Calculus 1

Dr. Yamar Hamwi

Al-Manara University

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Calculus 1

Lecture 5

Derivatives

Chapter 3
Derivatives

3.5 The Chain Rule

3.6 Implicit Differentiation

3.7 Linearization

3.8 Differentials

3.9 Leibnitz Theorem

The Chain Rule

How do we differentiate $F(x) = \sin(x^2 - 4)$? This function is the composition $f \circ g$ of two functions $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$ that we know how to differentiate. The answer, given by the *Chain Rule*, says that the derivative is the product of the derivatives of f and g. We develop the rule in this section.

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

The Chain Rule

EXAMPLE 3 Differentiate $\sin(x^2 + x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin(\underbrace{x^2 + x}_{\text{inside}}) = \cos(\underbrace{x^2 + x}_{\substack{\text{inside}\\ \text{left alone}}}) \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of}\\ \text{the inside}}}.$$

The Chain Rule

EXAMPLE 4 Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$$g'(t) = \frac{d}{dt}\tan(5 - \sin 2t)$$

$$= \sec^2(5 - \sin 2t)\cdot\frac{d}{dt}(5 - \sin 2t)$$

Derivative of tan u with $u = 5 - \sin 2t$

$$= \sec^2(5 - \sin 2t)\cdot\left(0 - \cos 2t\cdot\frac{d}{dt}(2t)\right)$$

Derivative of $5 - \sin u$ with $u = 2t$

$$= \sec^2(5 - \sin 2t)\cdot(-\cos 2t)\cdot 2$$

$$= -2(\cos 2t)\sec^2(5 - \sin 2t).$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}.$$

$$\frac{d}{du}(u^n) = nu^{n-1}$$

The Chain Rule with Powers of a Function

EXAMPLE 5 The Power Chain Rule simplifies computing the derivative of a power of an expression.

(a) $$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}(5x^3 - x^4)$$

Power Chain Rule with $u = 5x^3 - x^4, n = 7$

$$= 7(5x^3 - x^4)^6 (15x^2 - 4x^3)$$

(b) $$\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$

$$= -1(3x - 2)^{-2} \frac{d}{dx}(3x - 2)$$

Power Chain Rule with $u = 3x - 2, n = -1$

$$= -1(3x - 2)^{-2}(3)$$

$$= -\frac{3}{(3x - 2)^2}$$

In part (b) we could also find the derivative with the Quotient Rule.

(c) $$\frac{d}{dx}(\sin^5 x) = 5\sin^4 x \cdot \frac{d}{dx}\sin x$$

Power Chain Rule with $u = \sin x, n = 5$, because $\sin^n x$ means $(\sin x)^n$, $n \neq -1$.

$$= 5\sin^4 x \cos x$$

The Chain Rule with Powers of a Function

EXAMPLE 6

$$\begin{aligned}
\frac{d}{dx}(|x|) &= \frac{d}{dx}\sqrt{x^2} \\
&= \frac{1}{2\sqrt{x^2}} \cdot \frac{d}{dx}(x^2) && \text{Power Chain Rule with } u = x^2, n = 1/2, x \neq 0 \\
&= \frac{1}{2|x|} \cdot 2x && \sqrt{x^2} = |x| \\
&= \frac{x}{|x|}, \quad x \neq 0.
\end{aligned}$$

Implicit Differentiation

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encounter equations like

$$x^3 + y^3 - 9xy = 0, \qquad y^2 - x = 0, \qquad \text{or} \qquad x^2 + y^2 - 25 = 0.$$

When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find $dy\backslash dx$ by *implicit differentiation*. This section describes the technique

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

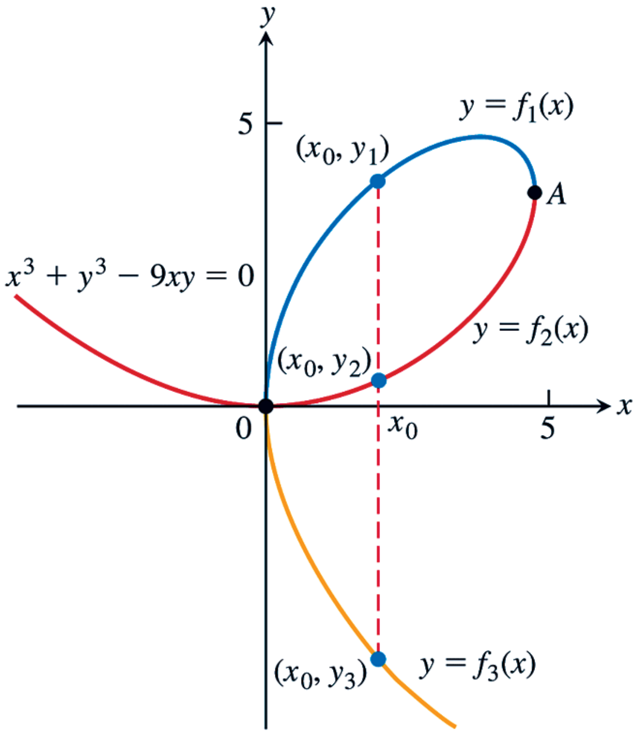

FIGURE 3.29 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

Implicit Differentiation

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EXAMPLE 1 Find dy/dx if $y^2 = x$.

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

The Chain Rule gives

$$\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}.$$

EXAMPLE 2 Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}.$

Implicit Differentiation

EXAMPLE 3 Find dy/dx if $y^2 = x^2 + \sin xy$

Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

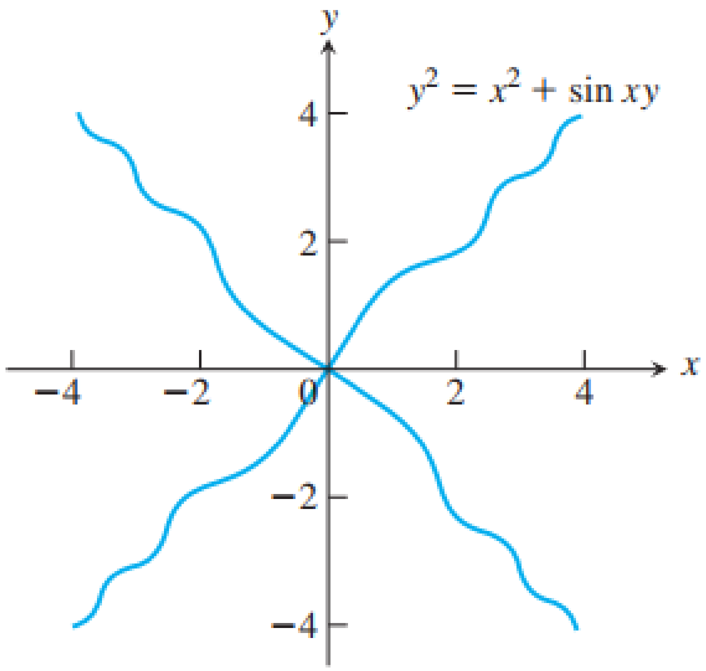
$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$



Derivatives of Higher Order

EXAMPLE 4 Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2} \cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

Tangent Lines

EXAMPLE 5 Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.32).

Solution The point $(2, 4)$ lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at $(2, 4)$, we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

Differentiate both sides with respect to x.

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

Treat xy as a product and y as a function of x.

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$

Solve for dy/dx.

Tangent Lines

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at $(2, 4)$ is the line through $(2, 4)$ with slope $4/5$:

$$y = 4 + \frac{4}{5}(x - 2)$$

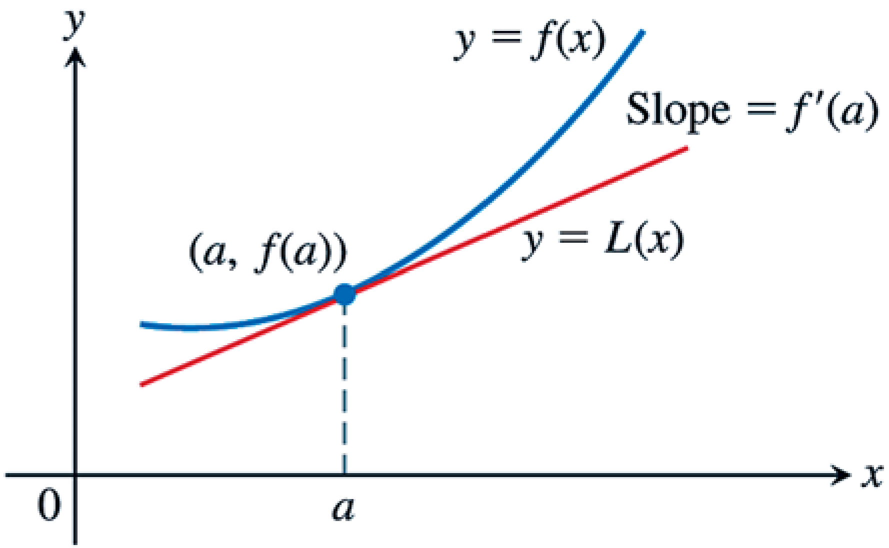
$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at $(2, 4)$ is the line perpendicular to the tangent there, the line through $(2, 4)$ with slope $-5/4$:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$

Linearization



DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

Linearization

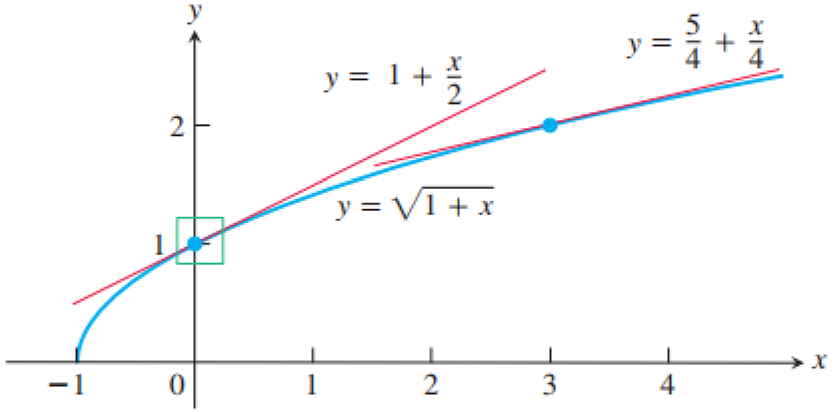
EXAMPLE 1 Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$

Solution Since

$$f'(x) = \frac{1}{2}(1 + x)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$



$$\sqrt{1 + x} \approx 1 + (x/2)$$

Linearization

Approximation	True value	$\lvert \text{True value} - \text{approximation} \rvert$
$\sqrt{1.005} \approx 1 + \dfrac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$
$\sqrt{1.05} \approx 1 + \dfrac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.2} \approx 1 + \dfrac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$

Linearization

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EXAMPLE 3 Find the linearization of $f(x) = \cos x$ at $x = \pi/2$

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we find the linearization at $a = \pi/2$ to be

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

Differentials

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$

EXAMPLE 4

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

Differentials

The geometric meaning of differentials

Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

The corresponding change in the tangent line L is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,dx.\end{aligned}$$

Differentials

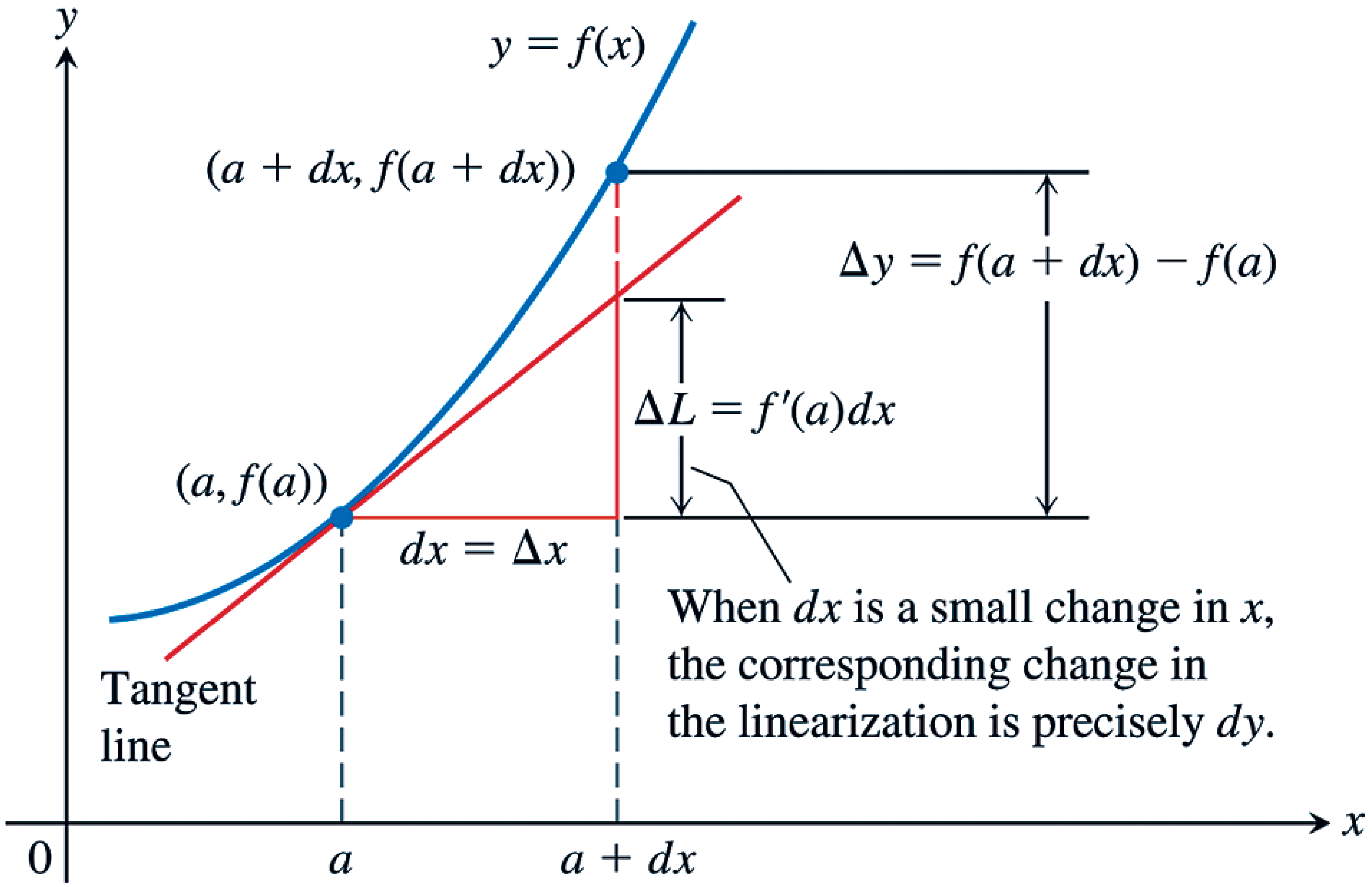

FIGURE 3.56 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

Differentials

dy **represents the amount the tangent line rises or falls when** ***x*** **changes by an amount** ***dx*** **=** ***Δx.***

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling df the **differential of** $\boldsymbol{f}$. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Differentials

Every differentiation formula like

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{or} \qquad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u+v) = du + dv \qquad \text{or} \qquad d(\sin u) = \cos u \, du.$$

EXAMPLE 5 We can use the Chain Rule and other differentiation rules to find differentials of functions.

(a) $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2 2x \, dx$

(b) $d\left(\frac{x}{x+1}\right) = \frac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \frac{x dx + dx - x\,dx}{(x+1)^2} = \frac{dx}{(x+1)^2}$ ■

Leibnitz Theorem Formula

The first derivative could be written as;

$$(uv)' = u'v+uv'$$

Now if we differentiate the above expression again, we get the second derivative;

$$\begin{aligned}(uv)'' &= [(uv)']' \\ &= (u'v+uv')' \\ &= (u'v)'+(uv')' \\ &= u''v + u'v' + u'\,v' + uv'' \\ &= u''v + 2u'v' + uv''\end{aligned}$$

Leibnitz Theorem

$$(uv)^n = \sum_{i=0}^{n} \binom{n}{i} u^{(n-i)} v^i$$

Where $\binom{n}{i}$ represents the number of i-combinations on n elements.

$$(f \cdot g)^{(n)} = f^{(n)} \cdot g + \binom{n}{1} f^{(n-1)} \cdot g^{(1)} + \cdots$$

$$\ldots + \binom{n}{k} f^{(n-k)} \cdot g^{(k)} + \cdots + f \quad \cdot g^{(n)}$$

Tangent Lines and the Derivative at a Point

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(f \cdot g)'' = f'' \cdot g + 2f' \cdot g' + f \cdot g''$$

Tangent Lines and the Derivative at a Point

Example : Find the second derivative of the product of the functions x^2, and Tanx, using lebiniz rule.

Solution:

The given functions are $f(x) = x^2$, and $g(x) = Tanx$.

The leibniz rule for the product of two functions is $(f(x).g(x))'' = f''(x).g(x) + 2f'(x).g'(x) + f(x).g'(x)$.

$$\frac{d^2}{dx^2}.x^2.Tanx = Tanx.\frac{d^2}{dx^2}.x^2 + 2\frac{d}{dx}.x^2.\frac{d}{dx}.Tanx + x^2.\frac{d^2}{dx^2}$$

$$\frac{d^2}{dx^2}.x^2.Tanx = Tanx.2 + 2.2x.Sec^2x + x^2.2Secx.Secx.Tan$$

$$\frac{d^2}{dx^2}.x^2.Tanx = 2Tanx + 4x.Sec^2x + 2x^2Sec^2x.Tanx$$

Therefore the derivative of the product of two functions using leibniz rule is $2Tanx + 4x.Sec^2x + 2x^2Sec^2x.Tanx$.

Exercises

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = e^{-x}$
5. $y = \sqrt{u}, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = \pi x^2$
8. $y = -\sec u, \quad u = \dfrac{1}{x} + 7x$

1. $f(u)=6u-9 \Rightarrow f'(u)=6 \Rightarrow f'(g(x))=6; g(x)=\frac{1}{2}x^4 \Rightarrow g'(x)=2x^3;$
therefore $\frac{dy}{dx}=f'(g(x))g'(x)=6\cdot 2x^3=12x^3$

2. $f(u)=2u^3 \Rightarrow f'(u)=6u^2 \Rightarrow f'(g(x))=6(8x-1)^2; g(x)=8x-1 \Rightarrow g'(x)=8;$
therefore $\frac{dy}{dx}=f'(g(x))g'(x)=6(8x-1)^2\cdot 8=48(8x-1)^2$

3. $f(u)=\sin u \Rightarrow f'(u)=\cos u \Rightarrow f'(g(x))=\cos(3x+1); g(x)=3x+1 \Rightarrow g'(x)=3;$
therefore $\frac{dy}{dx}=f'(g(x))g'(x)=(\cos(3x+1))(3)=3\cos(3x+1)$

4. $f(u)=\cos u \Rightarrow f'(u)=-\sin u \Rightarrow f'(g(x))=-\sin(e^{-x});\ \ g(x)=e^{-x} \Rightarrow g'(x)=-e^{-x};$ therefore,
$\frac{dy}{dx}=f'(g(x))g'(x)=-\sin(e^{-x})(-e^{-x})=e^{-x}\sin(e^{-x})$

5. $f(u)=\sqrt{u} \Rightarrow f'(u)=\frac{1}{2\sqrt{u}} \Rightarrow f'(g(x))=\frac{1}{2\sqrt{\sin x}};\ \ g(x)=\sin x \Rightarrow g'(x)=\cos x;$ therefore,
$\frac{dy}{dx}=f'(g(x))g'(x)=\frac{\cos x}{2\sqrt{\sin x}}$

6. $f(u)=\sin u \Rightarrow f'(u)=\cos u \Rightarrow f'(g(x))=\cos(x-\cos x); g(x)=x-\cos x \Rightarrow g'(x)=1+\sin x;$
therefore $\frac{dy}{dx}=f'g(x))g'(x)=(\cos(x-\cos x))(1+\sin x)$

7. $f(u)=\tan u \Rightarrow f'(u)=\sec^2 u \Rightarrow f'(g(x))=\sec^2(\pi x^2);\ g(x)=\pi x^2 \Rightarrow g'(x)=2\pi x;$

therefore $\frac{dy}{dx}=f'(g(x))g'(x)=\sec^2(\pi x^2)(2\pi x)=2\pi x\sec^2(\pi x^2)$

8. $f(u)=-\sec u \Rightarrow f'(u)=-\sec u\tan u \Rightarrow f'(g(x))=-\sec\left(\frac{1}{x}+7x\right)\tan(\frac{1}{x}+7x);\ g(x)=\frac{1}{x}+7x \Rightarrow$

$g'(x)=-\frac{1}{x^2}+7;$ therefore, $\frac{dy}{dx}=f'(g(x))g'(x)=\left(\frac{1}{x^2}-7\right)\sec(\frac{1}{x}+7x)\tan(\frac{1}{x}+7x)$

Exercises

In Exercises 9–22, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$

10. $y = (4 - 3x)^9$

11. $y = \left(1 - \frac{x}{7}\right)^{-7}$

12. $y = \left(\frac{\sqrt{x}}{2} - 1\right)^{-10}$

13. $y = \left(\frac{x^2}{8} + x - \frac{1}{x}\right)^4$

14. $y = \sqrt{3x^2 - 4x + 6}$

15. $y = \sec(\tan x)$

16. $y = \cot\left(\pi - \frac{1}{x}\right)$

17. $y = \tan^3 x$

18. $y = 5\cos^{-4} x$

9. With $u=(2x+1)$, $y=u^5$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=5u^4\cdot 2=10(2x+1)^4$

10. With $u=(4-3x)$, $y=u^9$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=9u^8\cdot(-3)=-27(4-3x)^8$

11. With $u=\left(1-\frac{x}{7}\right)$, $y=u^{-7}$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-7u^{-8}\cdot\left(-\frac{1}{7}\right)=\left(1-\frac{x}{7}\right)^{-8}$

12. With $u=\frac{\sqrt{x}}{2}-1$, $y=u^{-10}$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-10u^{-11}\cdot\left(\frac{1}{4\sqrt{x}}\right)=-\frac{1}{4\sqrt{x}}\left(\frac{\sqrt{x}}{2}-1\right)^{-11}$

13. With $u=\left(\frac{x^2}{8}+x-\frac{1}{x}\right)$, $y=u^4$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=4u^3\cdot\left(\frac{x}{4}+1+\frac{1}{x^2}\right)=4\left(\frac{x^2}{8}+x-\frac{1}{x}\right)^3\left(\frac{x}{4}+1+\frac{1}{x^2}\right)$

14. With $u=3x^2-4x+6$, $y=u^{1/2}$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{2}u^{-1/2}\cdot(6x-4)=\frac{3x-2}{\sqrt{3x^2-4x+6}}$

15. With $u=\tan x$, $y=\sec u$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(\sec u\tan u)(\sec^2 x)=(\sec(\tan x)\tan(\tan x))\sec^2 x$

16. With $u=\pi-\frac{1}{x}$, $y=\cot u$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(-\csc^2 u)\left(\frac{1}{x^2}\right)=-\frac{1}{x^2}\csc^2\left(\pi-\frac{1}{x}\right)$

17. With $u=\tan x$, $y=u^3$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=3u^2\sec^2 x=3\tan^2 x\sec^2 x$

18. With $u=\cos x$, $y=5u^{-4}$: $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=(-20u^{-5})(-\sin x)=20(\cos^{-5} x)(\sin x)$

Exercises

Find y'' in Exercises 71–78.

71. $y = \left(1 + \frac{1}{x}\right)^3$

72. $y = (1 - \sqrt{x})^{-1}$

73. $y = \frac{1}{9}\cot(3x - 1)$

74. $y = 9\tan\left(\frac{x}{3}\right)$

75. $y = x(2x + 1)^4$

76. $y = x^2(x^3 - 1)^5$

71. $y=\left(1+\frac{1}{x}\right)^3 \Rightarrow y'=3\left(1+\frac{1}{x}\right)^2\left(-\frac{1}{x^2}\right)=-\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2 \Rightarrow y''=\left(-\frac{3}{x^2}\right)\cdot\frac{d}{dx}\left(1+\frac{1}{x}\right)^2-\left(1+\frac{1}{x}\right)^2\cdot\frac{d}{dx}\left(\frac{3}{x^2}\right)$

$=\left(-\frac{3}{x^2}\right)\left(2\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\right)+\left(\frac{6}{x^3}\right)\left(1+\frac{1}{x}\right)^2=\frac{6}{x^4}\left(1+\frac{1}{x}\right)+\frac{6}{x^3}\left(1+\frac{1}{x}\right)^2=\frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(\frac{1}{x}+1+\frac{1}{x}\right)=\frac{6}{x^3}\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)$

72. $y=\left(1-\sqrt{x}\right)^{-1} \Rightarrow y'=-\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-1/2}\right)=\frac{1}{2}\left(1-\sqrt{x}\right)^{-2}x^{-1/2}$

$\Rightarrow y''=\frac{1}{2}\left[\left(1-\sqrt{x}\right)^{-2}\left(-\frac{1}{2}x^{-3/2}\right)+x^{-1/2}(-2)\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2}x^{-1/2}\right)\right]$

$=\frac{1}{2}\left[\frac{-1}{2}x^{-3/2}\left(1-\sqrt{x}\right)^{-2}+x^{-1}\left(1-\sqrt{x}\right)^{-3}\right]=\frac{1}{2}x^{-1}\left(1-\sqrt{x}\right)^{-3}\left[-\frac{1}{2}x^{-1/2}\left(1-\sqrt{x}\right)+1\right]$

$=\frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(-\frac{1}{2\sqrt{x}}+\frac{1}{2}+1\right)=\frac{1}{2x}\left(1-\sqrt{x}\right)^{-3}\left(\frac{3}{2}-\frac{1}{2\sqrt{x}}\right)$

73. $y=\frac{1}{9}\cot(3x-1) \Rightarrow y'=-\frac{1}{9}\csc^2(3x-1)(3)=-\frac{1}{3}\csc^2(3x-1) \Rightarrow y''=\left(-\frac{2}{3}\right)(\csc(3x-1)\cdot\frac{d}{dx}\csc(3x-1))$

$=-\frac{2}{3}\csc(3x-1)(-\csc(3x-1)\cot(3x-1)\cdot\frac{d}{dx}(3x-1))=2\csc^2(3x-1)\cot(3x-1)$

74. $y=9\tan\left(\frac{x}{3}\right) \Rightarrow y'=9\left(\sec^2\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right)=3\sec^2\left(\frac{x}{3}\right) \Rightarrow y''=3\cdot 2\sec\left(\frac{x}{3}\right)\left(\sec\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)\right)\left(\frac{1}{3}\right)=2\sec^2\left(\frac{x}{3}\right)\tan\left(\frac{x}{3}\right)$

75. $y=x(2x+1)^4 \Rightarrow y'=x\cdot 4(2x+1)^3(2)+1\cdot(2x+1)^4=(2x+1)^3(8x+(2x+1))=(2x+1)^3(10x+1)$

$\Rightarrow y''=(2x+1)^3(10)+3(2x+1)^2(2)(10x+1)=2(2x+1)^2(5(2x+1)+3(10x+1))=2(2x+1)^2(40x+8)$

$=16(2x+1)^2(5x+1)$

76. $y=x^2(x^3-1)^5 \Rightarrow y'=x^2\cdot 5(x^3-1)^4(3x^2)+2x(x^3-1)^5=x(x^3-1)^4[15x^3+2(x^3-1)]=(x^3-1)^4(17x^4-2x)$

$\Rightarrow y''=(x^3-1)^4(68x^3-2)+4(x^3-1)^3(3x^2)(17x^4-2x)=2(x^3-1)^3[(x^3-1)(34x^3-1)+6x^2(17x^4-2x)]$

$=2\left(x^3-1\right)^3\left(136x^6-47x^3+1\right)$

Exercises

In Exercises 79–84, find the value of $(f \circ g)'$ at the given value of x.

79. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

80. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

81. $f(u) = \cot \frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

82. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

83. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

84. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

79. $g(x)=\sqrt{x} \Rightarrow g'(x)=\frac{1}{2\sqrt{x}} \Rightarrow g(1)=1$ and $g'(1)=\frac{1}{2}$; $f(u)=u^5+1 \Rightarrow f'(u)=5u^4 \Rightarrow f'(g(1))=f'(1)=5$; therefore, $(f\circ g)'(1)=f'(g(1))\cdot g'(1)=5\cdot\frac{1}{2}=\frac{5}{2}$

80. $g(x)=(1-x)^{-1} \Rightarrow g'(x)=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2} \Rightarrow g(-1)=\frac{1}{2}$ and $g'(-1)=\frac{1}{4}$; $f(u)=1-\frac{1}{u} \Rightarrow f'(u)=\frac{1}{u^2}$ $\Rightarrow f'(g(-1))=f'\left(\frac{1}{2}\right)=4$; therefore, $(f\circ g)'(-1)=\ f'(g(-1))g'(-1)=4\cdot\frac{1}{4}=1$

81. $g(x)=5\sqrt{x} \Rightarrow g'(x)=\frac{5}{2\sqrt{x}} \Rightarrow g(1)=5$ and $g'(1)=\frac{5}{2}$; $f(u)=\cot\left(\frac{\pi u}{10}\right) \Rightarrow f'(u)=-\csc^2\left(\frac{\pi u}{10}\right)\left(\frac{\pi}{10}\right)=\frac{-\pi}{10}\csc^2\left(\frac{\pi u}{10}\right)$ $\Rightarrow f'(g(1))=f'(5)=-\frac{\pi}{10}\csc^2\left(\frac{\pi}{2}\right)=-\frac{\pi}{10}$; therefore, $(f\circ g)'(1)=f'(g(1))g'(1)=-\frac{\pi}{10}\cdot\frac{5}{2}=-\frac{\pi}{4}$

82. $g(x)=\pi\ x \Rightarrow g'(x)=\pi \Rightarrow g\left(\frac{1}{4}\right)=\frac{\pi}{4}$ and $g'\left(\frac{1}{4}\right)=\pi$; $f(u)=u+\sec^2 u \Rightarrow f'(u)=1+2\sec u\cdot\sec u\tan u$ $=1+2\sec^2 u\tan u \Rightarrow f'\left(g\left(\frac{1}{4}\right)\right)=f'\left(\frac{\pi}{4}\right)=1+2\sec^2\frac{\pi}{4}\tan\frac{\pi}{4}=5$; therefore, $(f\circ g)'\left(\frac{1}{4}\right)=f'\left(g\left(\frac{1}{4}\right)\right)g'\left(\frac{1}{4}\right)=5\pi$

83. $g(x)=10x^2+x+1 \Rightarrow g'(x)=20x+1 \Rightarrow g(0)=1$ and $g'(0)=1$; $f(u)=\frac{2u}{u^2+1} \Rightarrow f'(u)=\frac{\left(u^2+1\right)(2)-(2u)(2u)}{\left(u^2+1\right)^2}$ $=\frac{-2u^2+2}{(u^2+1)^2} \Rightarrow f'(g(0))=f'(1)=0$; therefore, $(f\circ g)'(0)=f'(g(0))g'(0)=0\cdot 1=0$

84. $g(x)=\frac{1}{x^2}-1 \Rightarrow g'(x)=-\frac{2}{x^3} \Rightarrow g(-1)=0$ and $g'(-1)=2$; $f(u)=\left(\frac{u-1}{u+1}\right)^2 \Rightarrow f'(u)=2\left(\frac{u-1}{u+1}\right)\frac{d}{du}\left(\frac{u-1}{u+1}\right)$ $=2\left(\frac{u-1}{u+1}\right)\cdot\frac{(u+1)(1)-(u-1)(1)}{(u+1)^2}=\frac{2(u-1)(2)}{(u+1)^3}=\frac{4(u-1)}{(u+1)^3} \Rightarrow f'(g(-1))=f'(0)=-4$; therefore, $(f\circ g)'(-1)=f'(g(-1))g'(-1)\ =(-4)(2)=-8$

Exercises

85. Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$?

86. If $r = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is dr/dt at $t = 0$?

85. $y = f(g(x)),\ f'(3) = -1,\ g'(2) = 5,\ g(2) = 3 \Rightarrow y' = f'(g(x))g'(x) \Rightarrow y'\big|_{x=2} = f'(g(2))g'(2) = f'(3)\cdot 5$
$= (-1)\cdot 5 = -5$

86. $r = \sin(f(t)),\ f(0) = \frac{\pi}{3},\ \ f'(0) = 4 \Rightarrow \frac{dr}{dt} = \cos(f(t))\cdot f'(t) \Rightarrow \frac{dr}{dt}\Big|_{t=0} = \cos(f(0))\cdot f'(0) = \cos\left(\frac{\pi}{3}\right)\cdot 4 = \left(\frac{1}{2}\right)\cdot 4 = 2$

Exercises

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 1–16.

1. $x^2y + xy^2 = 6$
2. $x^3 + y^3 = 18xy$
3. $2xy + y^2 = x + y$
4. $x^3 - xy + y^3 = 1$
5. $x^2(x - y)^2 = x^2 - y^2$
6. $(3xy + 7)^2 = 6y$
7. $y^2 = \dfrac{x - 1}{x + 1}$
8. $x^3 = \dfrac{2x - y}{x + 3y}$
9. $x = \sec y$
10. $xy = \cot(xy)$
11. $x + \tan(xy) = 0$
12. $x^4 + \sin y = x^3y^2$
13. $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$
14. $x \cos(2x + 3y) = y \sin x$

1. $x^2y + xy^2 = 6$:

Step 1: $\left(x^2\frac{dy}{dx} + y\cdot 2x\right) + \left(x\cdot 2y\frac{dy}{dx} + y^2\cdot 1\right) = 0$

Step 2: $x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} = -2xy - y^2$

Step 3: $\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$

Step 4: $\frac{dy}{dx} = \frac{-2xy-y^2}{x^2+2xy}$

2. $x^3 + y^3 = 18xy \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 18y + 18x\frac{dy}{dx} \Rightarrow (3y^2 - 18x)\frac{dy}{dx} = 18y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6y-x^2}{y^2-6x}$

3. $2xy + y^2 = x + y$:

Step 1: $\left(2x\frac{dy}{dx} + 2y\right) + 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$

Step 2: $2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$

Step 3: $\frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$

Step 4: $\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$

4. $x^3 - xy + y^3 = 1 \Rightarrow 3x^2 - y - x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0 \Rightarrow (3y^2 - x)\frac{dy}{dx} = y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y-3x^2}{3y^2-x}$

5. $x^2(x-y)^2 = x^2 - y^2$:

Step 1: $x^2\left[2(x-y)\left(1-\frac{dy}{dx}\right)\right] + (x-y)^2(2x) = 2x - 2y\frac{dy}{dx}$

Step 2: $-2x^2(x-y)\frac{dy}{dx} + 2y\frac{dy}{dx} = 2x - 2x^2(x-y) - 2x(x-y)^2$

Step 3: $\frac{dy}{dx}\left[-2x^2(x-y)+2y\right] = 2x[1 - x(x-y) - (x-y)^2]$

Step 4: $\frac{dy}{dx} = \frac{2x\left[1-x(x-y)-(x-y)^2\right]}{-2x^2(x-y)+2y} = \frac{x\left[1-x(x-y)-(x-y)^2\right]}{y-x^2(x-y)} = \frac{x\left(1-x^2+xy-x^2+2xy-y^2\right)}{x^2y-x^3+y} = \frac{x-2x^3+3x^2y-xy^2}{x^2y-x^3+y}$

6. $(3xy+7)^2 = 6y \Rightarrow 2(3xy+7)\cdot\left(3x\frac{dy}{dx}+3y\right) = 6\frac{dy}{dx} \Rightarrow 2(3xy+7)(3x)\frac{dy}{dx} - 6\frac{dy}{dx} = -6y(3xy+7)$

$\Rightarrow \frac{dy}{dx}[6x(3xy+7)-6] = -6y(3xy+7) \Rightarrow \frac{dy}{dx} = -\frac{y(3xy+7)}{x(3xy+7)-1} = \frac{3xy^2+7y}{1-3x^2y-7x}$

7. $y^2 = \frac{x-1}{x+1} \Rightarrow 2y\frac{dy}{dx} = \frac{(x+1)-(x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$

8. $x^3 = \frac{2x-y}{x+3y} \Rightarrow x^4 + 3x^3y = 2x - y \Rightarrow 4x^3 + 9x^2y + 3x^3y' = 2 - y' \Rightarrow (3x^3+1)y' = 2 - 4x^3 - 9x^2y$

$\Rightarrow y' = \frac{2-4x^3-9x^2y}{3x^3+1}$

9. $x = \tan y \Rightarrow 1 = (\sec^2 y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

10. $xy = \cot(xy) \Rightarrow x\frac{dy}{dx} + y = -\csc^2(xy)\left(x\frac{dy}{dx} + y\right) \Rightarrow x\frac{dy}{dx} + x\csc^2(xy)\frac{dy}{dx} = -y\csc^2(xy) - y$

$\Rightarrow \frac{dy}{dx}\left[x + x\csc^2(xy)\right] = -y\left[\csc^2(xy) + 1\right] \Rightarrow \frac{dy}{dx} = \frac{-y\left[\csc^2(xy)+1\right]}{x\left[1+\csc^2(xy)\right]} = -\frac{y}{x}$

11. $x + \tan(xy) = 0 \Rightarrow 1 + \left[\sec^2(xy)\right]\left(y + x\frac{dy}{dx}\right) = 0 \Rightarrow x\sec^2(xy)\frac{dy}{dx} = -1 - y\sec^2(xy) \Rightarrow \frac{dy}{dx} = \frac{-1-y\sec^2(xy)}{x\sec^2(xy)}$

$= \frac{-1}{x\sec^2(xy)} - \frac{y}{x} = \frac{-\cos^2(xy)}{x} - \frac{y}{x} = \frac{-\cos^2(xy)-y}{x}$

12. $x^4 + \sin y = x^3y^2 \Rightarrow 4x^3 + (\cos y)\frac{dy}{dx} = 3x^2y^2 + x^3 \cdot 2y\frac{dy}{dx} \Rightarrow (\cos y - 2x^3y)\frac{dy}{dx} = 3x^2y^2 - 4x^3 \Rightarrow \frac{dy}{dx} = \frac{3x^2y^2-4x^3}{\cos y - 2x^3y}$

13. $y\sin\left(\frac{1}{y}\right) = 1 - xy \Rightarrow y\left[\cos\left(\frac{1}{y}\right)\cdot(-1)\frac{1}{y^2}\cdot\frac{dy}{dx}\right] + \sin\left(\frac{1}{y}\right)\cdot\frac{dy}{dx} = -x\frac{dy}{dx} - y \Rightarrow \frac{dy}{dx}\left[-\frac{1}{y}\cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x\right] = -y$

$\Rightarrow \frac{dy}{dx} = \frac{-y}{-\frac{1}{y}\cos\left(\frac{1}{y}\right)+\sin\left(\frac{1}{y}\right)+x} = \frac{-y^2}{y\sin\left(\frac{1}{y}\right)-\cos\left(\frac{1}{y}\right)+xy}$

14. $x\cos(2x+3y) = y\sin x \Rightarrow -x\sin(2x+3y)(2+3y') + \cos(2x+3y) = y\cos x + y'\sin x$

$\Rightarrow -2x\sin(2x+3y) - 3xy'\sin(2x+3y) + \cos(2x+3y) = y\cos x + y'\sin x$

$\Rightarrow \cos(2x+3y) - 2x\sin(2x+3y) - y\cos x = (\sin x + 3x\sin(2x+3y))y'$

$\Rightarrow y' = \frac{\cos(2x+3y) - 2x\sin(2x+3y) - y\cos x}{\sin x + 3x\sin(2x+3y)}$

Exercises

In Exercises 21–26, use implicit differentiation to find dy/dx and then d^2y/dx^2.

21. $x^2 + y^2 = 1$

22. $x^{2/3} + y^{2/3} = 1$

23. $y^2 = e^{x^2} + 2x$

24. $y^2 - 2x = 1 - 2y$

25. $2\sqrt{y} = x - y$

26. $xy + y^2 = 1$

27. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.

28. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

21. $x^2+y^2=1 \Rightarrow 2x+2yy'=0 \Rightarrow 2yy'=-2x \Rightarrow \frac{dy}{dx}=y'=-\frac{x}{y}$; now to find $\frac{d^2y}{dx^2}$, $\frac{d}{dx}(y')=\frac{d}{dx}\left(-\frac{x}{y}\right)$

$\Rightarrow y''=\frac{y(-1)+xy'}{y^2}=\frac{-y+x\left(-\frac{x}{y}\right)}{y^2}$ since $y'=-\frac{x}{y} \Rightarrow \frac{d^2y}{dx^2}=y''=\frac{-y^2-x^2}{y^3}=\frac{-y^2-(1-y^2)}{y^3}=\frac{-1}{y^3}$

22. $x^{2/3}+y^{2/3}=1 \Rightarrow \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}\left[\frac{2}{3}y^{-1/3}\right]=-\frac{2}{3}x^{-1/3} \Rightarrow y'=\frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\left(\frac{y}{x}\right)^{1/3}$;

Differentiating again, $y''=\frac{x^{1/3}\cdot\left(-\frac{1}{3}y^{-2/3}\right)y'+y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}=\frac{x^{1/3}\left(-\frac{1}{3}y^{-2/3}\right)\left(-\frac{y^{1/3}}{x^{1/3}}\right)+y^{1/3}\left(\frac{1}{3}x^{-2/3}\right)}{x^{2/3}}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{3}x^{-2/3}y^{-1/3}+\frac{1}{3}y^{1/3}x^{-4/3}=\frac{y^{1/3}}{3x^{4/3}}+\frac{1}{3y^{1/3}x^{2/3}}$

23. $y^2=e^{x^2}+2x \Rightarrow 2yy'=2x+2=2xe^{x^2}+2 \Rightarrow \frac{dy}{dx}=\frac{xe^{x^2}+1}{y} \Rightarrow \frac{d^2y}{dx^2}=\frac{y\left(2x^2e^{x^2}+e^{x^2}\right)-\left(xe^{x^2}+1\right)y'}{y^2}$

$=\frac{y\left(2x^2e^{x^2}\right)-\left(xe^{x^2}+1\right)\cdot\frac{xe^{x^2}+1}{y}}{y^2}=\frac{\left(2x^2y^2+y^2-2x\right)e^{x^2}-x^2e^{2x^2}-1}{y^3}$

24. $y^2-2x=1-2y \Rightarrow 2y\cdot y'-2=-2y' \Rightarrow y'(2y+2)=2 \Rightarrow y'=\frac{1}{y+1}=(y+1)^{-1}$; then $y''=-(y+1)^{-2}\cdot y'$

$=-(y+1)^{-2}(y+1)^{-1} \Rightarrow \frac{d^2y}{dx^2}=y''=\frac{-1}{(y+1)^3}$

25. $2\sqrt{y} = x - y \Rightarrow y^{-1/2}y' = 1 - y' \Rightarrow y'\left(y^{-1/2} + 1\right) = 1 \Rightarrow \frac{dy}{dx} = y' = \frac{1}{y^{-1/2}+1} = \frac{\sqrt{y}}{\sqrt{y}+1}$; we can differentiate the equation $y'\left(y^{-1/2}+1\right) = 1$ again to find y'': $y'\left(-\frac{1}{2}y^{-3/2}y'\right) + \left(y^{-1/2}+1\right)y'' = 0 \Rightarrow \left(y^{-1/2}+1\right)y'' = \frac{1}{2}[y']^2 y^{-3/2}$

$$\Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{\frac{1}{2}\left(\frac{1}{y^{-1/2}+1}\right)^2 y^{-3/2}}{(y^{-1/2}+1)} = \frac{1}{2y^{3/2}(y^{-1/2}+1)^3} = \frac{1}{2\left(1+\sqrt{y}\right)^3}$$

26. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow xy' + 2yy' = -y \Rightarrow y'(x+2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)}$;

$$\frac{d^2y}{dx^2} = y'' = \frac{-(x+2y)y'+y(1+2y')}{(x+2y)^2} = \frac{-(x+2y)\left[\frac{-y}{(x+2y)}\right]+y\left[1+2\left(\frac{-y}{(x+2y)}\right)\right]}{(x+2y)^2} = \frac{\frac{1}{(x+2y)}[y(x+2y)+y(x+2y)-2y^2]}{(x+2y)^2} = \frac{2y(x+2y)-2y^2}{(x+2y)^3}$$
$$= \frac{2y^2+2xy}{(x+2y)^3} = \frac{2y(x+y)}{(x+2y)^3}$$

27. $x^3 + y^3 = 16 \Rightarrow 3x^2 + 3y^2y' = 0 \Rightarrow 3y^2y' = -3x^2 \Rightarrow y' = -\frac{x^2}{y^2}$; we differentiate $y^2y' = -x^2$ to find y'':

$$y^2y'' + y'[2y \cdot y'] = -2x \Rightarrow y^2y'' = -2x - 2y[y']^2 \Rightarrow y'' = \frac{-2x-2y\left(-\frac{x^2}{y^2}\right)^2}{y^2} = \frac{-2x-\frac{2x^4}{y^3}}{y^2} = \frac{-2xy^3-2x^4}{y^5}$$
$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{(2,2)} = \frac{-33-32}{32} = -2$$

28. $xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow y'(x+2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)} \Rightarrow y'' = \frac{(x+2y)(-y')-(-y)(1+2y')}{(x+2y)^2}$; since $y'\big|_{(0,-1)} = -\frac{1}{2}$ we obtain $y''\big|_{(0,-1)} = \frac{(-2)\left(\frac{1}{2}\right)-(-1)(0)}{4} = -\frac{1}{4}$

Exercises

In Exercises 1–5, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$
2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$
3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$
4. $f(x) = \sqrt[3]{x}, \quad a = -8$
5. $f(x) = \tan x, \quad a = \pi$
6. **Common linear approximations at $x = 0$** Find the linearizations of the following functions at $x = 0$.

   a. $\sin x$ **b.** $\cos x$ **c.** $\tan x$ **d.** e^x **e.** $\ln(1 + x)$

1. $f(x)=x^3-2x+3 \Rightarrow f'(x)=3x^2-2 \Rightarrow L(x)=f'(2)(x-2)+f(2)=10(x-2)+7 \Rightarrow L(x)=10x-13$ at $x=2$

2. $f(x)=\sqrt{x^2+9}=(x^2+9)^{1/2} \Rightarrow f'(x)=\left(\frac{1}{2}\right)(x^2+9)^{-1/2}(2x)=\frac{x}{\sqrt{x^2+9}} \Rightarrow L(x)=f'(-4)(x+4)+f(-4)$
$=-\frac{4}{5}(x+4)+5 \Rightarrow L(x)=-\frac{4}{5}x+\frac{9}{5}$ at $x=-4$

3. $f(x)=x+\frac{1}{x} \Rightarrow f'(x)=1-x^{-2} \Rightarrow L(x)=f(1)+f'(1)(x-1)=2+0(x-1)=2$

4. $f(x)=x^{1/3} \Rightarrow f'(x)=\frac{1}{3x^{2/3}} \Rightarrow L(x)=f'(-8)(x-(-8))+f(-8)=\frac{1}{12}(x+8)-2 \Rightarrow L(x)=\frac{1}{12}x-\frac{4}{3}$

5. $f(x)=\tan x \Rightarrow f'(x)=\sec^2 x \Rightarrow L(x)=f(\pi)+f'(\pi)(x-\pi)=0+1(x-\pi)=x-\pi$

6. (a) $f(x)=\sin x \Rightarrow f'(x)=\cos x \Rightarrow L(x)=f(0)+f'(0)(x-0)=x \Rightarrow L(x)=x$
(b) $f(x)=\cos x \Rightarrow f'(x)=-\sin x \Rightarrow L(x)=f(0)+f'(0)(x-0)=1 \Rightarrow L(x)=1$
(c) $f(x)=\tan x \Rightarrow f'(x)=\sec^2 x \Rightarrow L(x)=f(0)+f'(0)(x-0)=x \Rightarrow L(x)=x$
(d) $f(x)=e^x \Rightarrow f'(x)=e^x \Rightarrow L(x)=f(0)+f'(0)(x-0)=1+x \Rightarrow L(x)=1+x$
(e) $f(x)=\ln(1+x) \Rightarrow f'(x)=\frac{1}{1+x} \Rightarrow L(x)=f(0)+f'(0)(x-0)=x \Rightarrow L(x)=x$

Exercises

15. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

16. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

    a. $f(x) = (1 - x)^6$ b. $f(x) = \dfrac{2}{1 - x}$

15. $f'(x) = k(1+x)^{k-1}$. We have $f(0) = 1$ and $f'(0) = k$. $L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx$

16. (a) $f(x) = (1-x)^6 = [1+(-x)]^6 \approx 1 + 6(-x) = 1 - 6x$

(b) $f(x) = \frac{2}{1-x} = 2\left[1+(-x)\right]^{-1} \approx 2[1+(-1)(-x)] = 2 + 2x$

Exercises

In Exercises 19–38, find dy.

19. $y = x^3 - 3\sqrt{x}$

20. $y = x\sqrt{1 - x^2}$

21. $y = \dfrac{2x}{1 + x^2}$

22. $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$

23. $2y^{3/2} + xy - x = 0$

24. $xy^2 - 4x^{3/2} - y = 0$

25. $y = \sin(5\sqrt{x})$

26. $y = \cos(x^2)$

27. $y = 4\tan(x^3/3)$

28. $y = \sec(x^2 - 1)$

29. $y = 3\csc\left(1 - 2\sqrt{x}\right)$

30. $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

19. $y = x^3 - 3\sqrt{x} = x^3 - 3x^{1/2} \Rightarrow dy = \left(3x^2 - \frac{3}{2}x^{-1/2}\right)dx \Rightarrow dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right)dx$

20. $y = x\sqrt{1-x^2} = x(1-x^2)^{1/2} \Rightarrow dy = \left[(1)(1-x^2)^{1/2} + (x)\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)\right]dx$

$= \left(1-x^2\right)^{-1/2}\left[(1-x^2) - x^2\right]dx = \frac{\left(1-2x^2\right)}{\sqrt{1-x^2}}dx$

21. $y = \frac{2x}{1+x^2} \Rightarrow dy = \left(\frac{(2)(1+x^2)-(2x)(2x)}{(1+x^2)^2}\right)dx = \frac{2-2x^2}{(1+x^2)^2}\,dx$

22. $y = \frac{2\sqrt{x}}{3\left(1+\sqrt{x}\right)} = \frac{2x^{1/2}}{3\left(1+x^{1/2}\right)} \Rightarrow dy = \left(\frac{x^{-1/2}\left(3\left(1+x^{1/2}\right)\right) - 2x^{1/2}\left(\frac{3}{2}x^{-1/2}\right)}{9\left(1+x^{1/2}\right)^2}\right)dx = \frac{3x^{-1/2}+3-3}{9(1+x^{1/2})^2}\,dx \Rightarrow dy = \frac{1}{3\sqrt{x}(1+\sqrt{x})^2}\,dx$

23. $2y^{3/2} + xy - x = 0 \Rightarrow 3y^{1/2}dy + y\,dx + x\,dy - dx = 0 \Rightarrow (3y^{1/2} + x)\,dy = (1-y)\,dx \Rightarrow dy = \frac{1-y}{3\sqrt{y}+x}dx$

24. $xy^2 - 4x^{3/2} - y = 0 \Rightarrow y^2dx + 2xy\,dy - 6x^{1/2}\,dx - dy = 0 \Rightarrow (2xy-1)\,dy = (6x^{1/2} - y^2)dx \Rightarrow dy = \frac{6\sqrt{x}-y^2}{2xy-1}dx$

25. $y = \sin(5\sqrt{x}) = \sin(5x^{1/2}) \Rightarrow dy = (\cos(5x^{1/2}))\left(\frac{5}{2}x^{-1/2}\right)dx \Rightarrow dy = \frac{5\cos\left(5\sqrt{x}\right)}{2\sqrt{x}}dx$

26. $y = \cos(x^2) \Rightarrow dy = [-\sin(x^2)](2x)dx = -2x\sin(x^2)dx$

27. $y = 4\tan\left(\frac{x^3}{3}\right) \Rightarrow dy = 4\left(\sec^2\left(\frac{x^3}{3}\right)\right)(x^2)\,dx \Rightarrow dy = 4x^2\sec^2\left(\frac{x^3}{3}\right)dx$

28. $y = \sec\left(x^2-1\right) \Rightarrow dy = [\sec(x^2-1)\tan(x^2-1)](2x)\,dx = 2x[\sec(x^2-1)\tan(x^2-1)]dx$

29. $y = 3\csc(1-2\sqrt{x}) = 3\csc(1-2x^{1/2}) \Rightarrow dy = 3(-\csc(1-2x^{1/2}))\cot(1-2x^{1/2})\,(-x^{-1/2})\,dx$
$\Rightarrow dy = \frac{3}{\sqrt{x}}\csc(1-2\sqrt{x})\cot(1-2\sqrt{x})\,dx$

Thank you for your attention