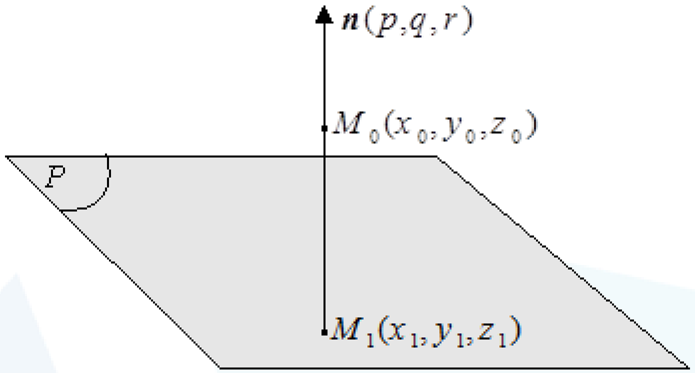
$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1 + t) = 6$$

$$t = -1$$

The point of intersection is $(x, y, z)\big|_{t=-1} = \left(\frac{8}{3} - 2, 2, 1 - 1\right) = \left(\frac{2}{3}, 2, 0\right)$

Lines and Planes in Space

The Distance from a Point to a Plane



$$|\overrightarrow{M_1M_0}\cdot\boldsymbol{n}| = |\overrightarrow{M_1M_0}|\cdot|\boldsymbol{n}| = d\cdot|\boldsymbol{n}|$$

$$d = \frac{\left|\overrightarrow{M_1M_0}\cdot\boldsymbol{n}\right|}{|\boldsymbol{n}|} = \left|\frac{px_0+qy_0+rz_0-(px_1+qy_1+rz_1)}{\sqrt{p^2+q^2+r^2}}\right|$$

$$d = \frac{\left|\overrightarrow{M_1M_0}\cdot\boldsymbol{n}\right|}{|\boldsymbol{n}|} = \left|\frac{px_0+qy_0+rz_0+h}{\sqrt{p^2+q^2+r^2}}\right|$$

$$d = \frac{|\boldsymbol{P}(x_0,y_0,z_0)|}{\sqrt{p^2+q^2+r^2}}$$

Lines and Planes in Space

The Distance from a Point to a Plane

EXAMPLE 11 Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

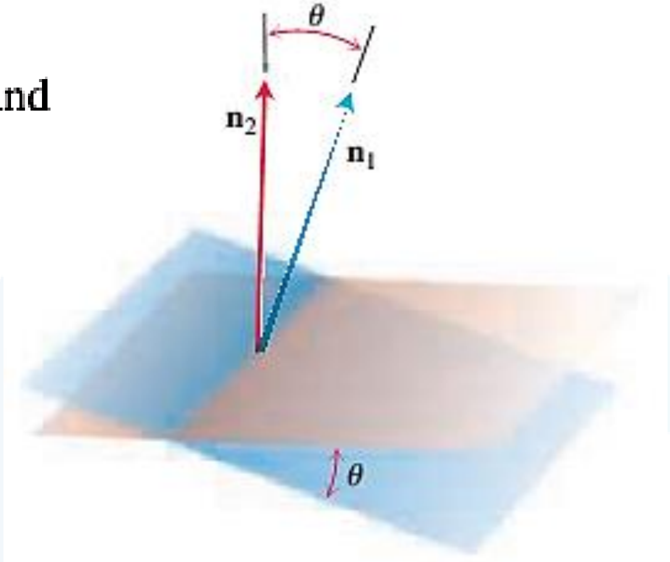
$$\boldsymbol{P}(x, y, z) \equiv 3x + 2y + 6z - 6 = 0 \qquad \boldsymbol{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}$$

$$d = \frac{|\boldsymbol{P}(1,1,3)|}{\sqrt{9+4+36}} = \left|\frac{3(1)+2(1)+6(3)-6}{\sqrt{49}}\right| = \frac{17}{7}$$

Lines and Planes in Space

Angles Between Planes

EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.



$$\mathbf{n}_1 = 3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}, \qquad \mathbf{n}_2 = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) = \cos^{-1}\left(\frac{4}{21}\right) \approx 1.38 \text{ radians.}$$

EXERCISES

Find equations for the planes

The plane through $(1, -1, 3)$ parallel to the plane

$$3x + y + z = 7$$

$3x + y + z = 5$

The plane through $(1, 1, -1)$, $(2, 0, 2)$, and $(0, -2, 1)$

$7x - 5y - 4z = 6$

Find the point of intersection of the lines $x = 2t + 1$, $y = 3t + 2$, $z = 4t + 3$, and $x = s + 2$, $y = 2s + 4$, $z = -4s - 1$, and then find the plane determined by these lines.

$P(1, 2, 3)$

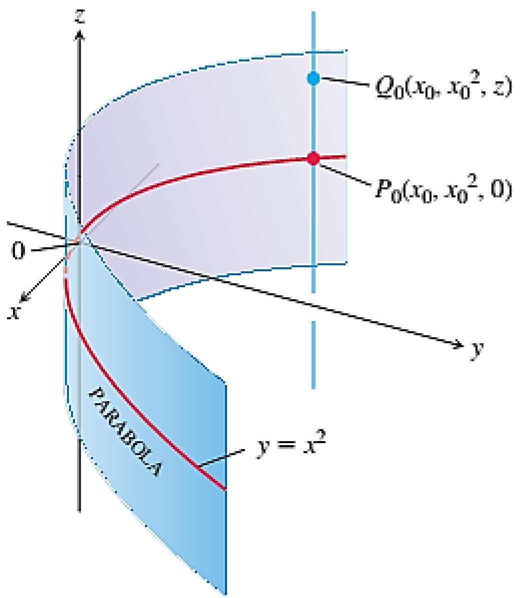
$-20x + 12y + z = 7.$

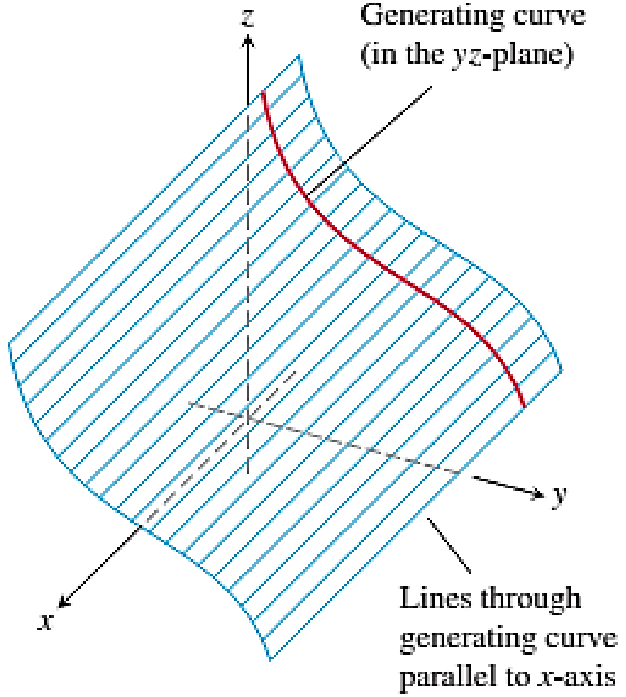
Find the distance from the plane $x + 2y + 6z = 1$ to the plane $x + 2y + 6z = 10$.

$\frac{9}{\sqrt{41}}$

Cylinders and Quadric Surfaces

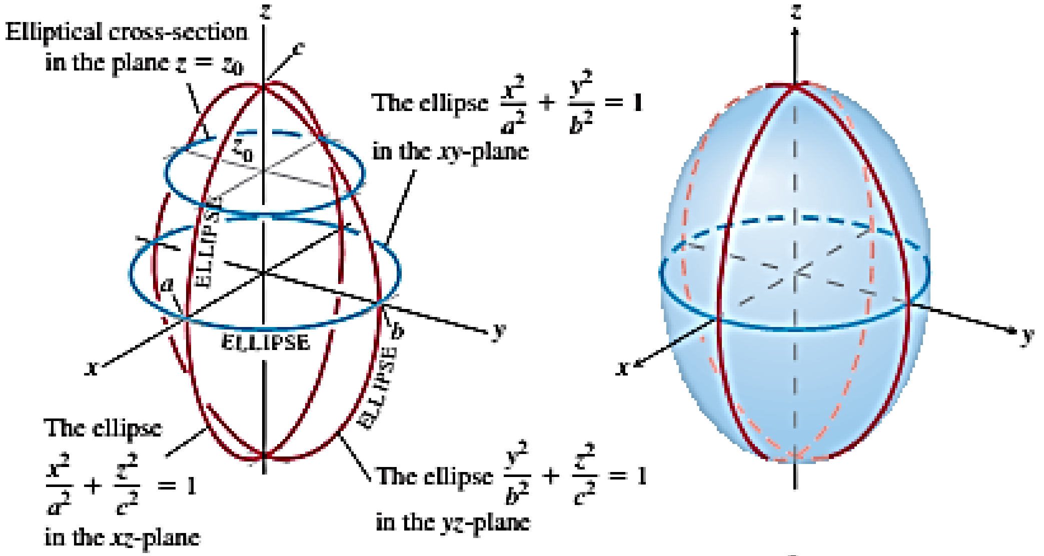
EXAMPLE 1 Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2$, $z = 0$ (Figure 12.45).



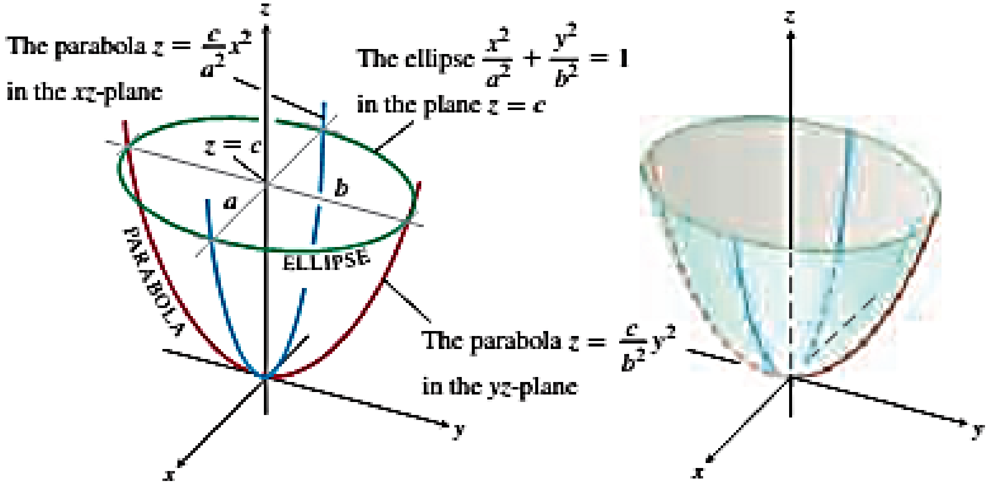


Cylinders and Quadric Surfaces

$$Ax^2 + By^2 + Cz^2 + Dz = E,$$

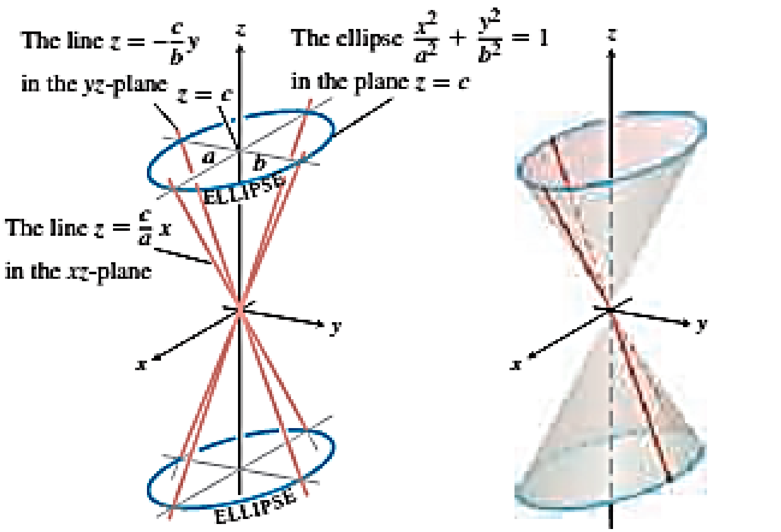


ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

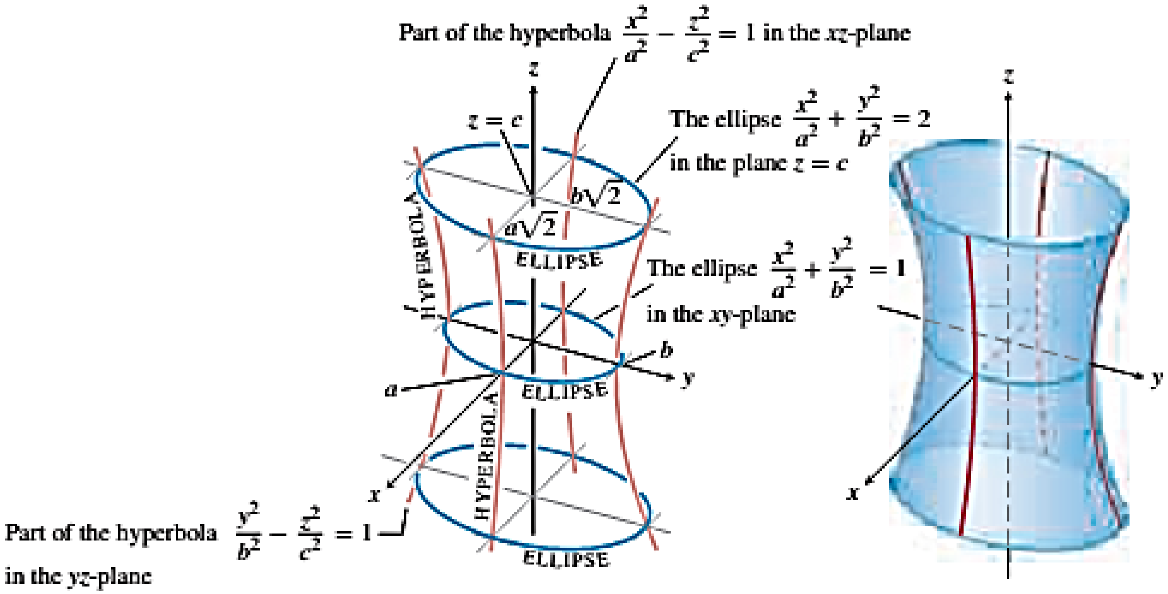


ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$

Cylinders and Quadric Surfaces

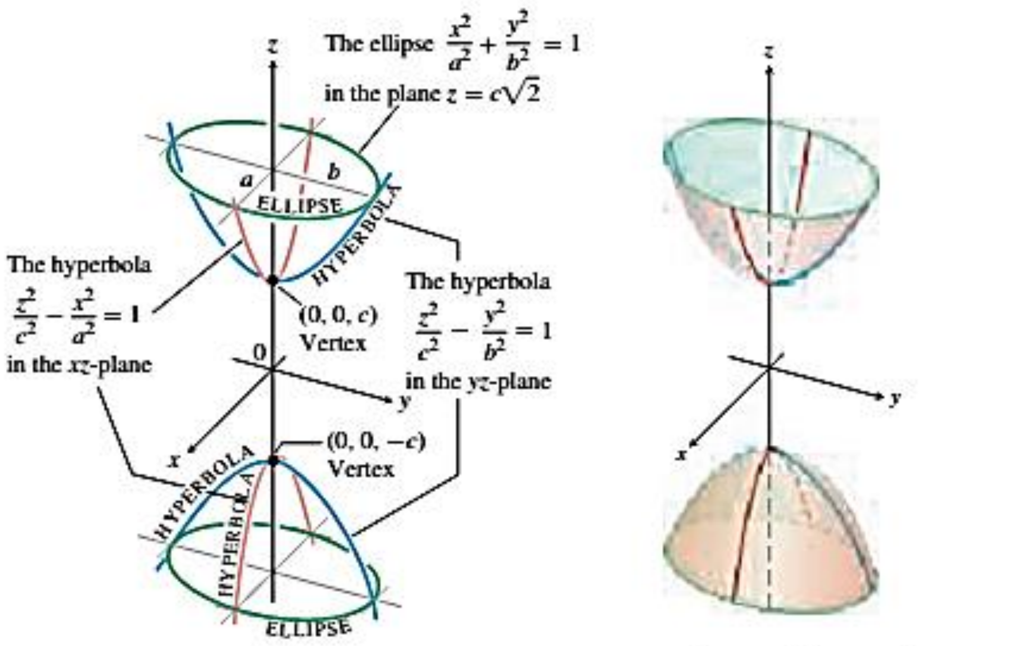


ELLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

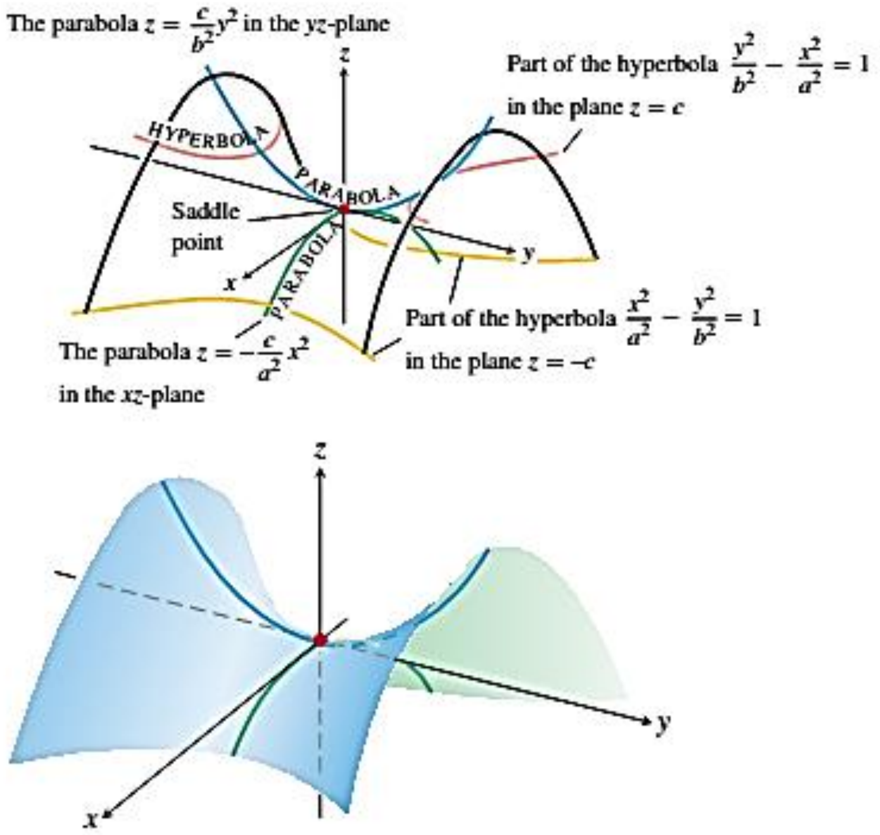


HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Cylinders and Quadric Surfaces



HYPERBOLOID OF TWO SHEETS $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, c > 0$

General Quadric Surfaces

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gz + Hy + Iz + J = 0,$$

EXAMPLE 4 Identify the surface given by the equation

$$x^2 + y^2 + 4z^2 - 2x + 4y + 1 = 0.$$

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{4} + \frac{z^2}{1} = 1.$$