



## المحاضرة الثالثة: المحددات

تحليل رياضي 2

جامعة المنارة

2023-2024

**المحددات**

**المحددات والعمليات الأولية**

**خواص المحددات**

## The Determinant of a Matrix محدد مصفوفة

$\det: M_n(K) \rightarrow K; A \mapsto \det(A) = |A|$  محدد مصفوفة عبارة عن تابع معرف بالشكل

$K: (R \text{ or } C)$  مجموعة عددية

- The determinant of a  $2 \times 2$  matrix  $2 \times 2$  :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- Ex. 1:

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$$

ملاحظة: محدد المصفوفة قد يكون موجب أو سالب أو صفر ▪

- **Minor of the entry  $a_{ij}$ :**

محدد المصفوفة الناتجة عن حذف السطر-i والعمود j من المصفوفة

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}$$

- **Cofactor of  $a_{ij}$ :**

$$C_{ij} = (-1)^{i+j} M_{ij}$$

■ Ex 2:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$

- Ex 3: Find all the minors and cofactors of  $A$

أوجد كافة المصغرات والمرافقات الجبرية

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Sol: (1) All the minors of  $A$

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,$$

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5,$$

$$M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2,$$

$$M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,$$

$$M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,$$

$$M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3,$$

$$M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

## (2) All the cofactors of $A$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = + \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,$$

$$C_{12} = - \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5,$$

$$C_{13} = + \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2,$$

$$C_{22} = + \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,$$

$$C_{23} = - \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,$$

$$C_{32} = - \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3,$$

$$C_{33} = + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$

▪ Theorem :

محدد المصفوفة المربعة  $A$  يحسب بالشكل

$$(a) \det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

( $i$ -th row,  $i = 1, 2, \dots, n$ )

or

$$(b) \det(A) = |A| = \sum_{i=1}^n a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

( $j$ -th column,  $j = 1, 2, \dots, n$ )

▪ Ex 4: 3 محدد مصفوفة من الحجم

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}\end{aligned}$$

- Ex 5: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} C_{11} &= -1, C_{12} = 5, C_{13} = 4 \\ \text{From Ex 3: } C_{21} &= -2, C_{22} = -4, C_{23} = 8 \\ C_{31} &= 5, C_{32} = 3, C_{33} = -6 \end{aligned}$$

Sol:

$$\begin{aligned}
 \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\
 &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\
 &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\
 &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\
 &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\
 &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14
 \end{aligned}$$

- Ex 6: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ -4 & 1 \end{vmatrix} = 7$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (-1)(-5) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix} = -8$$

$$\begin{aligned}\Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (0)(7) + (2)(5) + (1)(-8) = 2\end{aligned}$$

- Note:

أفضل خيار لحساب المحدد هو النشر وفق السطر أو العمود الذي يحوي أكبر عدد من الأصفار

- Ex 7: The determinant of a matrix of order 4

$$A = \begin{bmatrix} 1 & -2 & \boxed{3} & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$

Sol:

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) = 3C_{13}$$

$$= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3 \left[ (0)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right]$$

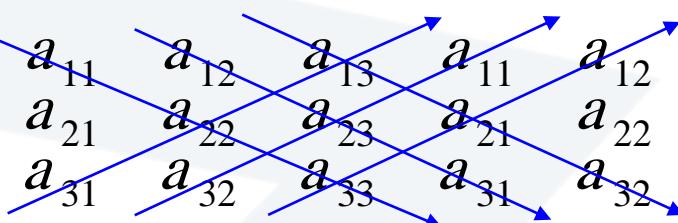
$$= 3[0 + (2)(1)(-4) + (3)(-1)(-7)]$$

$$= (3)(13)$$

$$= 39$$

- The determinant of a matrix of order 3 (Sarrus Rule)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



ثم طرح هذه الجداءات الثلاث

جمع هذه الجداءات الثلاث

$$\Rightarrow \det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

- Ex 8:

$$A = \begin{bmatrix} 0 & 2 & 1 & -4 & 0 & 6 \\ 3 & -1 & 2 & 0 & 2 & \\ 4 & -4 & 1 & 3 & -1 & \\ & & & 4 & -4 & \\ & & & 0 & 16 & -12 \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4 + 0 + 6) = 2$$

- **المصفوفة المثلثية العليا:** Upper triangular matrix

كافة العناصر الواقعة تحت قطر الرئيسي أصفار

- **المصفوفة المثلثية السفلى:** Lower triangular matrix

كافة العناصر فوق قطر الرئيسي أصفار

- **المصفوفة القطرية:** Diagonal matrix

كافة العناصر الواقعة تحت وفوف قطر الرئيسي أصفار

- Ex :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

diagonal

- Theorem :

If  $A$  is an  $n \times n$  triangular matrix (upper triangular, lower triangular, or diagonal),  
عندما تكون المصفوفة قطرية أو مثلثية العليا أو سفلية يكون المحدد جداء عناصر قطر الرئيسي

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdots a_{nn}$$

▪ Ex :

$$(a) \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

$$(b) \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$(a) \quad |A| = (2)(-2)(1)(3) = -12$$

$$(b) \quad |B| = (-1)(3)(2)(4)(-2) = 48$$

## Evaluation of a determinant using elementary operations

### ▪ Theorem : (العمليات الأولية على الأسطر والمحددات)

Let  $A$  and  $B$  be square matrices . مصفوفتين مربعتين .

$$(a) B = r_{ij}(A) \Rightarrow \det(B) = -\det(A) \quad (\text{i.e. } |r_{ij}(A)| = -|A|)$$

$$(b) B = r_i^{(k)}(A) \Rightarrow \det(B) = k \det(A) \quad (\text{i.e. } |r_i^{(k)}(A)| = k|A|)$$

$$(c) B = r_{ij}^{(k)}(A) \Rightarrow \det(B) = \det(A) \quad (\text{i.e. } |r_{ij}^{(k)}(A)| = |A|)$$

■ Ex :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$

- Notes:

$$\det(r_{ij}(A)) = -\det(A) \Rightarrow \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k \det(A) \Rightarrow \det(A) = \frac{1}{k} \det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \Rightarrow \det(A) = \det(r_{ij}^{(k)}(A))$$

- Ex :

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}, \quad \det(A) = ?$$

Sol:

$$\begin{aligned} \det(A) &= \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \xrightarrow{r_{12}} - \begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \xrightarrow{r_{12}^{(-2)}} - \begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix} \\ &\xrightarrow{r_2^{\left(\frac{-1}{7}\right)}} (-1)\left(\frac{1}{\frac{-1}{7}}\right) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} \xrightarrow{r_{23}^{(-1)}} (7) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} \xrightarrow{r_3^{(-1)}} (7)(-1) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} \\ &= (7)(-1)(1)(1)(1) = -7 \end{aligned}$$

## Determinants and elementary column operations

### ▪ Theorem : (العمليات الأولية على الأعمدة والمحددات )

Let  $A$  and  $B$  be square matrices . مصفوفتين مربعتين .

$$(a) B = c_{ij}(A) \Rightarrow \det(B) = -\det(A) \quad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b) B = c_i^{(k)}(A) \Rightarrow \det(B) = k \det(A) \quad (\text{i.e. } |c_i^{(k)}(A)| = k|A|)$$

$$(c) B = c_{ij}^{(k)}(A) \Rightarrow \det(B) = \det(A) \quad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$

- Ex :

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{\left(\frac{1}{2}\right)}(A) \Rightarrow \det(A_1) = \det(c_1^{\left(\frac{1}{2}\right)}(A)) = \frac{1}{2} \det(A) = \left(\frac{1}{2}\right)(-8) = -4$$

$$A_2 = c_{12}(A) \Rightarrow \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \Rightarrow \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$

- **Theorem : (Conditions that yield a zero determinant)**

إذا كانت المصفوفة مربعة وتحقق إحدى الشروط التالية عندئذ  $0 = \det(A)$

1. سطر (عمود) كل عناصره أصفار.
2. سطرين أو عمودين متساوين.
3. سطر (عمود) مضاعف لسطر (عمود) آخر

▪ Ex :

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$

- Ex : (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} c_{13}^{(2)} = \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ \boxed{-3 & 0 & 0} \end{vmatrix} = (-3)(-1)^{3+1} \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = (-3)(-1) = 3$$

$$\begin{aligned} \det(A) &= \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} r_{12}^{(\frac{4}{5})} = \begin{vmatrix} -3 & \boxed{5} & 2 \\ -2/5 & 0 & 3/5 \\ -3 & 0 & 6 \end{vmatrix} \\ &= (5)(-1)^{1+2} \begin{vmatrix} -2/5 & 3/5 \\ -3 & 6 \end{vmatrix} = (-5)(-\frac{3}{5}) = 3 \end{aligned}$$

- Ex : (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} r_{24}^{(1)} r_{25}^{(-1)} \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1)(-1)^{2+2} \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$

$$c_{41}^{(-3)} \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ \boxed{0 & 0 & 0 & 1} \end{vmatrix} = (1)(-1)^{4+4} \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} I_{21}^{(1)} \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3} \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$

## Properties of Determinants

- **Theorem 3.6:** (محدد جداء مصفوفة )

$$\det(AB) = \det(A)\det(B)$$

- **Notes:**

$$(1) \det(EA) = \det(E)\det(A)$$

$$(2) \det(A + B) \neq \det(A) + \det(B)$$

$$(3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

- Ex :

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

Find  $|A|$ ,  $|B|$ , and  $|AB|$

Sol:

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7, \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

تحقق:

$$|AB| = |A| |B|$$

$$-77 = -7 \times 11$$

- **Theorem : (محدد مصفوفة مضروبة بعده)**

If  $A$  is an  $n \times n$  matrix and  $c$  is a scalar, then

$$\det(cA) = c^n \det(A)$$

- **Ex 2:**

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5 \quad \text{Find } |A|$$

**Sol:**

$$A = 10 \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$

- **(محدد مصفوفة غير شاذة) Theorem :**

تكون المصفوفة المربعة قابلة للقلب إذا وفقط إذا كان  $\det(A) \neq 0$

- **حدد فيما إذا كانت المصفوفة شاذة أو لا Ex 3:**

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

**Sol:**

$|A| = 0 \Rightarrow A$  شاذة (it is singular).

$|B| = -12 \neq 0 \Rightarrow B$  قابلة للقلب (it is nonsingular).

- **(محدد مقلوب مصفوفة ) Theorem :**

If  $A$  is invertible, then  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

- **(محدد منقول مصفوفة ) Theorem :**

If  $A$  is a square matrix, then  $\det(A^T) = \det(A)$ .

- **Ex :**

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \quad (a) \quad |A^{-1}| = ? \quad (b) \quad |A^T| = ?$$

Sol:

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4 \Rightarrow |A^{-1}| = \frac{1}{|A|} = \frac{1}{4}, \quad |A^T| = |A| = 4$$

- **Equivalent conditions for a nonsingular matrix:**

If  $A$  is an  $n \times n$  matrix, عندئذ الشروط التالية متكافئة.

(1) قابلة للقلب  $A$ .

(2) تملك حل وحيد  $Ax = b$ .

(3) تملك الحل الصفرى  $Ax = 0$ .

(4)  $\det(A) \neq 0$

- Ex : Which of the following system has a unique solution?

$$(a) \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b) \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

Sol:

(a)  $A\mathbf{x} = \mathbf{b} \Rightarrow |A| = 0$ . هذه الجملة لا تملك حل وحيد.

(b)  $B\mathbf{x} = \mathbf{b} \Rightarrow |B| = -12 \neq 0$ . هذه الجملة تملك حل وحيد.