

MATHEMATICAL ANALYSIS 1

Lecture

1

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Basic Derivation Formulas

$$(c)' = 0$$

$$(cf)' = cf'$$

$$(f \pm g)' = f' \pm g'$$

$$(x^n)' = nx^{n-1}$$

$$(e^x)' = e^x$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{(g)^2}$$

Derivation of Trigonometric Functions

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

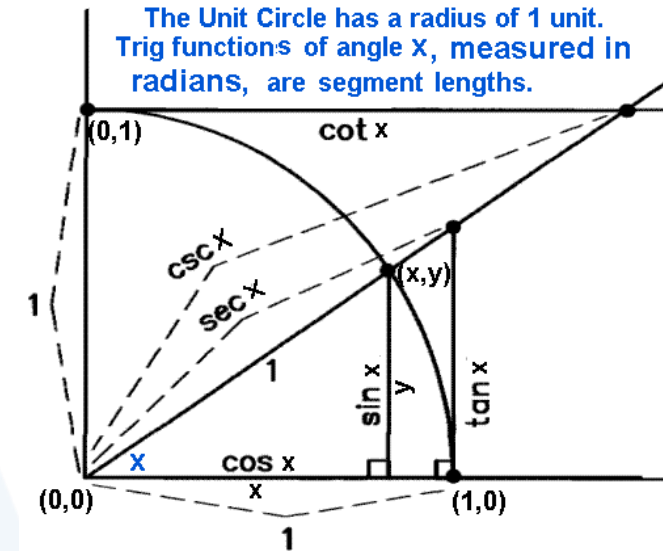
$$(\sec x)' = \sec x \tan x$$

$$(\csc x)' = -\csc x \cot x$$

$$\sec x = \frac{1}{\cos x}$$

$$\csc x = \frac{1}{\sin x}$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$



The Chain Rule

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x , then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x , and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

The Chain Rule

Find the derivative of the following function:

$$G(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$$

Solution

$$2) \quad G(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}}$$

$$G(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}} = (x^2 + x + 1)^{-1/3}$$

$$\left. \begin{array}{l} g(x) = x^2 + x + 1 \\ f(x) = (x)^{-1/3} \end{array} \right\} \Rightarrow \left. \begin{array}{l} g'(x) = 2x + 1 \\ f'(x) = \frac{-1}{3} x^{-4/3} \end{array} \right\} \Rightarrow G'(x) = f'(g(x)) \cdot g'(x) \\ = \frac{-1}{3} (x^2 + x + 1)^{-4/3} (2x + 1)$$

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

$$g'(t) = \frac{d}{dt} \tan(5 - \sin 2t)$$

$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t)$$

$$= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t) \right)$$

$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$

$$= -2(\cos 2t) \sec^2(5 - \sin 2t).$$

Derivative of $\tan u$ with
 $u = 5 - \sin 2t$

Derivative of $5 - \sin u$
with $u = 2t$

Exercises

Find the derivatives of the functions

$$h(x) = x \tan(2\sqrt{x}) + 7$$

$$k(x) = x^2 \sec\left(\frac{1}{x}\right)$$

$$g(t) = \left(\frac{1 + \sin 3t}{3 - 2t}\right)^{-1}$$

Assume that $f'(3) = -1$, $g'(2) = 5$, $g(2) = 3$, and $y = f(g(x))$. What is y' at $x = 2$? -5

Find the tangent to $y = ((x - 1)/(x + 1))^2$ at $x = 0$. $y = -4x + 1$

Running machinery too fast Suppose that a piston is moving straight up and down and that its position at time t sec is $s = A \cos(2\pi bt)$, with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why some machinery breaks when you run it too fast.)

$$v = \frac{ds}{dt}$$

2

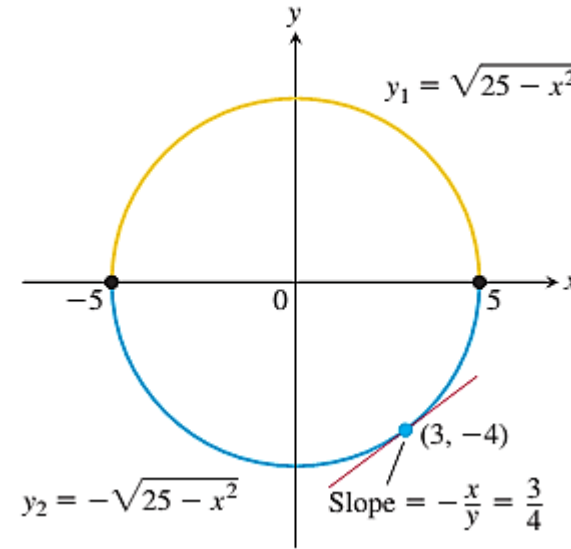
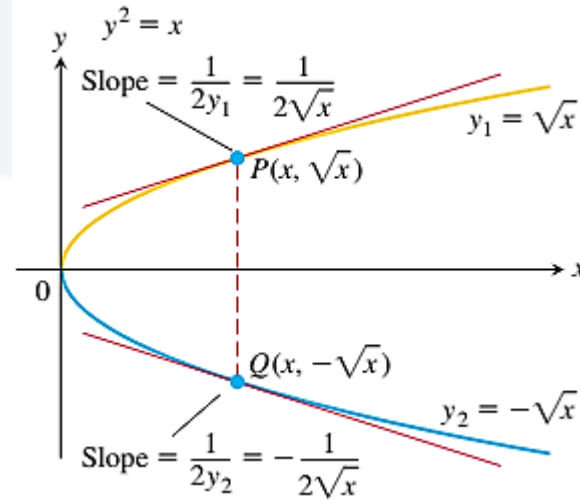
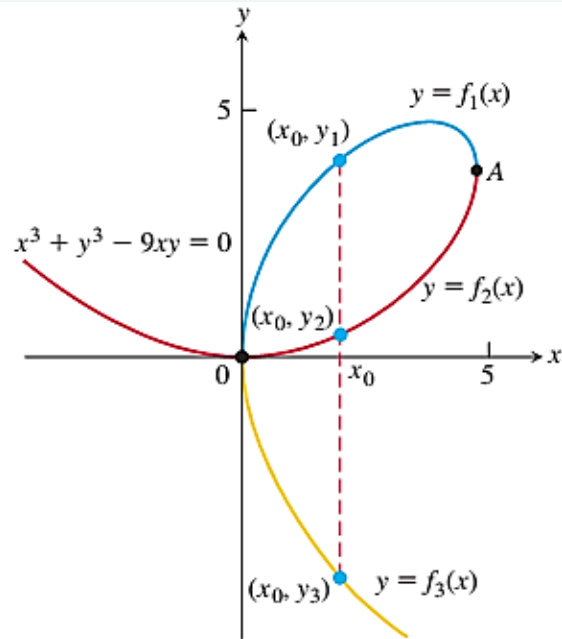
$$a = \frac{dv}{dt}$$

4

$$j = \frac{da}{dt}$$

8

Implicit Differentiation



$$x^3 + y^3 - 9xy = 0, \quad y^2 - x = 0, \quad \text{or} \quad x^2 + y^2 - 25 = 0.$$

Find the slope of the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

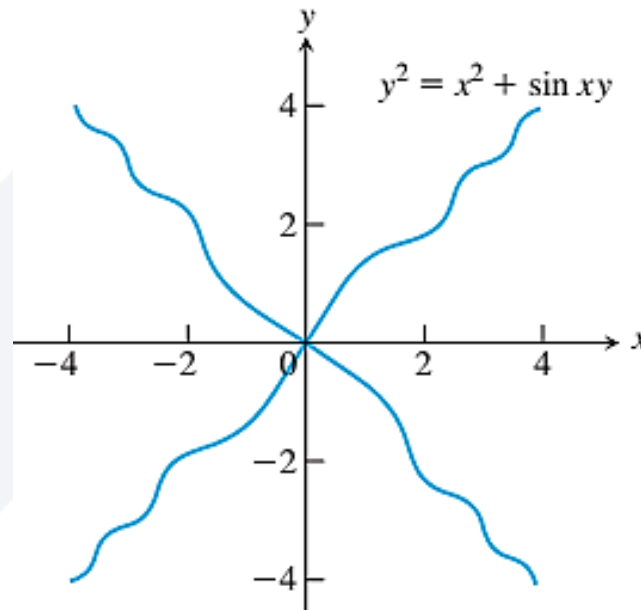
Implicit Differentiation

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx .

Find dy/dx if $y^2 = x^2 + \sin xy$

$$\frac{dy}{dx} = \frac{2x + y \cos xy}{2y - x \cos xy}$$



Implicit Differentiation

Derivatives of Higher Order

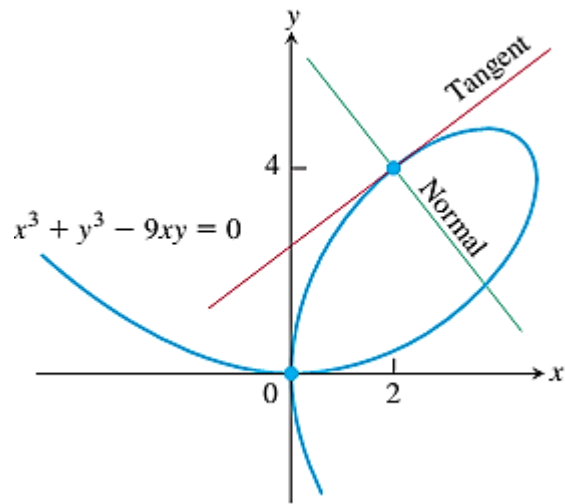
Implicit differentiation can also be used to find higher derivatives.

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \left(\frac{x^2}{y} \right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

Implicit Differentiation

EXAMPLE 5 Show that the point $(2, 4)$ lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.32).



Checking the given point lies to the curve

Finding the derivative

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$

Finding the slope

$$\left. \frac{dy}{dx} \right|_{(2,4)} = \left. \frac{3y - x^2}{y^2 - 3x} \right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}$$

Finding the tangent line equation

$$y = \frac{4}{5}x + \frac{12}{5}$$

Finding the normal line equation

$$y = -\frac{5}{4}x + \frac{13}{2}$$

Exercises

Use implicit differentiation to find dy/dx

$$x^3 = \frac{2x - y}{x + 3y} \quad y \sin\left(\frac{1}{y}\right) = 1 - xy \quad \sin y = x \cos y - 2$$

use implicit differentiation to find dy/dx and then d^2y/dx^2

If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x -axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

$$(-\sqrt{7}, 0) \text{ and } (\sqrt{7}, 0) \quad -2$$

Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

$$y = -2x + 3$$

$$y = -2x - 3$$

Related Rates

EXAMPLE 1 Water runs into a conical tank at the rate of $9 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

The variables

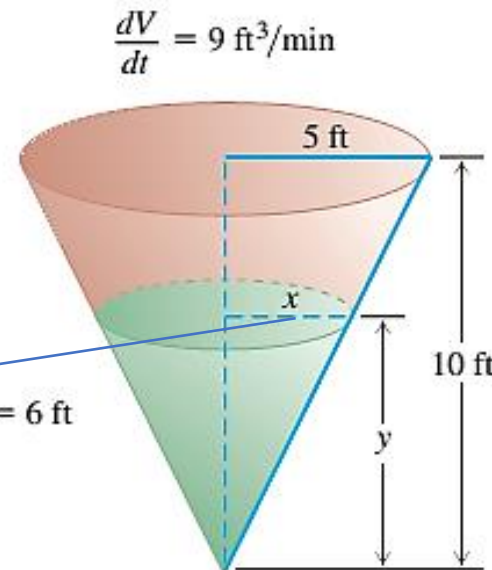
V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of the water in the tank at time t .

$$V = \frac{1}{3}\pi x^2 y. \quad x? \quad \frac{x}{y} = \frac{5}{10}$$

$$\frac{dy}{dt} = ? \text{ when } y = 6 \text{ ft}$$



$$V = \frac{1}{3}\pi \left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3 \quad \longrightarrow \quad \frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2 \frac{dy}{dt} = \frac{\pi}{4}y^2 \frac{dy}{dt}$$

$$y = 6 \text{ and } dV/dt = 9 \quad \longrightarrow \quad \frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

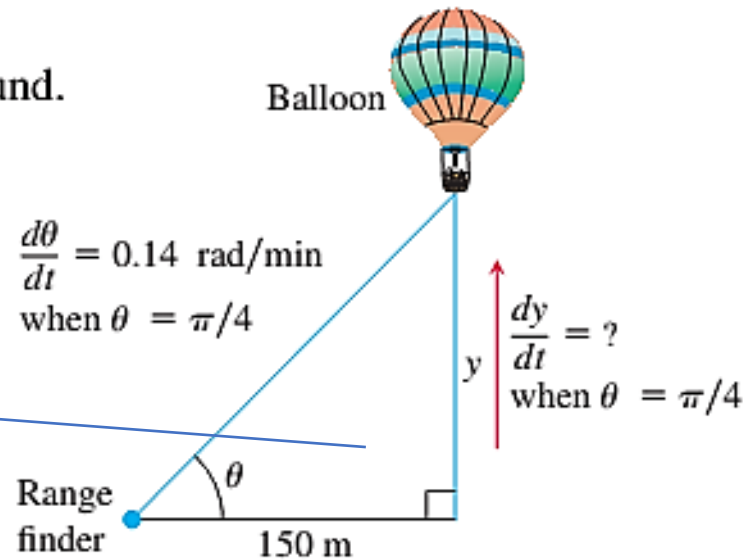
Related Rates

EXAMPLE 2 A hot air balloon rising straight up from a level field is tracked by a range finder 150 m from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

θ = the angle in radians the range finder makes with the ground.

y = the height in meters of the balloon above the ground.

$$\frac{y}{150} = \tan \theta \quad \text{or} \quad y = 150 \tan \theta$$



$$\frac{dy}{dt} = 150 (\sec^2 \theta) \frac{d\theta}{dt} \quad \Rightarrow \quad \frac{dy}{dt} = 150 (\sqrt{2})^2 (0.14) = 42$$

Related Rates

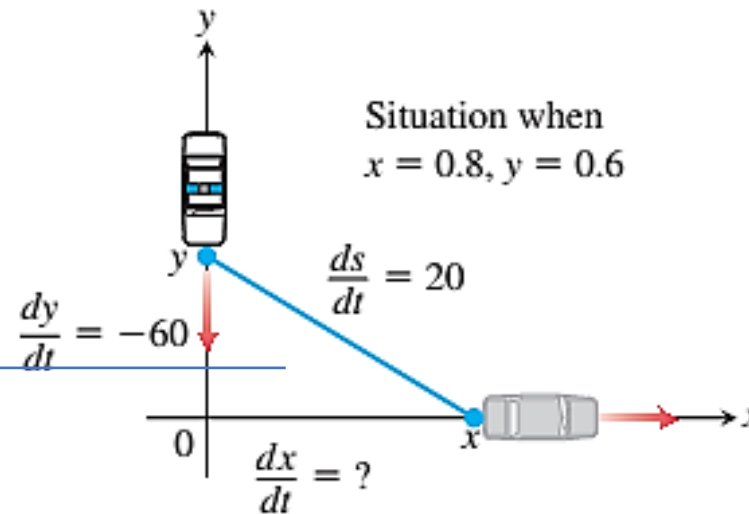
EXAMPLE 3 A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

x = position of car at time t

y = position of cruiser at time t

s = distance between car and cruiser at time t .

$$s^2 = x^2 + y^2$$



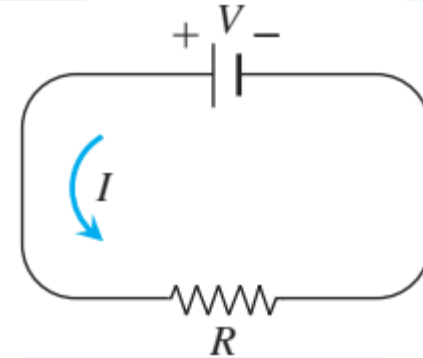
$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

Exercises

The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of $1/3$ amp/sec.

Let t denote time in seconds.

- What is the value of dV/dt ?
- What is the value of dI/dt ?
- What equation relates dR/dt to dV/dt and dI/dt ?
- Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amps. Is R increasing, or decreasing?



A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

20 ft/sec.

Linearization and Differentials

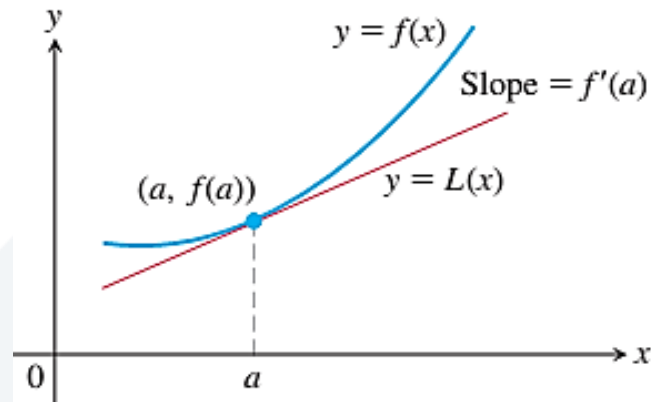
DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a . The approximation

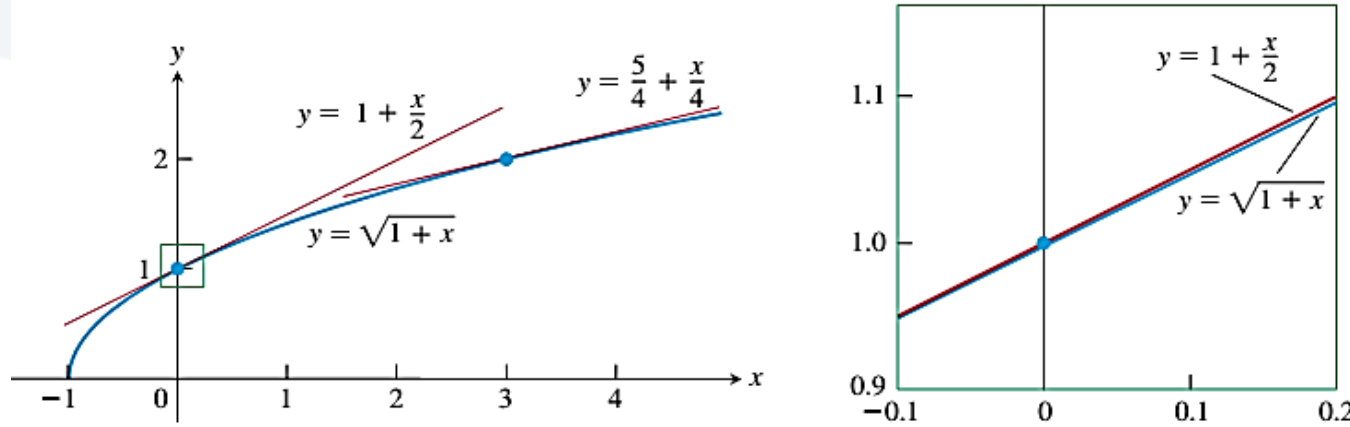
$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a . The point $x = a$ is the **center** of the approximation.



Linearization and Differentials

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$

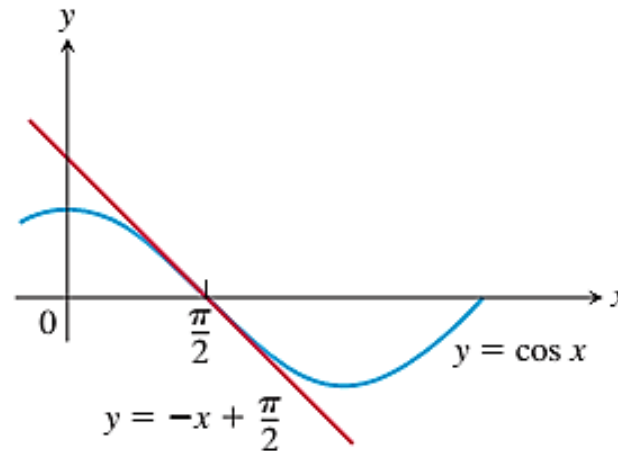


$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}.$$

Approximation	True value	True value - approximation
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$

Linearization and Differentials

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$



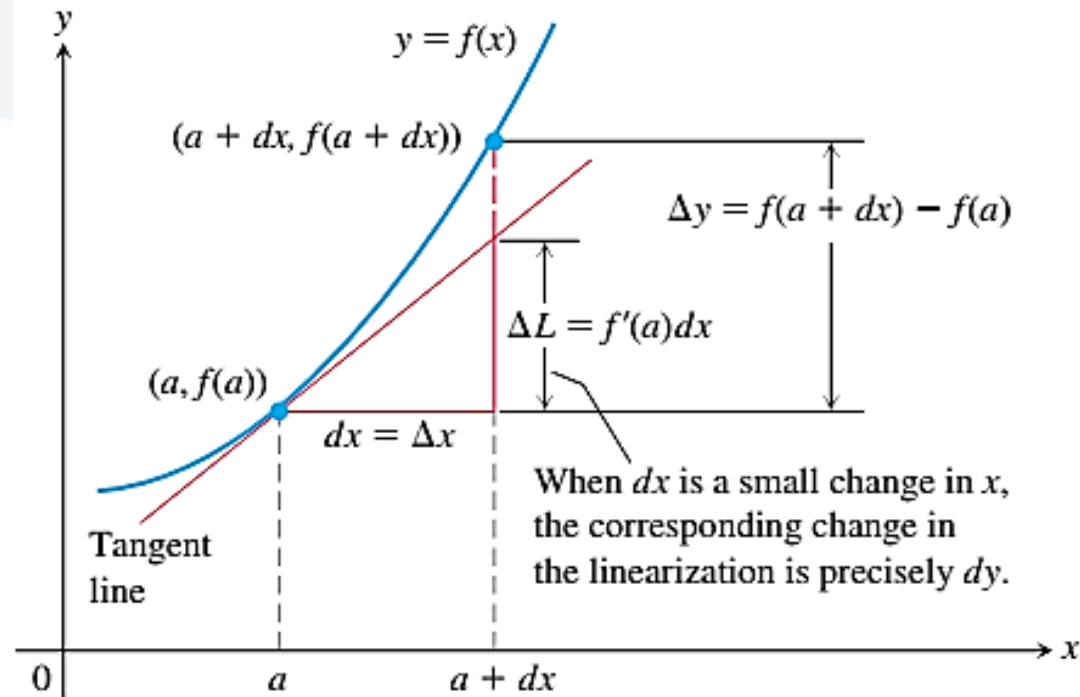
$$L(x) = f(a) + f'(a)(x - a) = -x + \frac{\pi}{2}.$$

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x) dx.$$

Linearization and Differentials

Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.



$$\Delta L = L(a + dx) - L(a) = \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a + dx)} - \underbrace{f(a)}_{L(a)} = f'(a)dx.$$

$$\Delta y = f(a + dx) - f(a) \approx f'(a)dx = dy$$

Linearization and Differentials

when $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to estimate $f(a + dx)$ when $f(a)$ is known, dx is small, and $dy = f'(a)dx$.

$$f(a + dx) = f(a) + \Delta y \quad \longrightarrow \quad f(a + dx) \approx f(a) + dy$$

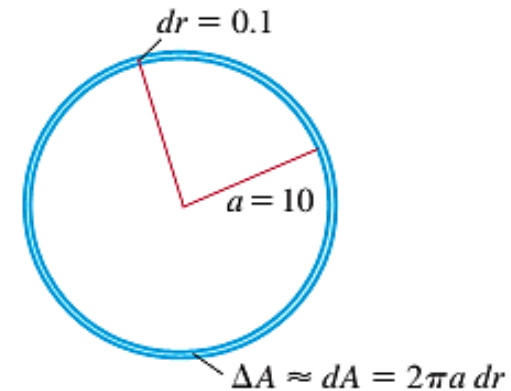
EXAMPLE 6 The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.45). Use dA to estimate the increase in the circle's area A . Estimate the area of the enlarged circle and compare your estimate to the true area found by direct calculation.

$$A = \pi r^2$$

$$dA = A'(a) dr = 2\pi a dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2$$

$$A(10 + 0.1) \approx A(10) + 2\pi = \pi(10)^2 + 2\pi = 102\pi.$$

$$A(10.1) = \pi(10.1)^2 = 102.01\pi \text{ m}^2$$



$$\text{Error} = \Delta A - dA = 0.01$$

Estimating with Differentials

EXAMPLE 7 Use differentials to estimate

(a) $7.97^{1/3}$

(b) $\sin(\pi/6 + 0.01)$.

(a) $7.97^{1/3}$

$$dy = \frac{1}{3x^{2/3}} dx$$

$$f(7.97) = f(8 - 0.03) \approx f(8) + dy = (8)^{1/3} + \frac{1}{3(8)^{2/3}}(-0.03) = 1.9975$$

$$7.97^{1/3} = 1.997497$$

(b) $\sin(\pi/6 + 0.01)$.

$$f\left(\frac{\pi}{6} + 0.01\right) \approx f\left(\frac{\pi}{6}\right) + dy = \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right) \times 0.01 = 0.5087$$

$$\sin\left(\frac{\pi}{6} + 0.01\right) = 0.508635$$

Estimating with Differentials

Error in Differential Approximation

The true change: $\Delta f = f(a + \Delta x) - f(a)$

The differential estimate: $df = f'(a) \Delta x$.

$$\begin{aligned} \text{Approximation error} &= \Delta f - df = \Delta f - f'(a)\Delta x = \underbrace{f(a + \Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\ &= \underbrace{\left(\frac{f(a + \Delta x) - f(a)}{\Delta x} - f'(a) \right)}_{\text{Call this part } \varepsilon} \cdot \Delta x = \varepsilon \cdot \Delta x. \end{aligned}$$

$$\underbrace{\Delta f}_{\text{true change}} = \underbrace{f'(a)\Delta x}_{\text{estimated change}} + \underbrace{\varepsilon \Delta x}_{\text{error}}$$

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a) \Delta x + \varepsilon \Delta x \quad (1)$$

in which $\varepsilon \rightarrow 0$ as $\Delta x \rightarrow 0$.

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a) dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$

Exercises

find a linearization at a suitably chosen integer near a at which the given function and its derivative are easy to evaluate.

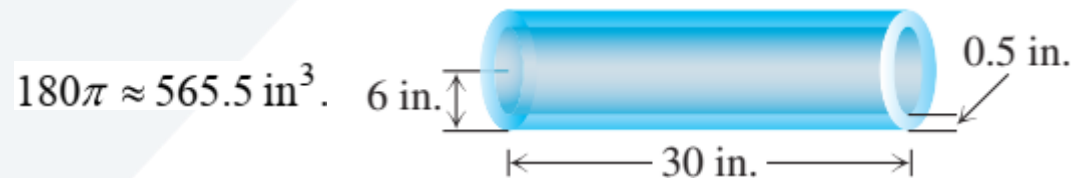
$$f(x) = x^{-1}, \quad a = 0.9$$

$$f(x) = 2x^2 + 3x - 3, \quad a = -0.9$$

Find the linearization of $f(x) = \sqrt{x+1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x+1}$ and $\sin x$ at $x = 0$?

$$L_f(x) = \frac{3}{2}x + 1 \quad L_g(x) = \frac{1}{2}x + 1, \quad L_h(x) = x.$$

Estimate the volume of material in a cylindrical shell with length 30 in., radius 6 in., and shell thickness 0.5 in.



$$180\pi \approx 565.5 \text{ in}^3.$$

Exercises

The edge x of a cube is measured with an error of at most 0.5%. What is the maximum corresponding percentage error in computing the cube's

a. surface area?

b. volume?

1%

1.5%