



Calculus 1

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Calculus 1

Lecture 3

Limits and Continuity

Chapter 2

Limits and Continuity

2.1 Limit of a Function and Limit Laws

2.2 One-Sided Limits

2.3 Continuity

**2.4 Limits Involving Infinity; Asymptotes of
Graph**



Limits of Function Values

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

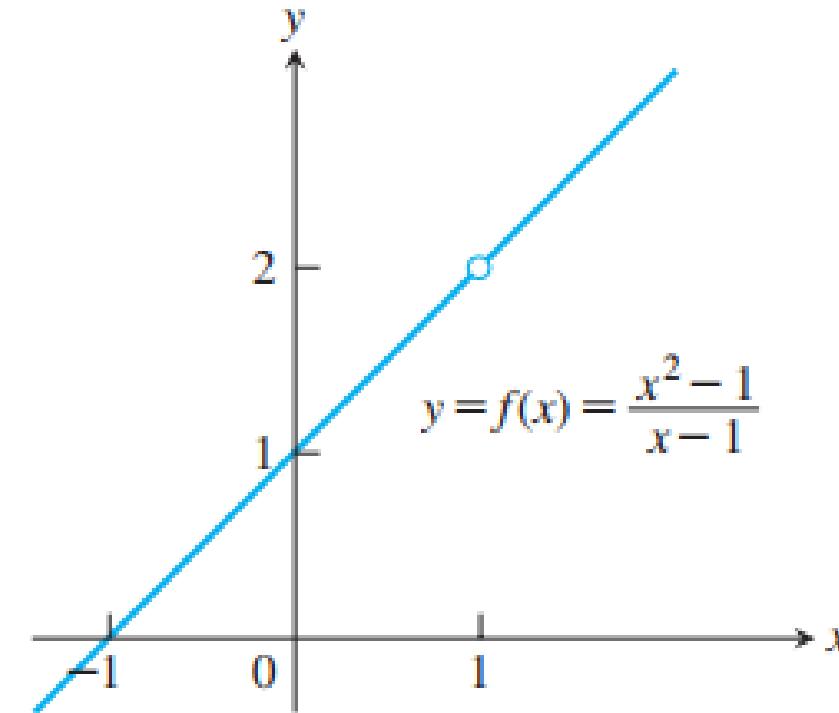
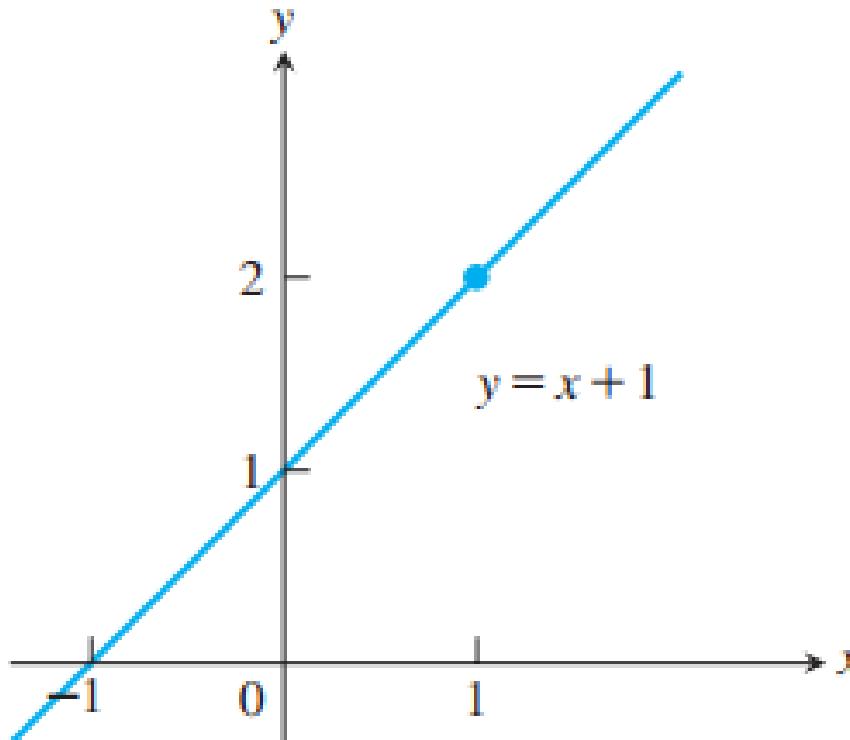
behave near $x = 1$?

Solution

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \quad \text{for } x \neq 1.$$



Limit of a Function and Limit Laws



Limit of a Function and Limit Laws

itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c , other than c itself, then we say that f approaches the **limit** L as x approaches c , and write

$$\lim_{x \rightarrow c} f(x) = L,$$

which is read “the limit of $f(x)$ as x approaches c is L .” In Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \rightarrow 1} f(x) = 2, \quad \text{or} \quad \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = 2.$$

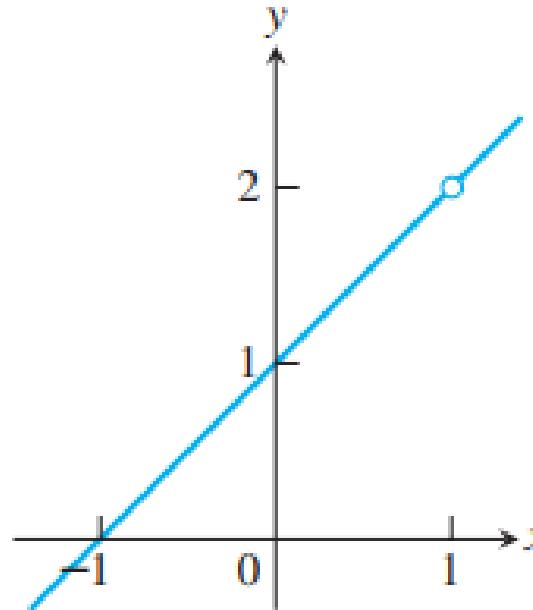
TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001



Limit of a Function and Limit Laws

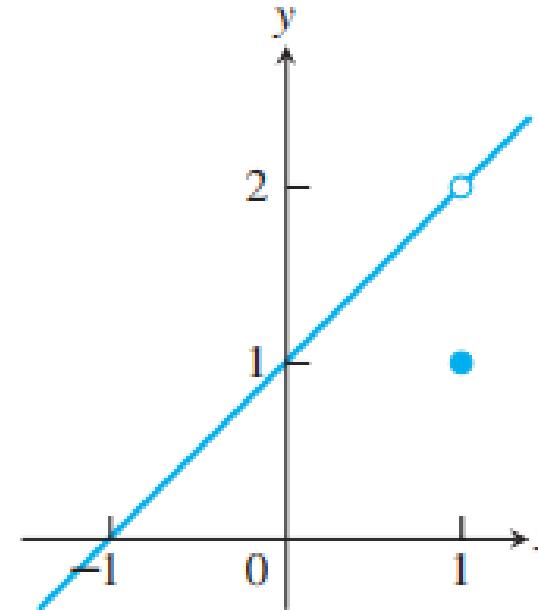
EXAMPLE 2



$$(a) f(x) = \frac{x^2 - 1}{x - 1}$$

$$\lim_{x \rightarrow 1} f(x) = 2$$

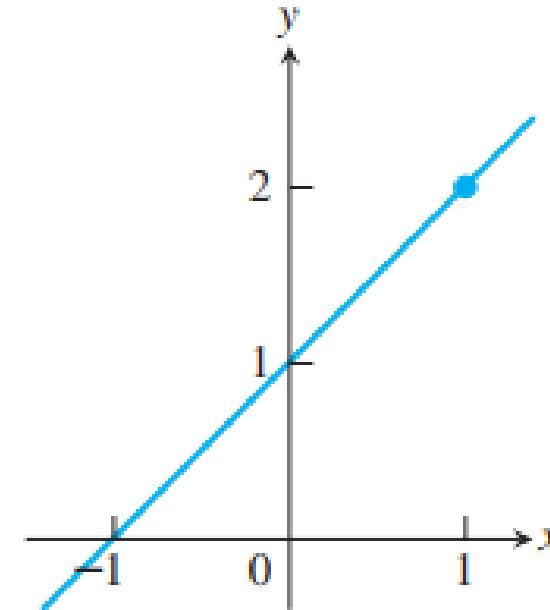
غير موجودة $f(1)$



$$(b) g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

$$\lim_{x \rightarrow 1} g(x) = 2$$

$$g(1) = 1$$



$$(c) h(x) = x + 1$$

$$\lim_{x \rightarrow 1} h(x) = 2$$

$$h(1) = 2$$



Limit of a Function and Limit Laws

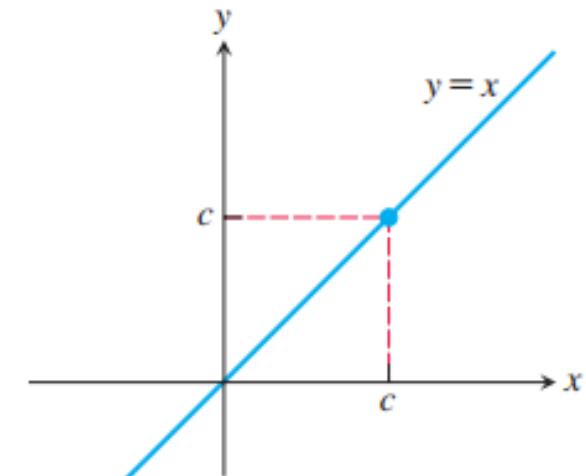
EXAMPLE 3

(a) If f is the **identity function** $f(x) = x$,

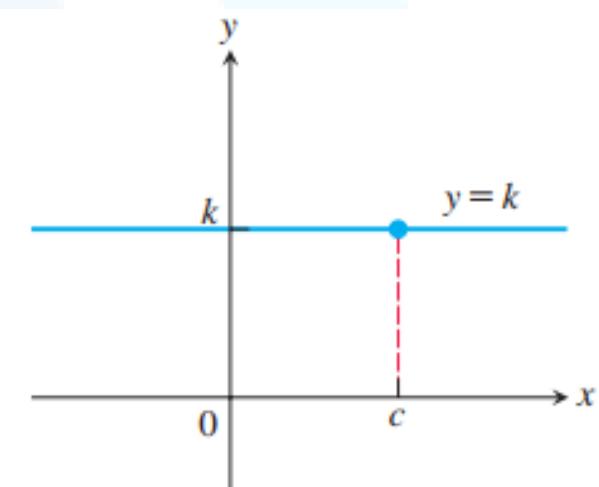
$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x = c.$$

(b) If f is the **constant function** $f(x) = k$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} k = k.$$

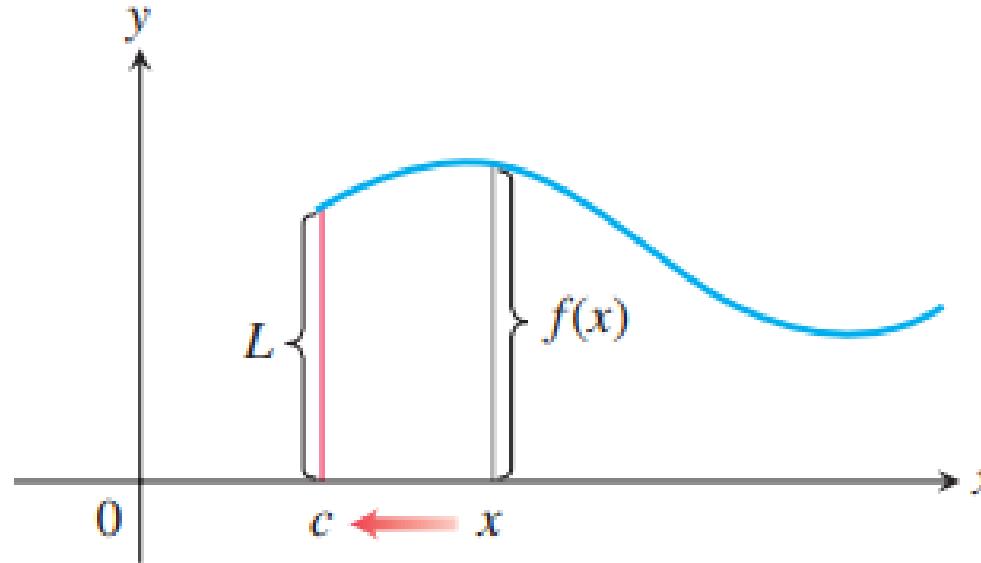


(a) Identity function



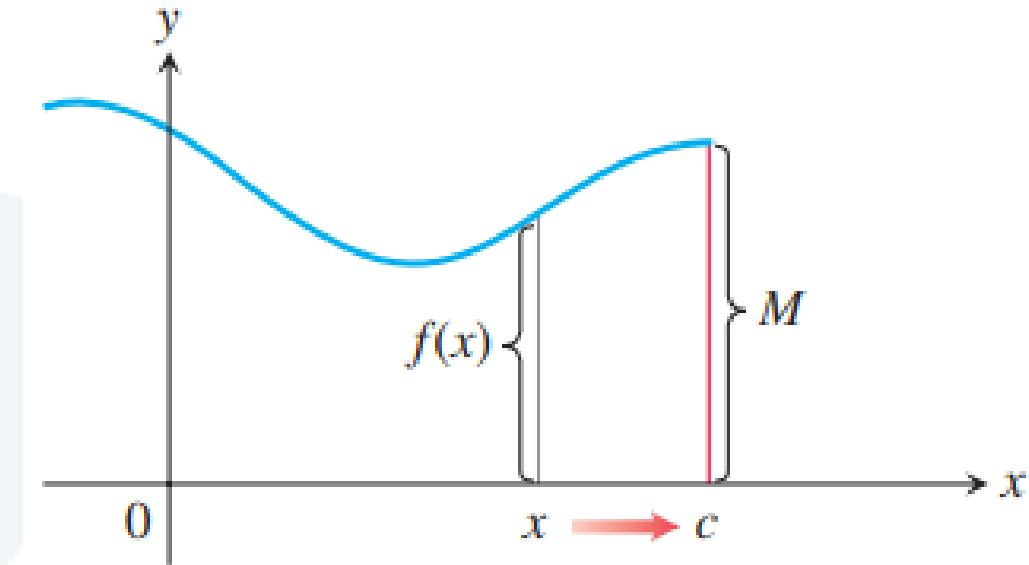


Approaching a Limit from One Side



$$(a) \lim_{x \rightarrow c^+} f(x) = L$$

(a) Right-hand limit as x approaches c .



$$(b) \lim_{x \rightarrow c^-} f(x) = M$$

(b) Left-hand limit as x approaches c .

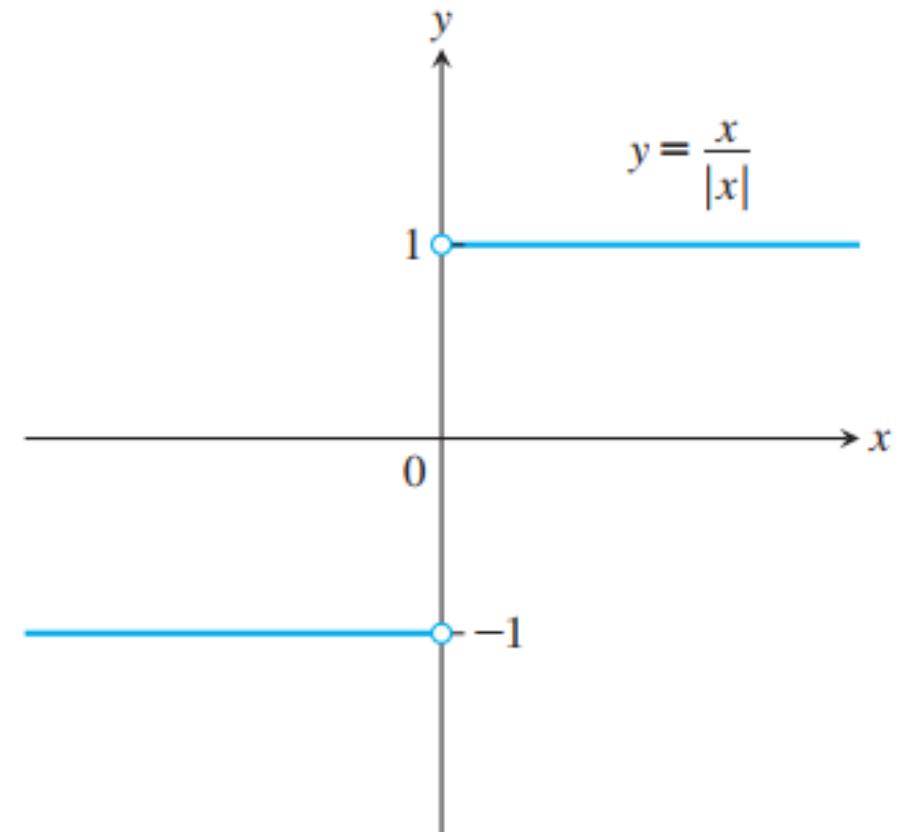


Approaching a Limit from One Side

EXAMPLE

$$f(x) = x/|x|$$

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0^-} f(x) = -1.$$

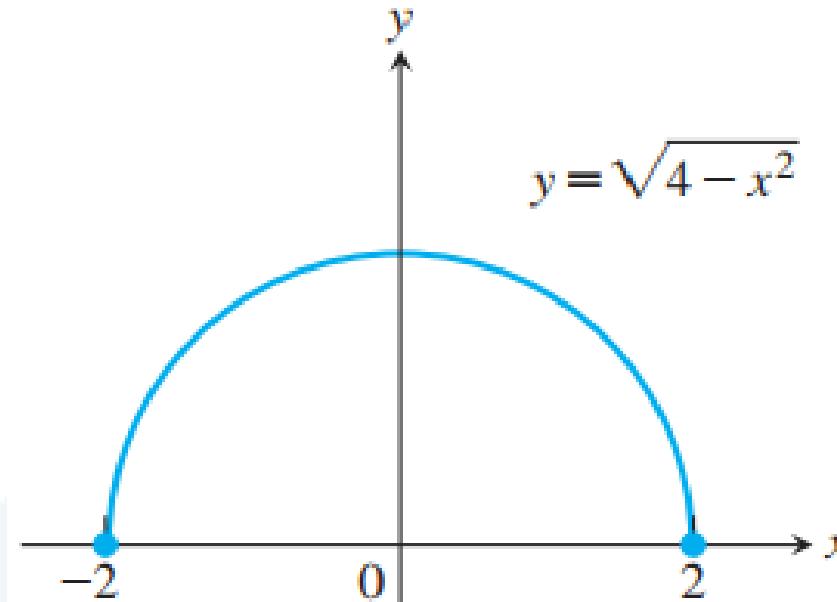




Approaching a Limit from One Side

EXAMPLE : The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x \rightarrow -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \rightarrow 2^-} \sqrt{4 - x^2} = 0.$$



Approaching a Limit from One Side

THEOREM 6 Suppose that a function f is defined on an open interval containing c , except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \rightarrow c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \rightarrow c^-} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c^+} f(x) = L.$$



Limit of a Function and Limit Laws

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \rightarrow 0$.

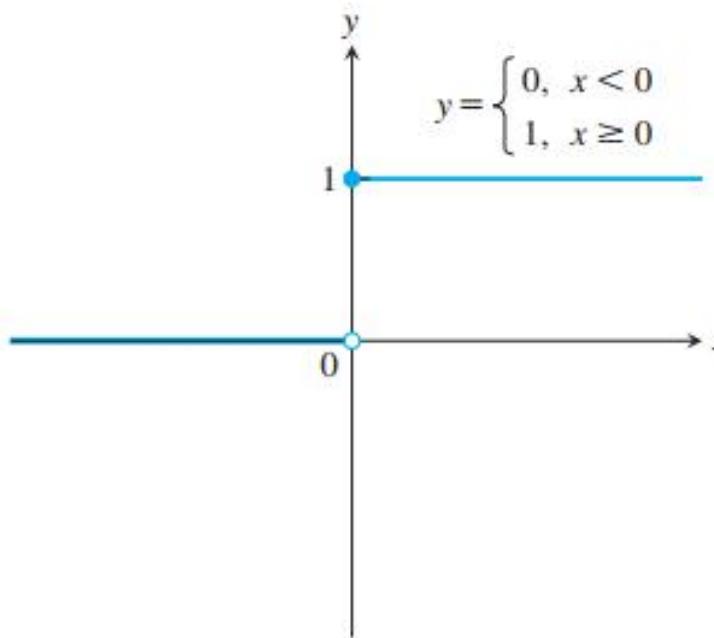
$$(a) U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

$$(b) g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$(c) f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \frac{1}{x}, & x > 0 \end{cases}$$



Limit of a Function and Limit Laws



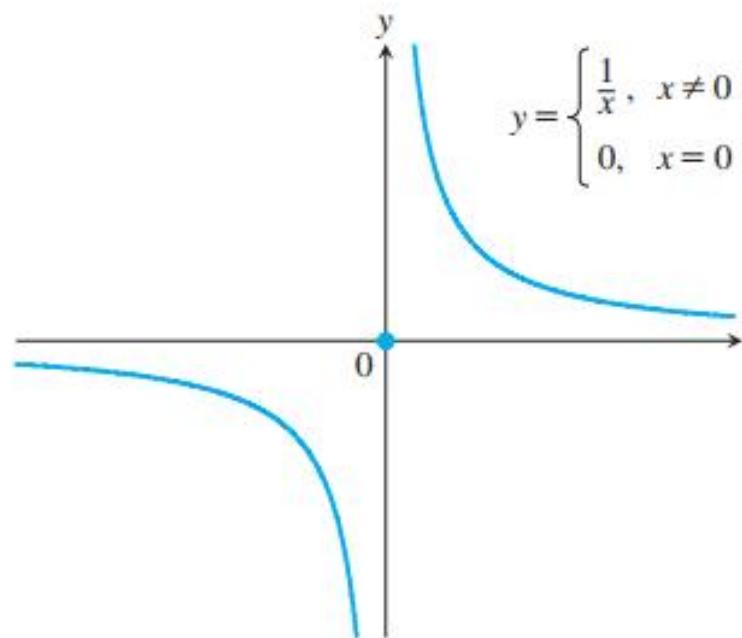
(a) Unit step function $U(x)$

$$\lim_{x \rightarrow 0^-} U(x) = 0$$

$$\lim_{x \rightarrow 0^+} U(x) = 1$$

$$\lim_{x \rightarrow 0^-} U(x) \neq \lim_{x \rightarrow 0^+} U(x)$$

النهاية غير موجودة



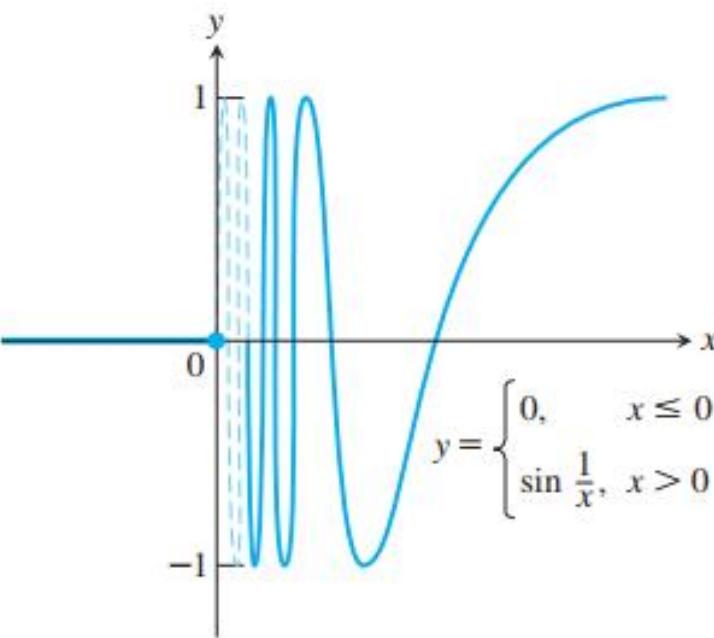
(b) $g(x)$

$$\lim_{x \rightarrow 0^-} g(x) = -\infty$$

$$\lim_{x \rightarrow 0^+} g(x) = +\infty$$

$$\lim_{x \rightarrow 0^-} g(x) \neq \lim_{x \rightarrow 0^+} g(x)$$

النهاية غير موجودة



(c) $f(x)$

$$\lim_{x \rightarrow 0^-} f(x) = 0$$

$$\text{النهاية غير موجودة} \quad \lim_{x \rightarrow 0^+} f(x)$$

$$-1 \leq \sin x \leq +1 \quad \text{النهاية غير موجودة}$$

the function's values between +1 and -1 in every open interval containing 0



THEOREM 1—Limit Laws

If L, M, c , and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:*

$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$$

2. *Difference Rule:*

$$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$$

3. *Constant Multiple Rule:*

$$\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$$

4. *Product Rule:*

$$\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$$

5. *Quotient Rule:*

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

6. *Power Rule:*

$$\lim_{x \rightarrow c} [f(x)]^n = L^n, \quad n \text{ a positive integer}$$

7. *Root Rule:*

$$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}, \quad n \text{ a positive integer}$$

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c .)



Limit of a Function and Limit Laws

EXAMPLE 5

(a) $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3)$

(b) $\lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x \rightarrow -2} \sqrt{4x^2 - 3}$

Solution

(a) $\lim_{x \rightarrow c} (x^3 + 4x^2 - 3) = \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3$
 $= c^3 + 4c^2 - 3$

(b) $\lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)}$
 $= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5}$
 $= \frac{c^4 + c^2 - 1}{c^2 + 5}$

(c) $\lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)}$
 $= \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3}$
 $= \sqrt{4(-2)^2 - 3}$
 $= \sqrt{16 - 3}$
 $= \sqrt{13}$



Evaluating Limits of Polynomials and Rational Functions

THEOREM 2—Limits of Polynomials

If $P(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_0$, then

$$\lim_{x \rightarrow c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6

$$\lim_{x \rightarrow -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$



Evaluating Limits

EXAMPLE 7 Evaluate

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \quad \text{if } x \neq 1.$$

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$



Evaluating Limits

EXAMPLE 8

Estimate the value of $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.

$$\begin{aligned}\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\&= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} \\&= \frac{1}{\sqrt{x^2 + 100} + 10}.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\&= \frac{1}{\sqrt{0^2 + 100} + 10} \\&= \frac{1}{20} = 0.05.\end{aligned}$$

Limit Quotient
not 0 at $x = 0$



The Sandwich Theorem

THEOREM 4 – The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c , except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L.$$

Then $\lim_{x \rightarrow c} f(x) = L$.

EXAMPLE 10

Given a function u that satisfies

$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad \text{for all } x \neq 0,$$

find $\lim_{x \rightarrow 0} u(x)$, no matter how complicated u is.



The Sandwich Theorem

Solution Since

$$\lim_{x \rightarrow 0} (1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} (1 + (x^2/2)) = 1,$$

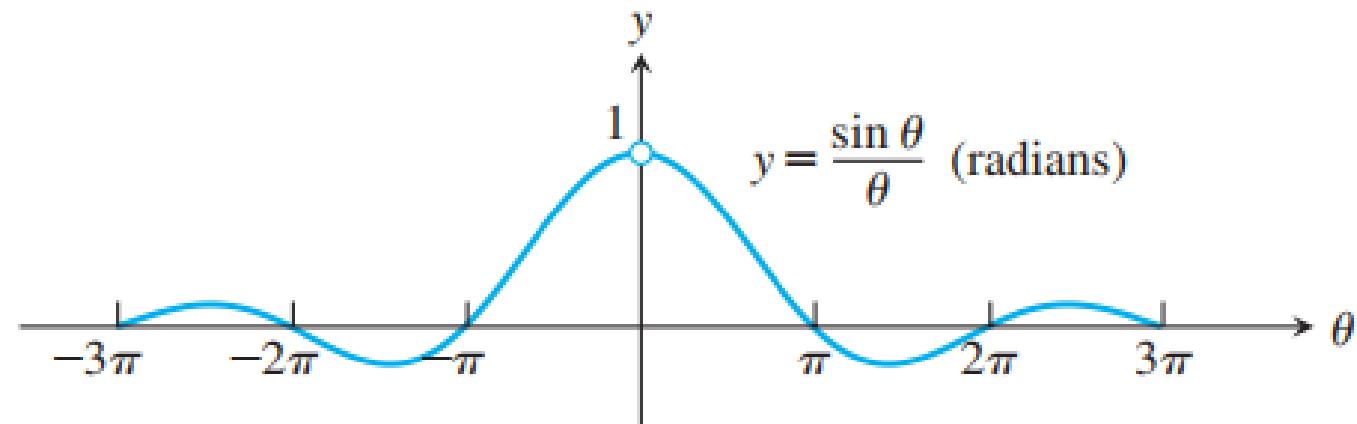
the Sandwich Theorem implies that $\lim_{x \rightarrow 0} u(x) = 1$



Limits Involving $(\sin x)/x$

THEOREM 7—Limit of the Ratio $\sin \theta/\theta$ as $\theta \rightarrow 0$

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians}) \quad (1)$$





Limits Involving $(\sin x)/x$

EXAMPLE Show that (a) $\lim_{y \rightarrow 0} \frac{\cos y - 1}{y} = 0$ and (b) $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2 \sin^2(y/2)$, we calculate

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\cos y - 1}{y} &= \lim_{h \rightarrow 0} -\frac{2 \sin^2(y/2)}{y} \\ &= -\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \sin \theta \quad \text{Let } \theta = y/2. \\ &= -(1)(0) = 0.\end{aligned}$$

Eq. (1) and Example 11a
in Section 2.2



Limits Involving $(\sin x)/x$

- (b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} &= \lim_{x \rightarrow 0} \frac{(2/5) \cdot \sin 2x}{(2/5) \cdot 5x} \\&= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} && \text{Eq. (1) applies with } \theta = 2x. \\&= \frac{2}{5}(1) = \frac{2}{5}.\end{aligned}$$



Limits Involving $(\sin x)/x$

EXAMPLE

Find $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\begin{aligned}\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} &= \lim_{t \rightarrow 0} \frac{1}{3} \cdot \frac{1}{t} \cdot \frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t} \\&= \frac{1}{3} \lim_{t \rightarrow 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \\&= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.\end{aligned}$$



EXAMPLE

Show that for nonzero constants A and B .

Solution

$$\begin{aligned}\lim_{\theta \rightarrow 0} \frac{\sin A\theta}{\sin B\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \frac{A\theta}{\sin B\theta} \frac{B\theta}{B\theta} \frac{1}{B\theta} \\&= \lim_{\theta \rightarrow 0} \frac{\sin A\theta}{A\theta} \frac{B\theta}{\sin B\theta} \frac{A}{B} \\&= \lim_{\theta \rightarrow 0} (1)(1) \frac{A}{B} \\&= \frac{A}{B}.\end{aligned}$$



Continuity at a Point

$$\lim_{x \rightarrow 1^-} f(x) = 0 \quad \lim_{x \rightarrow 1^+} f(x) = 1 \quad f(1) = 1$$

so the function is continuous from the right at $x = 1$.

$$\lim_{x \rightarrow 2} f(x) = 1 \quad f(2) = 2.$$

f is not continuous at $x = 2$.

$$\lim_{x \rightarrow 4^-} f(x) = 1 \quad f(4) = \frac{1}{2}$$

the function is not continuous from the left.

$$\lim_{x \rightarrow 3} f(x) = 2. \quad f(3) = 2.$$

The function is continuous at $x = 3$

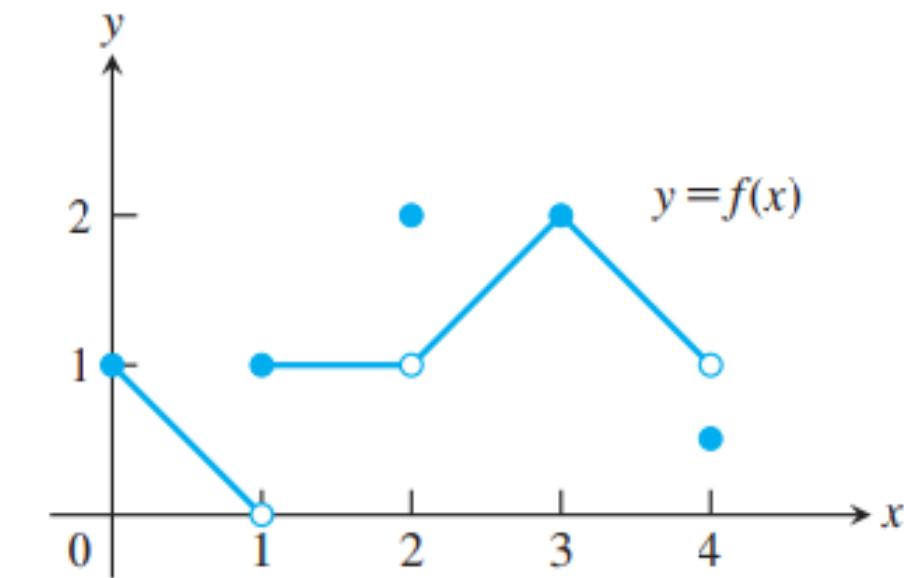


FIGURE : The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

$$\lim_{x \rightarrow 0^+} f(x) = 1 \quad f(0) = 1$$

The function is continuous from the right at $x = 0$.

DEFINITIONS Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f .

The function f is **continuous at c** if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x \rightarrow c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x \rightarrow c^-} f(x) = f(c).$$



Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \rightarrow c} f(x)$ exists (f has a limit as $x \rightarrow c$).
3. $\lim_{x \rightarrow c} f(x) = f(c)$ (the limit equals the function value).



THEOREM – Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

- | | |
|-------------------------------|--|
| 1. <i>Sums:</i> | $f + g$ |
| 2. <i>Differences:</i> | $f - g$ |
| 3. <i>Constant multiples:</i> | $k \cdot f$, for any number k |
| 4. <i>Products:</i> | $f \cdot g$ |
| 5. <i>Quotients:</i> | f/g , provided $g(c) \neq 0$ |
| 6. <i>Powers:</i> | f^n , n a positive integer |
| 7. <i>Roots:</i> | $\sqrt[n]{f}$, provided it is defined on an interval containing c , where n is a positive integer |



EXAMPLE

- (a) Every polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \rightarrow c} P(x) = P(c)$ by Theorem 2, Section 2.2.
- (b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2.

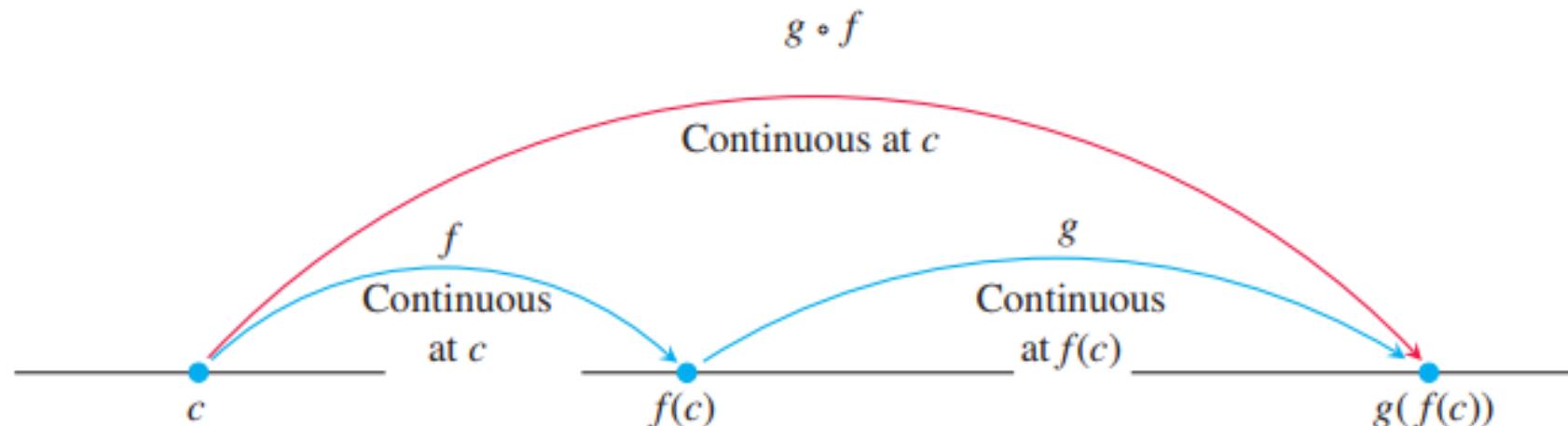
EXAMPLE The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \rightarrow 0} |x| = 0 = |0|$.



Continuity of Compositions of Functions

THEOREM C – Compositions of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c .





Continuity of Compositions of Functions

EXAMPLE 8

Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$

(b) $y = \frac{x^{2/3}}{1 + x^4}$

(c) $y = \left| \frac{x - 2}{x^2 - 2} \right|$

(d) $y = \left| \frac{x \sin x}{x^2 + 2} \right|$



Continuity of Compositions of Functions

Solution

- (a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7, Theorem 8). The given function is then the composition of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.
- (b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.
- (c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).
- (d) Because the sine function is everywhere-continuous (Exercise 64), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function



Continuity of Compositions of Functions

THEOREM 10—Limits of Continuous Functions

If $\lim_{x \rightarrow c} f(x) = b$ and g is continuous at the point b , then

$$\lim_{x \rightarrow c} g(f(x)) = g(b).$$

EXAMPLE 9 Applying Theorem 10, we have

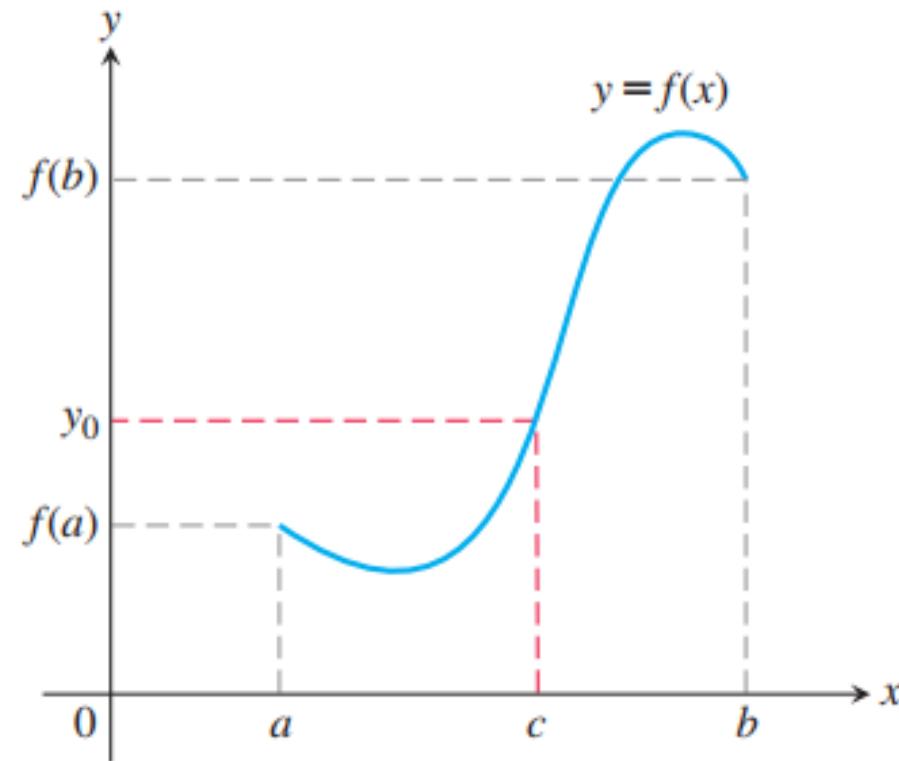
$$\begin{aligned}\lim_{x \rightarrow \pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) &= \cos\left(\lim_{x \rightarrow \pi/2} 2x + \lim_{x \rightarrow \pi/2} \sin\left(\frac{3\pi}{2} + x\right)\right) \\ &= \cos(\pi + \sin 2\pi) = \cos \pi = -1.\end{aligned}$$



Intermediate Value Theorem for Continuous Functions

THEOREM 11 – The Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

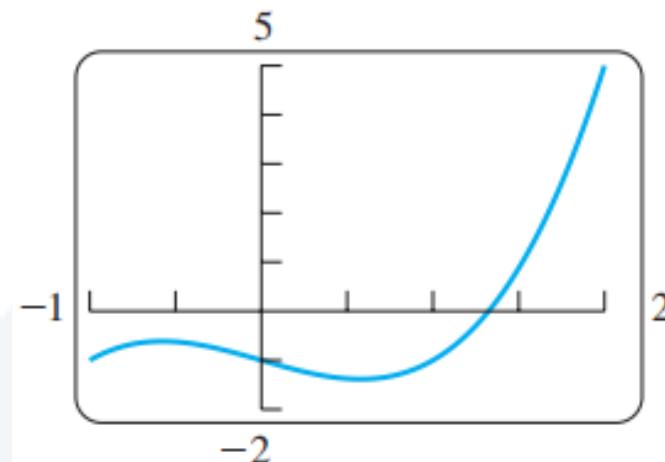




Intermediate Value Theorem for Continuous Functions

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.45 shows the result of zooming in to locate the root near $x = 1.32$.





EXAMPLE 11

Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

Solution We rewrite the equation as

$$\sqrt{2x + 5} + x^2 - 4 = 0,$$

and set $f(x) = \sqrt{2x + 5} + x^2 - 4$. Now $g(x) = \sqrt{2x + 5}$ is continuous on the interval $[-5/2, \infty)$ since it is formed as the composition of two continuous functions, the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2 - 4$, and the quadratic function is continuous for all values of x . It follows that $f(x) = \sqrt{2x + 5} + x^2 - 4$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} - 4 \approx -1.76$ and $f(2) = \sqrt{9} = 3$. Note that f is continuous on the finite closed interval $[0, 2] \subset [-5/2, \infty)$. Since the value $y_0 = 0$ is between the numbers $f(0) = -1.76$ and $f(2) = 3$, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 0$. The number c solves the original equation. ■



Limits Involving Infinity; Asymptotes of Graphs

Finite Limits as $x \rightarrow \pm\infty$

EXAMPLE 2 The properties in Theorem 12 are used to calculate limits in the same way as when x approaches a finite number c .

$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) &= \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} && \text{Sum Rule} \\ &= 5 + 0 = 5 && \text{Known limits} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \lim_{x \rightarrow -\infty} \frac{\pi\sqrt{3}}{x^2} &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \frac{1}{x} \cdot \frac{1}{x} && \text{Product Rule} \\ &= \lim_{x \rightarrow -\infty} \pi\sqrt{3} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \cdot \lim_{x \rightarrow -\infty} \frac{1}{x} \\ &= \pi\sqrt{3} \cdot 0 \cdot 0 = 0 && \text{Known limits} \end{aligned}$$



Limits Involving Infinity; Asymptotes of Graphs

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

(a)
$$\lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x \rightarrow \infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$

Divide numerator and denominator by x^3 .

See Fig. 2.51.

(b)
$$\lim_{x \rightarrow -\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x \rightarrow -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$

$$= \frac{0 + 0}{2 - 0} = 0$$

Divide numerator and denominator by x^3 .

See Fig. 2.52.





Horizontal Asymptotes

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = b \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = b.$$

EXAMPLE 4

Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$



Horizontal Asymptotes

Solution We calculate the limits as $x \rightarrow \pm\infty$.

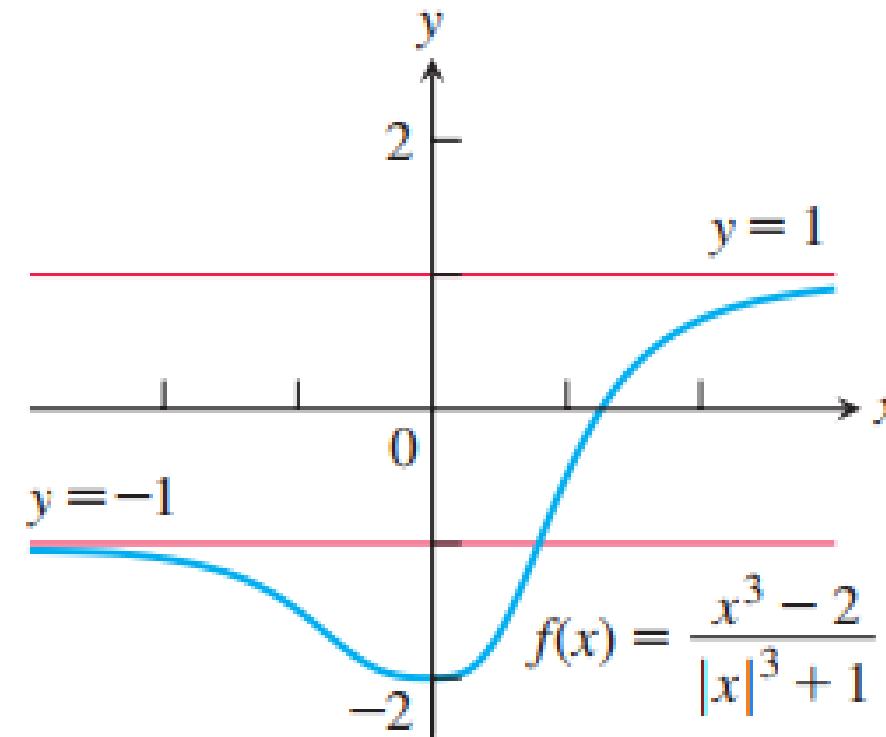
$$\text{For } x \geq 0: \lim_{x \rightarrow \infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow \infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x \rightarrow \infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$$

$$\text{For } x < 0: \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x \rightarrow -\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x \rightarrow -\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.53. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x . ■



Horizontal Asymptotes



Horizontal Asymptotes

EXAMPLE 5

Find (a) $\lim_{x \rightarrow \infty} \sin(1/x)$ and (b) $\lim_{x \rightarrow \pm\infty} x \sin(1/x)$.

$$\lim_{x \rightarrow \infty} \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \rightarrow \infty$ and $x \rightarrow -\infty$:

$$\lim_{x \rightarrow \infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x \rightarrow -\infty} x \sin \frac{1}{x} = \lim_{t \rightarrow 0^-} \frac{\sin t}{t} = 1.$$



Horizontal Asymptotes

EXAMPLE 7

Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. Since

$$0 \leq \left| \frac{\sin x}{x} \right| \leq \left| \frac{1}{x} \right|$$

and $\lim_{x \rightarrow \pm\infty} |1/x| = 0$, we have $\lim_{x \rightarrow \pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x \rightarrow \pm\infty} \left(2 + \frac{\sin x}{x} \right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.56).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■



Oblique Asymptotes

EXAMPLE 9 Find the oblique asymptote of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

Solution We are interested in the behavior as $x \rightarrow \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{)x^2 + 0x - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1 \right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4} \right)}_{\text{remainder}}.$$

$$g(x) = \frac{x}{2} + 1$$



Infinite Limits

EXAMPLE 10

Find $\lim_{x \rightarrow 1^+} \frac{1}{x - 1}$ and $\lim_{x \rightarrow 1^-} \frac{1}{x - 1}$.

$$\lim_{x \rightarrow 1^+} \frac{1}{x - 1} = \infty \quad \text{and} \quad \lim_{x \rightarrow 1^-} \frac{1}{x - 1} = -\infty.$$

EXAMPLE 13

Find $\lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7}$.

$$\begin{aligned}\lim_{x \rightarrow -\infty} \frac{2x^5 - 6x^4 + 1}{3x^2 + x - 7} &= \lim_{x \rightarrow -\infty} \frac{2x^3 - 6x^2 + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= \lim_{x \rightarrow -\infty} \frac{2x^2(x - 3) + x^{-2}}{3 + x^{-1} - 7x^{-2}} \\ &= -\infty,\end{aligned}$$



Vertical Asymptotes

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \rightarrow a^-} f(x) = \pm\infty.$$

EXAMPLE 16 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Vertical Asymptotes

Solution We are interested in the behavior as $x \rightarrow \pm\infty$ and as $x \rightarrow \pm 2$, where the denominator is zero. Notice that f is an even function of x , so its graph is symmetric with respect to the y -axis.

- (a) *The behavior as $x \rightarrow \pm\infty$.* Since $\lim_{x \rightarrow \infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.65). Notice that the curve approaches the x -axis from only the negative side (or from below). Also, $f(0) = 2$.
- (b) *The behavior as $x \rightarrow \pm 2$.* Since

$$\lim_{x \rightarrow 2^+} f(x) = -\infty \quad \text{and} \quad \lim_{x \rightarrow 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

Exercises

Graph the functions in Exercises 9 and 10. Then answer these questions.

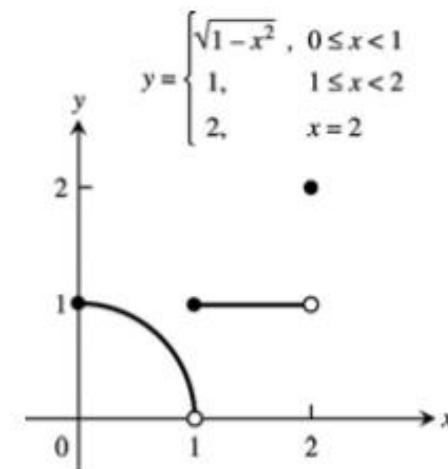
- What are the domain and range of f ?
- At what points c , if any, does $\lim_{x \rightarrow c} f(x)$ exist?
- At what points does the left-hand limit exist but not the right-hand limit?
- At what points does the right-hand limit exist but not the left-hand limit?

$$9. \quad f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \\ 2, & x = 2 \end{cases}$$

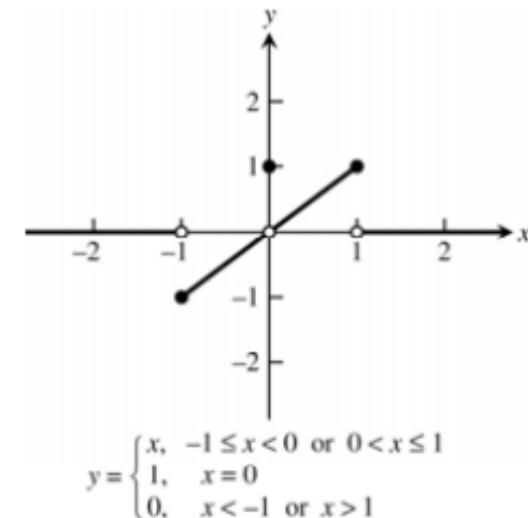
$$10. \quad f(x) = \begin{cases} x, & -1 \leq x < 0, \text{ or } 0 < x \leq 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$$

Solution

9. (a) domain: $0 \leq x \leq 2$
range: $0 < y \leq 1$ and $y = 2$
- (b) $\lim_{x \rightarrow c} f(x)$ exists for c belonging to $(0, 1) \cup (1, 2)$
- (c) $x = 2$
- (d) $x = 0$



10. (a) domain: $-\infty < x < \infty$
range: $-1 \leq y \leq 1$
- (b) $\lim_{x \rightarrow c} f(x)$ exists for c belonging to
 $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
- (c) none
- (d) none



Exercises Find the limits in Exercises

$$14. \lim_{x \rightarrow 1^-} \left(\frac{1}{x+1} \right) \left(\frac{x+6}{x} \right) \left(\frac{3-x}{7} \right)$$

$$15. \lim_{h \rightarrow 0^+} \frac{\sqrt{h^2 + 4h + 5} - \sqrt{5}}{h}$$

$$16. \lim_{h \rightarrow 0^-} \frac{\sqrt{6} - \sqrt{5h^2 + 11h + 6}}{h}$$

$$17. \text{ a. } \lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} \quad \text{ b. } \lim_{x \rightarrow -2^-} (x+3) \frac{|x+2|}{x+2}$$



Solution

$$14. \lim_{x \rightarrow 1^-} \left(\frac{1}{x+1} \right) \left(\frac{x+6}{x} \right) \left(\frac{3-x}{7} \right) = \left(\frac{1}{1+1} \right) \left(\frac{1+6}{1} \right) \left(\frac{3-1}{7} \right) = \left(\frac{1}{2} \right) \left(\frac{7}{1} \right) \left(\frac{2}{7} \right) = 1$$

$$\begin{aligned} 15. \lim_{h \rightarrow 0^+} \frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} &= \lim_{h \rightarrow 0^+} \left(\frac{\sqrt{h^2+4h+5}-\sqrt{5}}{h} \right) \left(\frac{\sqrt{h^2+4h+5}+\sqrt{5}}{\sqrt{h^2+4h+5}+\sqrt{5}} \right) = \lim_{h \rightarrow 0^+} \frac{(h^2+4h+5)-5}{h(\sqrt{h^2+4h+5}+\sqrt{5})} \\ &= \lim_{h \rightarrow 0^+} \frac{h(h+4)}{h(\sqrt{h^2+4h+5}+\sqrt{5})} = \frac{0+4}{\sqrt{5}+\sqrt{5}} = \frac{2}{\sqrt{5}} \end{aligned}$$

$$\begin{aligned} 16. \lim_{h \rightarrow 0^-} \frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} &= \lim_{h \rightarrow 0^-} \left(\frac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h} \right) \left(\frac{\sqrt{6}+\sqrt{5h^2+11h+6}}{\sqrt{6}+\sqrt{5h^2+11h+6}} \right) \\ &= \lim_{h \rightarrow 0^-} \frac{6-(5h^2+11h+6)}{h(\sqrt{6}+\sqrt{5h^2+11h+6})} = \lim_{h \rightarrow 0^-} \frac{-h(5h+11)}{h(\sqrt{6}+\sqrt{5h^2+11h+6})} = \frac{-(0+11)}{\sqrt{6}+\sqrt{6}} = -\frac{11}{2\sqrt{6}} \end{aligned}$$

$$\begin{aligned} 17. (a) \lim_{x \rightarrow -2^+} (x+3) \frac{|x+2|}{x+2} &= \lim_{x \rightarrow -2^+} (x+3) \frac{(x+2)}{(x+2)} && (|x+2| = (x+2) \text{ for } x > -2) \\ &= \lim_{x \rightarrow -2^+} (x+3) = ((-2)+3) = 1 \end{aligned}$$

$$\begin{aligned} (b) \lim_{x \rightarrow -2^-} (x+3) \frac{|x+2|}{x+2} &= \lim_{x \rightarrow -2^-} (x+3) \left[\frac{-(x+2)}{(x+2)} \right] && (|x+2| = -(x+2) \text{ for } x < -2) \\ &= \lim_{x \rightarrow -2^-} (x+3)(-1) = -(-2+3) = -1 \end{aligned}$$

Exercices

$$23. \lim_{y \rightarrow 0} \frac{\sin 3y}{4y}$$

$$25. \lim_{x \rightarrow 0} \frac{\tan 2x}{x}$$

$$27. \lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x}$$

$$29. \lim_{x \rightarrow 0} \frac{x + x \cos x}{\sin x \cos x}$$

$$31. \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta}$$

$$33. \lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t}$$

$$35. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$$

$$37. \lim_{\theta \rightarrow 0} \theta \cos \theta$$

$$39. \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x}$$

$$24. \lim_{h \rightarrow 0^-} \frac{h}{\sin 3h}$$

$$26. \lim_{t \rightarrow 0} \frac{2t}{\tan t}$$

$$28. \lim_{x \rightarrow 0} 6x^2(\cot x)(\csc 2x)$$

$$30. \lim_{x \rightarrow 0} \frac{x^2 - x + \sin x}{2x}$$

$$32. \lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 3x}$$

$$34. \lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h}$$

$$36. \lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x}$$

$$38. \lim_{\theta \rightarrow 0} \sin \theta \cot 2\theta$$

$$40. \lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y}$$

Solution

$$23. \lim_{y \rightarrow 0} \frac{\sin 3y}{4y} = \frac{1}{4} \lim_{y \rightarrow 0} \frac{3\sin 3y}{3y} = \frac{3}{4} \lim_{y \rightarrow 0} \frac{\sin 3y}{3y} = \frac{3}{4} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \frac{3}{4} \quad (\text{where } \theta = 3y)$$

$$24. \lim_{h \rightarrow 0^-} \frac{h}{\sin 3h} = \lim_{h \rightarrow 0^-} \left(\frac{1}{3} \cdot \frac{3h}{\sin 3h} \right) = \frac{1}{3} \lim_{h \rightarrow 0^-} \frac{1}{\left(\frac{\sin 3h}{3h} \right)} = \frac{1}{3} \left(\frac{1}{\lim_{\theta \rightarrow 0^-} \frac{\sin \theta}{\theta}} \right) = \frac{1}{3} \cdot 1 = \frac{1}{3} \quad (\text{where } \theta = 3h)$$

$$25. \lim_{x \rightarrow 0} \frac{\tan 2x}{x} = \lim_{x \rightarrow 0} \frac{\left(\frac{\sin 2x}{\cos 2x} \right)}{x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{x \cos 2x} = \left(\lim_{x \rightarrow 0} \frac{1}{\cos 2x} \right) \left(\lim_{x \rightarrow 0} \frac{2 \sin 2x}{2x} \right) = 1 \cdot 2 = 2$$

$$26. \lim_{t \rightarrow 0} \frac{2t}{\tan t} = 2 \lim_{t \rightarrow 0} \frac{t}{\left(\frac{\sin t}{\cos t} \right)} = 2 \lim_{t \rightarrow 0} \frac{t \cos t}{\sin t} = 2 \left(\lim_{t \rightarrow 0} \cos t \right) \left(\frac{1}{\lim_{t \rightarrow 0} \frac{\sin t}{t}} \right) = 2 \cdot 1 \cdot 1 = 2$$

$$27. \lim_{x \rightarrow 0} \frac{x \csc 2x}{\cos 5x} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin 2x} \cdot \frac{1}{\cos 5x} \right) = \left(\frac{1}{2} \lim_{x \rightarrow 0} \frac{2x}{\sin 2x} \right) \left(\lim_{x \rightarrow 0} \frac{1}{\cos 5x} \right) = \left(\frac{1}{2} \cdot 1 \right) (1) = \frac{1}{2}$$

$$28. \lim_{x \rightarrow 0} 6x^2 (\cot x)(\csc 2x) = \lim_{x \rightarrow 0} \frac{6x^2 \cos x}{\sin x \sin 2x} = \lim_{x \rightarrow 0} \left(3 \cos x \cdot \frac{x}{\sin x} \cdot \frac{2x}{\sin 2x} \right) = 3 \cdot 1 \cdot 1 = 3$$

$$\begin{aligned} 29. \lim_{x \rightarrow 0} \frac{x+x \cos x}{\sin x \cos x} &= \lim_{x \rightarrow 0} \left(\frac{x}{\sin x \cos x} + \frac{x \cos x}{\sin x \cos x} \right) = \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \cdot \frac{1}{\cos x} \right) + \lim_{x \rightarrow 0} \frac{x}{\sin x} \\ &= \lim_{x \rightarrow 0} \left(\frac{\frac{1}{\sin x}}{x} \right) \cdot \lim_{x \rightarrow 0} \left(\frac{1}{\cos x} \right) + \lim_{x \rightarrow 0} \left(\frac{\frac{1}{\sin x}}{x} \right) = (1)(1) + 1 = 2 \end{aligned}$$

30.
$$\lim_{x \rightarrow 0} \frac{x^2 - x + \sin x}{2x} = \lim_{x \rightarrow 0} \left(\frac{x}{2} - \frac{1}{2} + \frac{1}{2} \left(\frac{\sin x}{x} \right) \right) = 0 - \frac{1}{2} + \frac{1}{2}(1) = 0$$

31.
$$\begin{aligned} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta} &= \lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{(2 \sin \theta \cos \theta)(1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{(2 \cos \theta)(1 + \cos \theta)} = \frac{0}{(2)(2)} = 0 \end{aligned}$$

32.
$$\lim_{x \rightarrow 0} \frac{x - x \cos x}{\sin^2 3x} = \lim_{x \rightarrow 0} \frac{x(1 - \cos x)}{\sin^2 3x} = \lim_{x \rightarrow 0} \frac{\frac{x(1 - \cos x)}{9x^2}}{\frac{\sin^2 3x}{9x^2}} = \lim_{x \rightarrow 0} \frac{\frac{1 - \cos x}{9x}}{\left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9} \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{x}\right)}{\lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x}\right)^2} = \frac{\frac{1}{9}(0)}{1^2} = 0$$

33.
$$\lim_{t \rightarrow 0} \frac{\sin(1 - \cos t)}{1 - \cos t} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = 1 - \cos t \rightarrow 0 \text{ as } t \rightarrow 0$$

34.
$$\lim_{h \rightarrow 0} \frac{\sin(\sin h)}{\sin h} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{ since } \theta = \sin h \rightarrow 0 \text{ as } h \rightarrow 0$$

35.
$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\sin 2\theta} \cdot \frac{2\theta}{2\theta} \right) = \frac{1}{2} \lim_{\theta \rightarrow 0} \left(\frac{\sin \theta}{\theta} \cdot \frac{2\theta}{\sin 2\theta} \right) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$$

36.
$$\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 4x} = \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{\sin 4x} \cdot \frac{4x}{5x} \cdot \frac{5}{4} \right) = \frac{5}{4} \lim_{x \rightarrow 0} \left(\frac{\sin 5x}{5x} \cdot \frac{4x}{\sin 4x} \right) = \frac{5}{4} \cdot 1 \cdot 1 = \frac{5}{4}$$

$$37. \lim_{\theta \rightarrow 0} \theta \cos \theta = 0 \cdot 1 = 0$$

$$38. \lim_{\theta \rightarrow 0} \sin \theta \cot 2\theta = \lim_{\theta \rightarrow 0} \sin \theta \frac{\cos 2\theta}{\sin 2\theta} = \lim_{\theta \rightarrow 0} \sin \theta \frac{\cos 2\theta}{2 \sin \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\cos 2\theta}{2 \cos \theta} = \frac{1}{2}$$

$$\begin{aligned} 39. \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 8x} &= \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \right) = \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 8x} \cdot \frac{8x}{3x} \cdot \frac{3}{8} \right) \\ &= \frac{3}{8} \lim_{x \rightarrow 0} \left(\frac{1}{\cos 3x} \right) \left(\frac{\sin 3x}{3x} \right) \left(\frac{8x}{\sin 8x} \right) = \frac{3}{8} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{8} \end{aligned}$$

$$\begin{aligned} 40. \lim_{y \rightarrow 0} \frac{\sin 3y \cot 5y}{y \cot 4y} &= \lim_{y \rightarrow 0} \frac{\sin 3y \sin 4y \cos 5y}{y \cos 4y \sin 5y} = \lim_{y \rightarrow 0} \left(\frac{\sin 3y}{y} \right) \left(\frac{\sin 4y}{\cos 4y} \right) \left(\frac{\cos 5y}{\sin 5y} \right) \left(\frac{3 \cdot 4 \cdot 5y}{3 \cdot 4 \cdot 5y} \right) \\ &= \lim_{y \rightarrow 0} \left(\frac{\sin 3y}{3y} \right) \left(\frac{\sin 4y}{4y} \right) \left(\frac{5y}{\sin 5y} \right) \left(\frac{\cos 5y}{\cos 4y} \right) \left(\frac{3 \cdot 4}{5} \right) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{12}{5} = \frac{12}{5} \end{aligned}$$

Exercises

$$33. \lim_{y \rightarrow 1} \sec(y \sec^2 y - \tan^2 y - 1)$$

$$34. \lim_{x \rightarrow 0} \tan\left(\frac{\pi}{4} \cos(\sin x^{1/3})\right)$$

$$35. \lim_{t \rightarrow 0} \cos\left(\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right)$$

$$36. \lim_{x \rightarrow \pi/6} \sqrt{\csc^2 x + 5\sqrt{3} \tan x}$$

Solution

33. $\lim_{y \rightarrow 1} \sec(y \sec^2 y - \tan^2 y - 1) = \lim_{y \rightarrow 1} \sec(y \sec^2 y - \sec^2 y) = \lim_{y \rightarrow 1} \sec((y-1)\sec^2 y) = \sec((1-1)\sec^2 1) = \sec 0 = 1$, and function continuous at $y = 1$.

34. $\lim_{x \rightarrow 0} \tan\left[\frac{\pi}{4} \cos(\sin x^{1/3})\right] = \tan\left[\frac{\pi}{4} \cos(\sin(0))\right] = \tan\left(\frac{\pi}{4} \cos(0)\right) = \tan\left(\frac{\pi}{4}\right) = 1$, and function continuous at $x = 0$.

35. $\lim_{t \rightarrow 0} \cos\left[\frac{\pi}{\sqrt{19 - 3 \sec 2t}}\right] = \cos\left[\frac{\pi}{\sqrt{19 - 3 \sec 0}}\right] = \cos \frac{\pi}{\sqrt{16}} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, and function continuous at $t = 0$.

Exercises

45. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x ?

Solution

45. As defined, $\lim_{x \rightarrow 2^-} f(x) = 12$ and $\lim_{x \rightarrow 2^+} f(x) = a^2(2) - 2a = 2a^2 - 2a$. For $f(x)$ to be continuous we must have $12 = 2a^2 - 2a \Rightarrow a = 3$ or $a = -2$.

Exercises

47. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

Solution

47. As defined, $\lim_{x \rightarrow -1^-} f(x) = -2$ and $\lim_{x \rightarrow -1^+} f(x) = a(-1) + b = -a + b$, and $\lim_{x \rightarrow 1^-} f(x) = a(1) + b = a + b$ and $\lim_{x \rightarrow 1^+} f(x) = 3$. For $f(x)$ to be continuous we must have $-2 = -a + b$ and $a + b = 3 \Rightarrow a = \frac{5}{2}$ and $b = \frac{1}{2}$.

Exercises

48. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x ?

Solution

48. As defined, $\lim_{x \rightarrow 0^-} g(x) = a(0) + 2b = 2b$ and $\lim_{x \rightarrow 0^+} g(x) = (0)^2 + 3a - b = 3a - b$, and $\lim_{x \rightarrow 2^-} g(x) = (2)^2 + 3a - b = 4 + 3a - b$ and $\lim_{x \rightarrow 2^+} g(x) = 3(2) - 5 = 1$. For $g(x)$ to be continuous we must have $2b = 3a - b$ and $4 + 3a - b = 1 \Rightarrow a = -\frac{3}{2}$ and $b = -\frac{3}{2}$.

Exercices

$$25. \lim_{x \rightarrow -\infty} \left(\frac{1 - x^3}{x^2 + 7x} \right)^5$$

$$27. \lim_{x \rightarrow \infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$$

$$29. \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$$

$$31. \lim_{x \rightarrow \infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$$

$$33. \lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + 1}}{x + 1}$$

$$35. \lim_{x \rightarrow \infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$$

$$26. \lim_{x \rightarrow \infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$$

$$28. \lim_{x \rightarrow \infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$$

$$30. \lim_{x \rightarrow \infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$$

$$32. \lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$$

$$34. \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$$

$$36. \lim_{x \rightarrow -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$$



Solution

$$25. \lim_{x \rightarrow -\infty} \left(\frac{1-x^3}{x^2-7x} \right)^5 = \lim_{x \rightarrow -\infty} \left(\frac{\frac{1}{x^2}-x}{1-\frac{7}{x}} \right)^5 = \left(\lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2}-x}{1-\frac{7}{x}} \right)^5 = \left(\frac{0+\infty}{1-0} \right)^5 = \infty$$

$$26. \lim_{x \rightarrow \infty} \sqrt{\frac{x^2-5x}{x^3+x-2}} = \lim_{x \rightarrow \infty} \sqrt{\frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}} = \sqrt{\lim_{x \rightarrow \infty} \frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{1}{x^2}-\frac{2}{x^3}}} = \sqrt{\frac{0-0}{1+0-0}} = \sqrt{0} = 0$$

$$27. \lim_{x \rightarrow \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{x^{1/2}}\right)+\left(\frac{1}{x^2}\right)}{3-\frac{7}{x}} = 0$$

$$28. \lim_{x \rightarrow \infty} \frac{2+\sqrt{x}}{2-\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{\left(\frac{2}{x^{1/2}}\right)+1}{\left(\frac{2}{x^{1/2}}\right)-1} = -1$$

$$29. \lim_{x \rightarrow \infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}} = \lim_{x \rightarrow \infty} \frac{1-x^{(1/5)-(1/3)}}{1+x^{(1/5)-(1/3)}} = \lim_{x \rightarrow \infty} \frac{1-\left(\frac{1}{x^{2/15}}\right)}{1+\left(\frac{1}{x^{2/15}}\right)} = 1$$

$$30. \lim_{x \rightarrow \infty} \frac{x^{-1}+x^{-4}}{x^{-2}-x^{-3}} = \lim_{x \rightarrow \infty} \frac{\frac{x+1}{x^2}}{1-\frac{1}{x}} = \infty$$

$$31. \lim_{x \rightarrow \infty} \frac{2x^{5/3}-x^{1/3}+7}{x^{8/5}+3x+\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{2x^{1/15}-\frac{1}{x^{19/15}}+\frac{7}{x^{8/5}}}{1+\frac{3}{x^{3/5}}+\frac{1}{x^{11/10}}} = \infty$$

32. $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x-5x+3}}{2x+x^{2/3}-4} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^{2/3}}-5+\frac{3}{x}}{2+\frac{1}{x^{1/3}}-\frac{4}{x}} = -\frac{5}{2}$

33. $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}} = \lim_{x \rightarrow \infty} \frac{\sqrt{(x^2+1)/x^2}}{(x+1)/x} = \lim_{x \rightarrow \infty} \frac{\sqrt{1+1/x^2}}{(1+1/x)} = \frac{\sqrt{1+0}}{(1+0)} = 1$

34. $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+1}}{x+1} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+1}/\sqrt{x^2}}{(x+1)/\sqrt{x^2}} = \lim_{x \rightarrow -\infty} \frac{\sqrt{(x^2+1)/x^2}}{(x+1)/(-x)} = \lim_{x \rightarrow \infty} \frac{\sqrt{1+1/x^2}}{(-1-1/x)} = \frac{\sqrt{1+0}}{(-1-0)} = -1$

35. $\lim_{x \rightarrow \infty} \frac{x-3}{\sqrt{4x^2+25}} = \lim_{x \rightarrow \infty} \frac{(x-3)/\sqrt{x^2}}{\sqrt{4x^2+25}/\sqrt{x^2}} = \lim_{x \rightarrow \infty} \frac{(x-3)/x}{\sqrt{(4x^2+25)/x^2}} = \lim_{x \rightarrow \infty} \frac{(1-3/x)}{\sqrt{4+25/x^2}} = \frac{(1-0)}{\sqrt{4+0}} = \frac{1}{2}$

36. $\lim_{x \rightarrow -\infty} \frac{4-3x^3}{\sqrt{x^6+9}} = \lim_{x \rightarrow -\infty} \frac{(4-3x^3)/\sqrt{x^6}}{\sqrt{x^6+9}/\sqrt{x^6}} = \lim_{x \rightarrow -\infty} \frac{(4-3x^3)/(-x^3)}{\sqrt{(x^6+9)/x^6}} = \lim_{x \rightarrow \infty} \frac{(-4/x^3+3)}{\sqrt{1+9/x^6}} = \frac{(0+3)}{\sqrt{1+0}} = 3$

Find the limits in Exercises 80–86.

$$80. \lim_{x \rightarrow \infty} (\sqrt{x+9} - \sqrt{x+4})$$

$$81. \lim_{x \rightarrow \infty} (\sqrt{x^2 + 25} - \sqrt{x^2 - 1})$$

$$82. \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 3} + x)$$

$$83. \lim_{x \rightarrow -\infty} (2x + \sqrt{4x^2 + 3x - 2})$$

$$84. \lim_{x \rightarrow \infty} (\sqrt{9x^2 - x} - 3x)$$

$$85. \lim_{x \rightarrow \infty} (\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x})$$

$$86. \lim_{x \rightarrow \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$$

$$\begin{aligned}
 80. \quad \lim_{x \rightarrow \infty} (\sqrt{x+9} - \sqrt{x+4}) &= \lim_{x \rightarrow \infty} [\sqrt{x+9} - \sqrt{x+4}] \cdot \left[\frac{\sqrt{x+9} + \sqrt{x+4}}{\sqrt{x+9} + \sqrt{x+4}} \right] = \lim_{x \rightarrow \infty} \frac{(x+9)-(x+4)}{\sqrt{x+9} + \sqrt{x+4}} \\
 &= \lim_{x \rightarrow \infty} \frac{5}{\sqrt{x+9} + \sqrt{x+4}} = \lim_{x \rightarrow \infty} \frac{\frac{5}{\sqrt{x}}}{\sqrt{1+\frac{9}{x}} + \sqrt{1+\frac{4}{x}}} = \frac{0}{1+1} = 0
 \end{aligned}$$

$$\begin{aligned}
 81. \quad \lim_{x \rightarrow \infty} (\sqrt{x^2 + 25} - \sqrt{x^2 - 1}) &= \lim_{x \rightarrow \infty} [\sqrt{x^2 + 25} - \sqrt{x^2 - 1}] \cdot \left[\frac{\sqrt{x^2 + 25} + \sqrt{x^2 - 1}}{\sqrt{x^2 + 25} + \sqrt{x^2 - 1}} \right] = \lim_{x \rightarrow \infty} \frac{(x^2 + 25) - (x^2 - 1)}{\sqrt{x^2 + 25} + \sqrt{x^2 - 1}} \\
 &= \lim_{x \rightarrow \infty} \frac{26}{\sqrt{x^2 + 25} + \sqrt{x^2 - 1}} = \lim_{x \rightarrow \infty} \frac{\frac{26}{x}}{\sqrt{1 + \frac{25}{x^2}} + \sqrt{1 - \frac{1}{x^2}}} = \frac{0}{1+1} = 0
 \end{aligned}$$

$$\begin{aligned}
 82. \quad \lim_{x \rightarrow -\infty} (\sqrt{x^2 + 3} + x) &= \lim_{x \rightarrow -\infty} [\sqrt{x^2 + 3} + x] \cdot \left[\frac{\sqrt{x^2 + 3} - x}{\sqrt{x^2 + 3} - x} \right] = \lim_{x \rightarrow -\infty} \frac{(x^2 + 3) - (x^2)}{\sqrt{x^2 + 3} - x} = \lim_{x \rightarrow -\infty} \frac{3}{\sqrt{x^2 + 3} - x} \\
 &= \lim_{x \rightarrow -\infty} \frac{\frac{3}{\sqrt{x^2}}}{\sqrt{1 + \frac{3}{x^2}} - \frac{x}{\sqrt{x^2}}} = \lim_{x \rightarrow -\infty} \frac{\frac{3}{x}}{\sqrt{1 + \frac{3}{x^2}} + 1} = \frac{0}{1+1} = 0
 \end{aligned}$$

$$\begin{aligned}
 83. \quad \lim_{x \rightarrow -\infty} \left(2x + \sqrt{4x^2 + 3x - 2} \right) &= \lim_{x \rightarrow -\infty} \left[2x + \sqrt{4x^2 + 3x - 2} \right] \cdot \left[\frac{2x - \sqrt{4x^2 + 3x - 2}}{2x - \sqrt{4x^2 + 3x - 2}} \right] = \lim_{x \rightarrow -\infty} \frac{(4x^2) - (4x^2 + 3x - 2)}{2x - \sqrt{4x^2 + 3x - 2}} \\
 &= \lim_{x \rightarrow -\infty} \frac{-3x + 2}{2x - \sqrt{4x^2 + 3x - 2}} = \lim_{x \rightarrow -\infty} \frac{\frac{-3x + 2}{\sqrt{x^2}}}{\frac{2x}{\sqrt{x^2}} - \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} = \lim_{x \rightarrow -\infty} \frac{\frac{-3x + 2}{-x}}{\frac{2x}{-x} - \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} \\
 &= \lim_{x \rightarrow -\infty} \frac{\frac{3 - \frac{2}{x}}{-x}}{-2 - \sqrt{4 + \frac{3}{x} - \frac{2}{x^2}}} = \frac{3 - 0}{-2 - 2} = -\frac{3}{4}
 \end{aligned}$$

$$\begin{aligned}
 84. \quad \lim_{x \rightarrow \infty} \left(\sqrt{9x^2 - x} - 3x \right) &= \lim_{x \rightarrow \infty} \left[\sqrt{9x^2 - x} - 3x \right] \cdot \left[\frac{\sqrt{9x^2 - x} + 3x}{\sqrt{9x^2 - x} + 3x} \right] = \lim_{x \rightarrow \infty} \frac{(9x^2 - x) - (9x^2)}{\sqrt{9x^2 - x} + 3x} = \lim_{x \rightarrow \infty} \frac{-x}{\sqrt{9x^2 - x} + 3x} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{-x}{x}}{\sqrt{\frac{9x^2 - x}{x^2} + \frac{3x}{x}}} = \lim_{x \rightarrow \infty} \frac{\frac{-1}{\sqrt{9 - \frac{1}{x}} + 3}}{\sqrt{9 - \frac{1}{x}} + 3} = \frac{-1}{3+3} = -\frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 85. \quad \lim_{x \rightarrow \infty} \left(\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right) &= \lim_{x \rightarrow \infty} \left[\sqrt{x^2 + 3x} - \sqrt{x^2 - 2x} \right] \cdot \left[\frac{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \right] = \lim_{x \rightarrow \infty} \frac{(x^2 + 3x) - (x^2 - 2x)}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{5x}{x}}{\sqrt{x^2 + 3x} + \sqrt{x^2 - 2x}} = \lim_{x \rightarrow \infty} \frac{5}{\sqrt{1 + \frac{3}{x}} + \sqrt{1 - \frac{2}{x}}} = \frac{5}{1+1} = \frac{5}{2}
 \end{aligned}$$

$$\begin{aligned}
 86. \quad \lim_{x \rightarrow \infty} \sqrt{x^2 + x} - \sqrt{x^2 - x} &= \lim_{x \rightarrow \infty} \left[\sqrt{x^2 + x} - \sqrt{x^2 - x} \right] \cdot \left[\frac{\sqrt{x^2 + x} + \sqrt{x^2 - x}}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} \right] = \lim_{x \rightarrow \infty} \frac{(x^2 + x) - (x^2 - x)}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} = \lim_{x \rightarrow \infty} \frac{2x}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} \\
 &= \lim_{x \rightarrow \infty} \frac{\frac{2}{\sqrt{1 + \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}}}{\sqrt{1 + \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}} = \frac{2}{1+1} = 1
 \end{aligned}$$



Thank you for your attention