Applications 1

CEDC102: Linear Algebra

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- Graphs and networks.
- Incidence matrices.
- Kirchhoff's laws

The model consists of nodes connected by edges. This is called a graph.

Graphs of this node-edge kind lead to matrices.

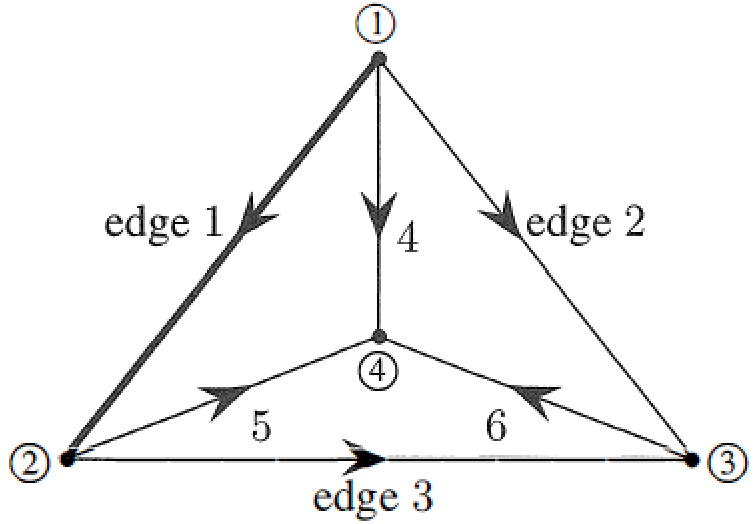
Incidence matrix of a graph- tells how the $\boldsymbol{n}$ nodes are connected by the $\boldsymbol{m}$ edges. Normally $\boldsymbol{m} > \boldsymbol{n}$, there are more edges than nodes

For any $\boldsymbol{m}$ by $\boldsymbol{n}$ matrix there are two fundamental subspaces in $\mathbb{R}^{\boldsymbol{n}}$ and two in $\mathbb{R}^{\boldsymbol{m}}$. They are the column spaces and nullspaces of $\boldsymbol{A}$ and $\boldsymbol{A}^{\boldsymbol{T}}$. Their dimensions r, $\boldsymbol{n} - \boldsymbol{r}$ and $\boldsymbol{r}, \boldsymbol{m} - \boldsymbol{r}$ come from the most important theorem in linear algebra.

Figure 1 displays a graph with $\boldsymbol{m} = \mathbf{6}$ edges and $\boldsymbol{n} = \mathbf{4}$ nodes. The $\mathbf{6}$ by $\mathbf{4}$ matrix $\boldsymbol{A}$ tells which nodes are connected by which edges.

The first row $-\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}$ shows that the first edge goes from node $\mathbf{1}$ to node $\mathbf{2}$ ($-\mathbf{1}$ for node $\mathbf{1}$ because the arrow goes out, $+\mathbf{1}$ for node $\mathbf{2}$ with arrow in).

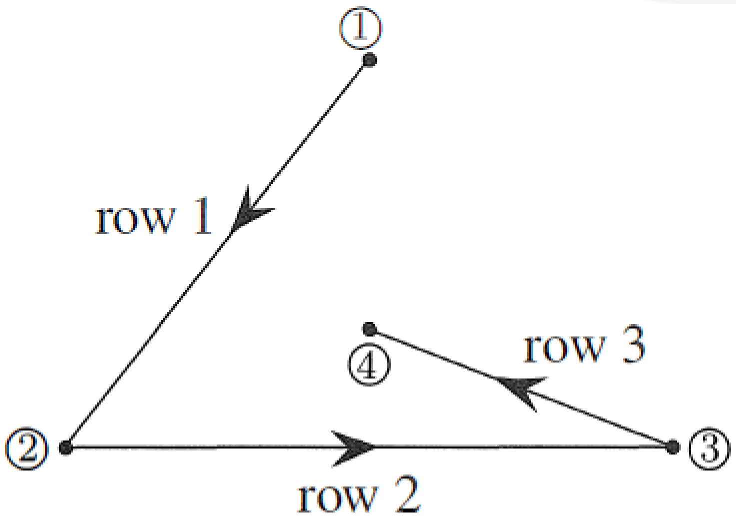
Row numbers in $\boldsymbol{A}$ are edge numbers, column numbers $\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$ are node numbers!



$$
\begin{array}{c}
\textbf{node} \\
\begin{array}{ccccc}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & & \\
A = & \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \textbf{edge}
\end{array}
\end{array}
$$

Figure 1: Complete graph with $m=6$ edges and $n=4$ nodes: 6 by 4 incidence matrix A.

You can write down the matrix by looking at the graph. The second graph has the same four nodes but only three edges. Its incidence matrix B is 3 by 4.



$$B = \begin{array}{c} \begin{matrix} ① & ② & ③ & ④ \end{matrix} \\ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad \text{edge}$$

Figure .1*: Tree with 3 edges and 4 nodes and no loops. Then B has independent rows.

The first graph is complete-every pair of nodes is connected by an edge.

The second graph is a tree-the graph has no closed loops

- The maximum number of edges is $\frac{1}{2}n(n - 1) = 6$
- and the minimum to stay connected is $n - l = 3$.

Elimination reduces every graph to a tree.

Rows are dependent when edges form a loop.

When x_1, x_2, x_3, x_4 are voltages at the nodes, Ax gives voltage differences:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix}.$$

If the voltages are equal, the differences are zero. This tells us the nullspace of A.

1. The nullspace contains the solutions to $Ax = 0$. All six voltage differences are zero.

This means: All four voltages are equal. Every x in the nullspace is a constant vector:

$x = (c, c, c, c)$. The nullspace of A is a line in $\mathbb{R}^{n}$-its dimension is $n - r = 1$.

The second incidence matrix B has the same nullspace. It contains $(1, 1, 1, 1)$:

1−dimensional nullspace: same for the tree

$$Bx = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can raise or lower all voltages by the same amount c, without changing the differences. There is an "arbitrary constant" in the voltages.

2. The left nullspace contains the solutions to $\boldsymbol{A^T}$y = 0. Its dimension is $m - r = 6 - 3$:

$$\textbf{Current Law} \qquad A^{\mathrm{T}}y = \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (2)$$

The true number of equations is $r =$ 3 and not $n =$ 4. Reason: The four equations add to 0 = 0.

The fourth equation follows automatically from the first three.
What do the equations mean?

The first equation says that $-y_1 - y_2 - y_4 = 0$. The net flow into node 1 is zero.

The fourth equation says that $y_4 + y_5 + y_6 = 0$.
Flow into node 4 minus flow out is zero

The equations $\boldsymbol{A^T}y = 0$ are famous and fundamental:

Kirchhoff's Current Law: $A^T y = 0$ ***Flow in equals flow out at each node.***

This law expresses "conservation" and "continuity" and "balance." Nothing is lost, nothing is gained.

What are the actual solutions to $\boldsymbol{A^T}$y = 0 ?

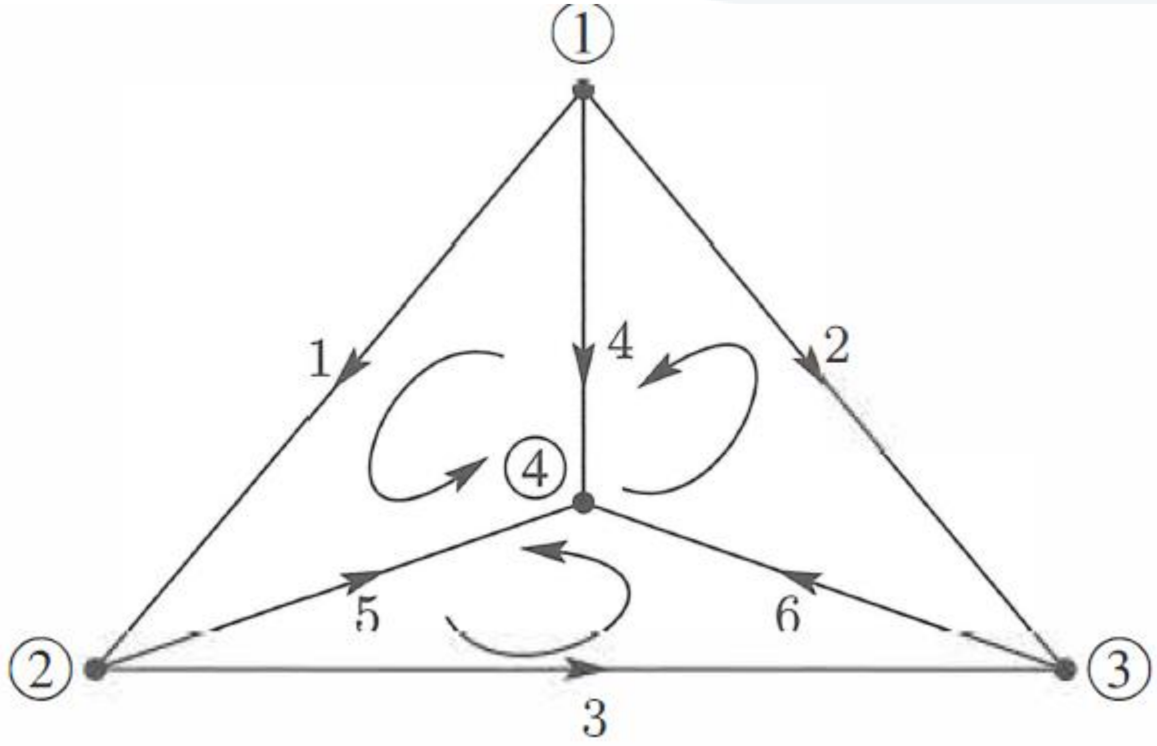
The easiest way is to flow around a loop. If a unit of current goes around the big triangle (forward on edge 1 and 3, backward on 2), the six currents are $y = (1, -1, 1, 0, 0, 0)$.

Every loop current is a solution to the Current Law.

Flow in equals flow out at every node. A smaller loop goes forward on edge 1, forward on 5, back on 4. Then $y = (1, 0, 0, -1, 1, 0)$ is also in the left nullspace.

We expect three independent $y's$: $m - r = 6 - 3 = 3$. The three small loops in the graph are independent.

The big triangle seems to give a fourth y, but that flow is the sum of flows around the small loops. Flows around the 3 small loops are a basis for the left nullspace.



$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{\text{3 small loops}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{big loop}}$$

The incidence matrix A comes from a connected graph with n nodes and m edges. The row space and column space have dimensions $r = n - 1$. The nullspaces of A and A^{T} have dimensions 1 and $m - n + 1$:

$\boldsymbol{N}(A)$ The constant vectors $(c, c, \ldots, c)$ make up the nullspace of A : $\dim = 1$.

$\boldsymbol{C}(A^{\mathrm{T}})$ The edges of any tree give r independent rows of A : $r = n - 1$.

$\boldsymbol{C}(A)$ ***Voltage Law***: The components of $A\boldsymbol{x}$ add to zero around all loops: $\dim = n - 1$.

$\boldsymbol{N}(A^{\mathrm{T}})$ ***Current Law***: $A^{\mathrm{T}}\boldsymbol{y} = (\textbf{flow in}) - (\textbf{flow out}) = \mathbf{0}$ is solved by loop currents.

There are $m - r = m - n + 1$ independent small loops in the graph.

For every graph in a plane, linear algebra yields ***Euler's formula***: Theorem 1 in topology!

$$(\textit{number of nodes}) - (\textit{number of edges}) + (\textit{number of small loops}) = \mathbf{1}.$$

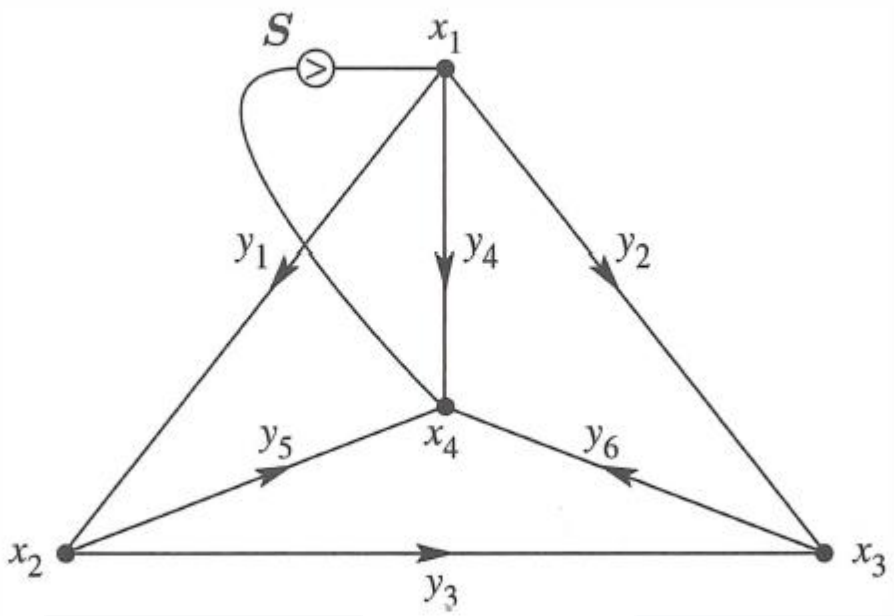
Voltages and Currents and $A^T Ax = f$

- We started with voltages $x = (x_1, \ldots, x_n)$ at the nodes.
- So far we have Ax to find voltage differences $x_i - x_j$ along edges.
- And we have the Current Law $A^T y = 0$ to find edge currents $y = (y_1, \ldots y_n)$

If all resistances in the network are 1, Ohm's Law will match $y = Ax$.
Then $A^T y = A^T Ay = 0$.

Without any sources, the solution to $A^T Ax = 0$ will just be no flow: x = 0 and y = 0.

We can produce $x \neq 0$ and $y \neq 0$ by adding current sources going into one or more nodes.



The currents y_1 to y_6 in a network with a source S from node 4 to node 1.