

Exercises 1: Systems of Linear Equations

CEDC102 : Linear Algebra and Matrix Theory

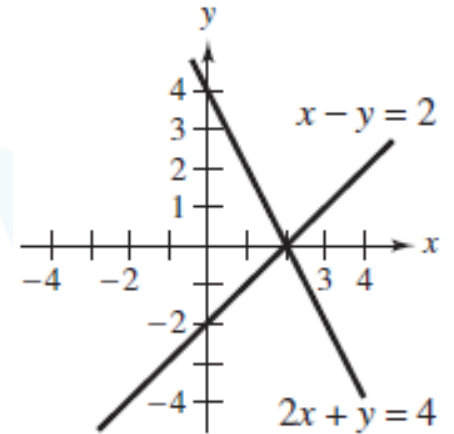
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Graph the system of linear equations. Solve the system and interpret your answer

$$\textcircled{1} \quad \begin{aligned} 2x + y &= 4 \\ x - y &= 2 \end{aligned}$$

Adding the first equation to the second produces a new equation, $3x = 6$, or $x = 2$. So, $y = 0$, and the solution is $x = 2$, $y = 0$.

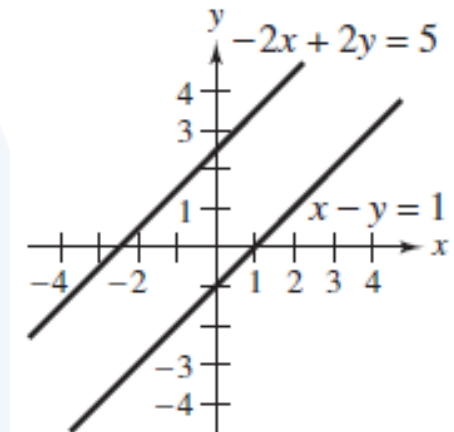


$$\textcircled{2} \quad \begin{aligned} x - y &= 1 \\ -2x + 2y &= 5 \end{aligned}$$

Adding 2 times the first equation to the second produces

$$\begin{aligned} x - y &= 1 \\ 0 &= 7 \end{aligned}$$

The second equation is a false statement, therefore the original system has no solution. The two lines are parallel



$$\textcircled{3} \quad \begin{aligned} \frac{1}{2}x - \frac{1}{3}y &= 1 \\ -2x + \frac{4}{3}y &= -4 \end{aligned}$$

Multiplying the first equation by 2 produces

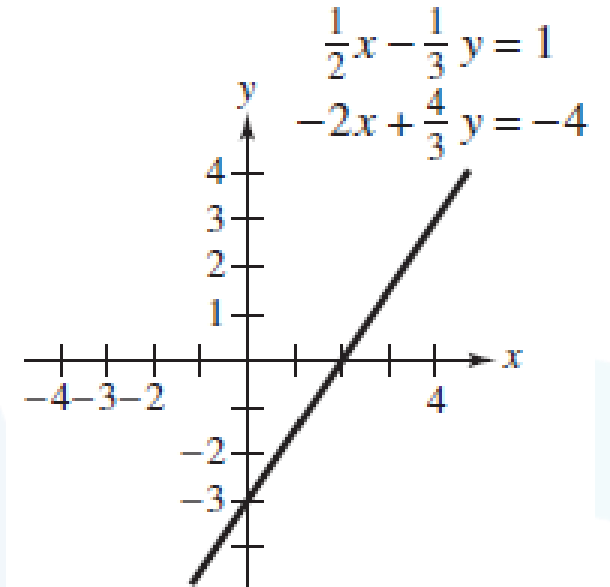
$$\begin{aligned} x - \frac{2}{3}y &= 1 \\ -2x + \frac{4}{3}y &= -4 \end{aligned}$$

Adding 2 times the first equation to the second equation produces

$$\begin{aligned} x - \frac{2}{3}y &= 1 \\ 0 &= 0 \end{aligned}$$

Choosing $y = t$ as the free variable, $x = (2/3)t + 2$.

So, the solution set is $x = (2/3)t + 2$ and $y = t$, where t is any real number.



Solve the system of linear equations

$$\textcircled{1} \quad \begin{array}{rcl} x_1 & - & x_2 = 0 \\ 3x_1 & - & 2x_2 = -1 \end{array}$$

Adding -3 times the first equation to the second equation produces

$$\begin{array}{rcl} x_1 & - & x_2 = 0 \\ & & x_2 = -1 \end{array}$$

Using back-substitution you can conclude that the system has exactly one solution: $x_1 = -1$ and $x_2 = -1$

$$\textcircled{2} \quad \begin{array}{rcl} 3x & + & 2y = 2 \\ 6x & + & 4y = 14 \end{array}$$

Adding -2 times the first equation to the second equation produces

$$\begin{array}{rcl} 3x + 2y & = & 2 \\ 0 & = & 10 \end{array}$$

Because the second equation is a false statement, the original system of equations has no solution.

$$\textcircled{3} \quad \begin{array}{rcl} \frac{2}{3}x + \frac{1}{6}y & = & 0 \\ 4x + y & = & 0 \end{array}$$

Multiplying the first equation by $3/2$ produces

$$\begin{array}{rcl} x + \frac{1}{4}y & = & 0 \\ 4x + y & = & 0 \end{array}$$

Adding -4 times the first equation to the second produces

$$\begin{array}{rcl} x + \frac{1}{4}y & = & 0 \\ 0 & = & 0 \end{array}$$

Choosing $x = t$ as the free variable, $y = -(1/4)t$

So the solution set is $x = t$ and $y = -(1/4)t$, where t is any real number

$$\begin{array}{rcl} x & + & y + z = 6 \\ \textcircled{4} \quad 2x & - & y + z = 3 \\ 3x & & - z = 0 \end{array}$$

Adding -2 times the first equation to the second produces

$$\begin{array}{rcl} x & + & y + z = 6 \\ & -3y & - z = -9 \\ & -3y & - 4z = -18 \end{array}$$

Dividing the second equation by -3 produces

$$\begin{array}{rcl} x & + & y + z = 6 \\ & y & + \frac{1}{3}z = 3 \\ & -3y & - 4z = -18 \end{array}$$

Adding 3 times the second equation to the third equation produces

$$\begin{array}{rcl} x & + & y + z = 6 \\ & y & + \frac{1}{3}z = 3 \\ & - & 3z = -9 \end{array}$$

$$x + y + z = 6$$

Dividing the third equation by -3 produces

$$y + \frac{1}{3}z = 3$$

$$z = 3$$

Using back-substitution you can conclude that the system has exactly one solution: $x=1$, $y=2$, and $z=3$

$$3x - 2y + 4z = 1$$

$$\textcircled{5} \quad x + y - 2z = 3$$

$$2x - 3y + 6z = 8$$

Dividing the first equation by 3 produces

$$x - \frac{2}{3}y + \frac{4}{3}z = \frac{1}{3}$$

$$x + y - 2z = 3$$

$$2x - 3y + 6z = 8$$

Subtracting the first equation from the second equation produces

$$\begin{aligned}x - \frac{2}{3}y + \frac{4}{3}z &= \frac{1}{3} \\ \frac{5}{3}y - \frac{10}{3}z &= \frac{8}{3} \\ 2x - 3y + 6z &= 8\end{aligned}$$

Adding -2 times the first equation to the third equation produces

$$\begin{aligned}x - \frac{2}{3}y + \frac{4}{3}z &= \frac{1}{3} \\ \frac{5}{3}y - \frac{10}{3}z &= \frac{8}{3} \\ -\frac{5}{3}y + \frac{10}{3}z &= \frac{22}{3}\end{aligned}$$

Equations 2 and 3 cannot both be satisfied. So, the original system of equations has no solution

$$\textcircled{6} \quad \begin{aligned}2x_1 + x_2 - 3x_3 &= 4 \\ 4x_1 + 2x_3 &= 10 \\ -2x_1 + 3x_2 - 13x_3 &= -8\end{aligned}$$

Dividing the first equation by 2 produces

$$\begin{array}{rclcrcl} x_1 & + & \frac{1}{2}x_2 & - & \frac{3}{2}x_3 & = & 2 \\ 4x_1 & & & + & 2x_3 & = & 10 \\ -2x_1 & + & 3x_2 & - & 13x_3 & = & -8 \end{array}$$

Adding -4 times the first equation to the second equation produces

$$\begin{array}{rclcrcl} x_1 & + & \frac{1}{2}x_2 & - & \frac{3}{2}x_3 & = & 2 \\ & & -2x_2 & + & 8x_3 & = & 2 \\ -2x_1 & + & 3x_2 & - & 13x_3 & = & -8 \end{array}$$

Adding 2 times the first equation to the third equation produces

$$\begin{array}{rclcrcl} x_1 & + & \frac{1}{2}x_2 & - & \frac{3}{2}x_3 & = & 2 \\ & & -2x_2 & + & 8x_3 & = & 2 \\ & & 4x_2 & - & 16x_3 & = & -4 \end{array}$$

Dividing the second equation by -2 produces

$$\begin{array}{rclcl} x_1 & + & \frac{1}{2}x_2 & - & \frac{3}{2}x_3 & = & 2 \\ & & x_2 & - & 4x_3 & = & -1 \\ & & 4x_2 & - & 16x_3 & = & -4 \end{array}$$

Adding -4 times the second equation to the third equation produces

$$\begin{array}{rclcl} x_1 & + & \frac{1}{2}x_2 & - & \frac{3}{2}x_3 & = & 2 \\ & & x_2 & - & 4x_3 & = & -1 \\ & & 0 & = & 0 \end{array}$$

Choosing $x_3 = t$ as the free variable

The solution is $x_1 = (5/2)t - 1/2$, $x_2 = 4t - 1$, $x_3 = t$, where t is any real number

Determine the value(s) of k such that the system of linear equations has exactly one solution, no solution, infinitely many solutions equations

$$\textcircled{1} \quad \begin{aligned} 4x + ky &= 6 \\ kx + y &= -3 \end{aligned}$$

Dividing the first equation by 4 produces

$$\begin{aligned} x + \frac{k}{4}y &= \frac{3}{2} \\ kx + y &= -3 \end{aligned}$$

Adding $-k$ times the first equation to the second equation produces

$$\begin{aligned} x + \frac{k}{4}y &= \frac{3}{2} \\ (1 - \frac{k^2}{4})y &= -\frac{3}{2}k - 3 \end{aligned}$$

$$(1 - \frac{k^2}{4}) = 0 \Rightarrow k = \pm 2$$

$$k = 2 \Rightarrow \text{produces} \quad \begin{aligned} x + \frac{1}{2}y &= \frac{3}{2} \\ 0 &= -6 \end{aligned} \Rightarrow \text{No solution}$$

$$k = -2 \Rightarrow \text{produces} \quad \begin{array}{rcl} x + \frac{1}{2}y & = & \frac{3}{2} \\ 0 & = & 0 \end{array} \Rightarrow \text{Infinitely many solutions}$$

$$k \neq \pm 2 \Rightarrow \text{exactly one solution}$$

$$\textcircled{2} \quad \begin{array}{rcl} x + ky & = & 0 \\ kx + y & = & 0 \end{array}$$

Adding $-k$ times the first equation to the second equation produces

$$x + ky = 0$$

$$(1 - k^2)y = 0$$

$$(1 - k^2) = 0 \Rightarrow k = \pm 1$$

$$k = \pm 1 \text{ produces } \begin{array}{rcl} x + y & = & 0 \\ 0 & = & 0 \end{array} \Rightarrow \text{Infinitely many solutions}$$

$$k \neq \pm 1 \Rightarrow \text{exactly one solution (trivial solution } x = y = 0)$$

$$\textcircled{3} \quad \begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 8z &= 4 \end{aligned}$$

Reduce the system to row-echelon form

$$\begin{aligned} x + 2y + kz &= 6 \\ (8-3k)z &= -14 \end{aligned}$$

$k = 8/3 \Rightarrow$ no solution

$k \neq 8/3 \Rightarrow$ infinitely many solutions

Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination

$$\textcircled{1} \quad \begin{array}{rcrcrcrcl} x & + & 2y & = & 7 \\ 2x & + & y & = & 8 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 7 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-2)}} \begin{bmatrix} 1 & 2 & 7 \\ 0 & -3 & -6 \end{bmatrix} \xrightarrow{r_2^{(-1/3)}} \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcl} x & + & 2y & = & 7 \\ & & y & = & 2 \end{array}$$

Using back-substitution you find that $x=3$ and $y=2$. Or using Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow x=3 \text{ and } y=2$$

$$\begin{aligned} -3x + 5y &= -22 \\ \textcircled{2} \quad 3x + 4y &= 4 \\ 4x - 8y &= 32 \end{aligned}$$

$$\begin{bmatrix} -3 & 5 & -22 \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} r_1^{(-1/3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 3 & 4 & 4 \\ 4 & -8 & 32 \end{bmatrix} r_{12}^{(-3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 4 & -8 & 32 \end{bmatrix} r_{13}^{(-4)} \rightarrow$$

$$\begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 9 & -18 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix} r_2^{(1/9)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & -\frac{4}{3} & \frac{8}{3} \end{bmatrix} r_{23}^{(4/3)} \rightarrow \begin{bmatrix} 1 & -\frac{5}{3} & \frac{22}{3} \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x - (5/3)y &= 22/3 \\ y &= -2 \end{aligned}$$

Using back-substitution you find that $x=4$ and $y=-2$

$$\begin{aligned} x - 3z &= -2 \\ \textcircled{3} \quad 3x + y - 2z &= 5 \\ 2x + 2y + z &= 4 \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 3 & 1 & -2 & 5 \\ 2 & 2 & 1 & 4 \end{bmatrix} &\xrightarrow{r_{12}^{(-3)}} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 2 & 2 & 1 & 4 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 2 & 7 & 8 \end{bmatrix} \\ &\xrightarrow{r_{23}^{(-2)}} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & -7 & -14 \end{bmatrix} \xrightarrow{r_3^{(-1/7)}} \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{aligned}$$

Using back-substitution you find that $x=4$, $y=-3$ and $z=2$
Or using Gauss-Jordan elimination

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 7 & 11 \\ 0 & 0 & 1 & 2 \end{bmatrix} r_{32}^{(-7)} \rightarrow \begin{bmatrix} 1 & 0 & -3 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} r_{31}^{(3)} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow x=4, y=-3 \text{ and } z=2$$

④

$$\begin{aligned} x + y - 5z &= 3 \\ x - 2z &= 1 \\ 2x - y - z &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{bmatrix} r_{12}^{(-1)} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 2 & -1 & -1 & 0 \end{bmatrix} r_3^{(-1)} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 2 & -1 & -1 & 0 \end{bmatrix}$$

$$r_{13}^{(-2)} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{bmatrix} r_{23}^{(3)} \rightarrow \begin{bmatrix} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}x + y - 5z &= 3 \\y - 3z &= 2 \\0 &= 0\end{aligned}$$

Choosing $z = t$ as the free variable

The solution is $x = 1 + 2t$, $y = 2 + 3t$, $z = t$, where t is any real number

$$\begin{aligned}&2x + 3z = 3 \\⑤ \quad &4x - 3y + 7z = 5 \\&8x - 9y + 15z = 10\end{aligned}$$

$$\begin{bmatrix} 2 & 0 & 3 & 3 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} r_1^{(1/2)} \rightarrow \begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 4 & -3 & 7 & 5 \\ 8 & -9 & 15 & 10 \end{bmatrix} r_{12}^{(-4)} \rightarrow \begin{bmatrix} 1 & 0 & 3/2 & 3/2 \\ 0 & -3 & 1 & -1 \\ 8 & -9 & 15 & 10 \end{bmatrix}$$

$$r_{13}^{(-8)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -3 & 1 & -1 \\ 0 & -9 & 3 & -2 \end{bmatrix} r_{23}^{(-3)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} r_2^{(-1/3)} \rightarrow \begin{bmatrix} 1 & 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Because the third row corresponds to the equation $0 = 1$, there is no solution to the original system

Graph the system of linear equations. Solve the system and interpret your answer

①
$$\begin{aligned} x + 3y &= 2 \\ -x + 2y &= 3 \end{aligned}$$

②
$$\begin{aligned} x + 3y &= 17 \\ 4x + 3y &= 7 \end{aligned}$$

③
$$\begin{aligned} \frac{x}{4} + \frac{y}{6} &= 1 \\ x - y &= 3 \end{aligned}$$

Solve the system of linear equations

①
$$\begin{aligned} 3u + v &= 240 \\ u + 3v &= 240 \end{aligned}$$

②
$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$$

③
$$\begin{aligned} x - y - z &= 0 \\ x + 2y - z &= 6 \\ 2x - z &= 5 \end{aligned}$$

④
$$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

⑤
$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$

Determine the value(s) of k such that the system of linear equations has exactly one solution, no solution, infinitely many solutions equations

①
$$\begin{aligned} x + ky &= 2 \\ kx + y &= 4 \end{aligned}$$

②
$$\begin{aligned} 4x + ky &= 0 \\ kx + y &= 0 \end{aligned}$$

③
$$\begin{aligned} x + 2y + kz &= 6 \\ 3x + 6y + 9z &= -1 \end{aligned}$$

Solve the system using either Gaussian elimination with back-substitution or Gauss-Jordan elimination

①
$$\begin{aligned} x + 3y &= 11 \\ 3x + y &= 9 \end{aligned}$$

②
$$\begin{aligned} x + 2y &= 0 \\ x + y &= 6 \\ 3x - 2y &= 8 \end{aligned}$$

③
$$\begin{aligned} 2x - 2y + 3z &= 22 \\ 3y - z &= 24 \\ 6x - 7y &= -22 \end{aligned}$$

④
$$\begin{aligned} x + y + z &= 2 \\ -x + 3y + 2z &= 8 \\ 4x + y &= 4 \end{aligned}$$

⑤
$$\begin{aligned} 5x_1 - 3x_2 + 2x_3 &= 3 \\ 2x_1 + 4x_2 - x_3 &= 7 \\ x_1 - 11x_2 + 4x_3 &= 3 \end{aligned}$$