Finite Element Concepts & Software

طرائق العناصر المنتهية وبرمجياتها

What is this class?

1. "Direct Stiffness" approach for springs
2. Bar elements and truss analysis
3. Beams and Frames Elements and Analysis
4. Displacement-based finite element formulation in 2D: formation of stiffness matrix and load vector for CST and quadrilateral elements.

1. “Direct Stiffness” approach for springs

CHAPTER OBJECTIVES

At the conclusion of this chapter, you will be able to:

■ Define the stiffness matrix.

■ Derive the stiffness matrix for a spring element.

■ Demonstrate how to assemble stiffness matrices into a global stiffness matrix.

■ Illustrate the concept of direct stiffness method to obtain the global stiffness matrix and solve a spring assemblage problem.

■ Describe and apply the different kinds of boundary conditions relevant for spring assemblages.

■ Show how the potential energy approach can be used to both derive the stiffness matrix for a spring and solve a spring assemblage problem

“Direct Stiffness” approach for springs

Introduction:

This chapter introduces some of the basic concepts on which the direct stiffness method is founded. The linear spring is introduced first because it provides a simple yet generally instructive tool to illustrate the basic concepts. We begin with a general definition of the stiffness matrix and then consider the derivation of the stiffness matrix for a linear-elastic spring element. We next illustrate how to assemble the total stiffness matrix for a structure comprising an assemblage of spring elements by using elementary concepts of equilibrium and compatibility. We then show how the total stiffness matrix for an assemblage can be obtained by superimposing the stiffness matrices of the individual elements in a direct manner. The term direct stiffness method evolved in reference to this technique.

After establishing the total structure stiffness matrix, we illustrate how to impose boundary conditions—both homogeneous and nonhomogeneous. A complete solution including the nodal displacements and reactions is thus obtained.

“Direct Stiffness” approach for springs

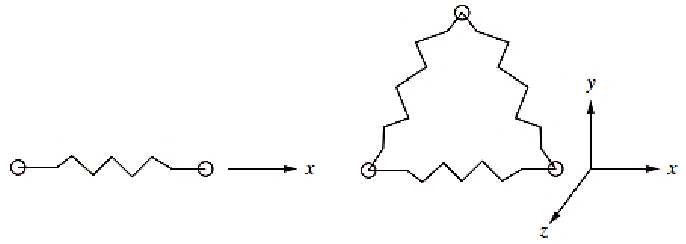
1.1: Definition of the Stiffness Matrix:

Familiarity with the stiffness matrix is essential to understanding the stiffness method. We define the stiffness matrix as follows: For *an element*, a stiffness matrix [k] is a matrix such that $\{f\} = [k]\{d\}$

where [k] relates nodal displacements {d} to nodal forces { f } of a single element, such as the spring shown in Figure (a).

For a continuous medium or structure comprising a series of elements, such as shown for the spring assemblage in Figure (b) , stiffness matrix [K] relates global-coordinate (x, y, z) nodal displacements {d} to global forces {F} of the whole medium or structure. such that $\{F\} = [K]\{d\}$

where [K] represents the stiffness matrix of the whole spring assemblage.

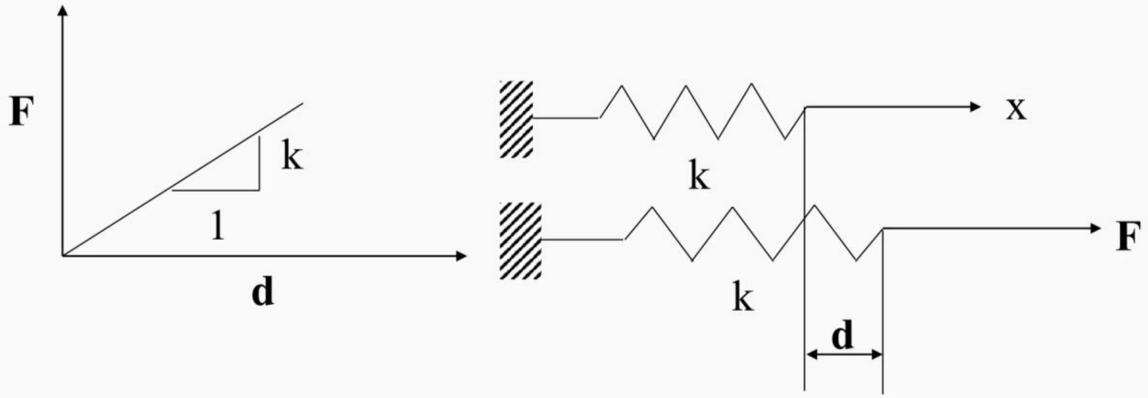


(a) (b)

"Direct Stiffness" approach for springs

Behavior of linear spring (recap):

Familiarity with the stiffness matrix is essential to understanding the stiffness method. We define the



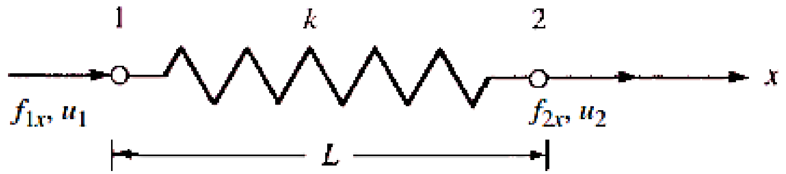
F = Force in the spring
d = deflection of the spring
k = "stiffness" of the spring

Hooke's Law
F = kd

"Direct Stiffness" approach for springs

1.2 Derivation of the Stiffness Matrix for a Spring Element:

- Using the direct equilibrium approach, we will now derive the stiffness matrix for a one- dimensional linear spring—that is, a spring that obeys Hooke's law and resists forces only in the direction of the spring. Consider the linear spring element shown in next Figure. Reference points 1 and 2 are located at the ends of the element. These reference points are called the nodes of the spring element.
- The local nodal forces are f_{1x} and f_{2x} for the spring element associated with the local axis x. The local axis acts in the direction of the spring so that we can directly measure displacements and forces along the spring.
- The local nodal displacements are u_1 and u_2 for the spring element. These nodal displacements are called the degrees of freedom at each node. Positive directions for the forces and displacements at each node are taken in the positive x direction as shown from node 1 to node 2 in the figure. The symbol k is called the spring constant or stiffness of the spring.

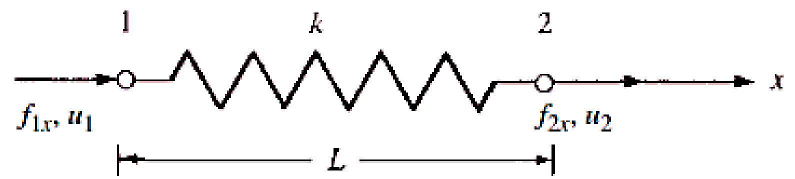


"Direct Stiffness" approach for springs

1.2 Derivation of the Stiffness Matrix for a Spring Element:

We now want to develop a relationship between nodal forces and nodal displacements for a spring element. This relationship will be the stiffness matrix. Therefore, we want to relate the nodal force matrix to the nodal displacement matrix as follows:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{1.2.1}$$



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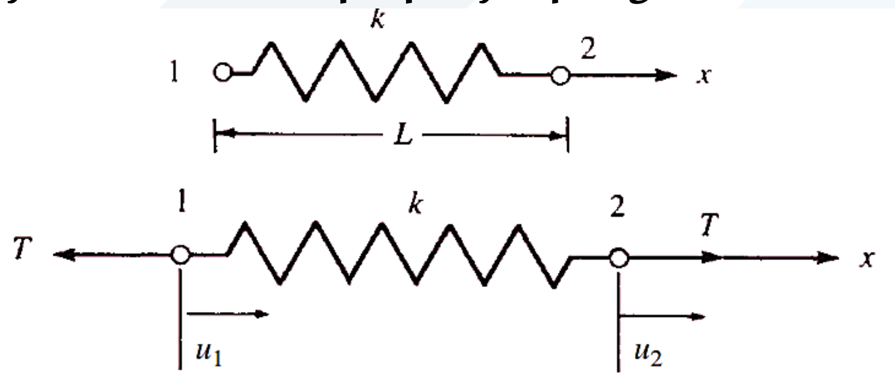
1.2 Derivation of the Stiffness Matrix for a Spring Element:

Step 1: Select the element type:

Consider the linear spring element (which can be an element in a system of springs) subjected to resulting nodal tensile forces T (which may result from the action of adjacent springs) directed along the spring axial direction x as shown in Figure, so as to be in equilibrium. The local x axis is directed from node 1 to node 2. We represent the spring by labeling nodes at each end and by labeling the element number. The original distance between nodes before deformation is denoted by L. The material property (spring constant) of the element is k.



"Direct Stiffness" approach for springs

1.2 Derivation of the Stiffness Matrix for a Spring Element:

Step 2 Define the Strain/Displacement and Stress/Strain Relationships

The tensile forces T produce a total elongation (deformation) d of the spring. The typical total elongation of the spring is shown in Figure 4. Here u_1 is a negative value because the direction of displacement is opposite the positive x direction, whereas u_2 is a positive value. The total deformation of the spring is represented by the difference in nodal displacements as $\delta = u_2 - u_1$ (1.2.2)

For a spring element, we can relate the force in the spring directly to the deformation. Therefore, the strain/displacement relationship is not necessary here. The stress/strain relationship can be expressed in terms of the force/deformation relation- ship instead as $T = k\delta$ (1.2.3)

Now, using Eq. (1.2.2) in Eq. (1.2.3), we obtain $T = k(u_2 - u_1)$ (1.2.4)

“Direct Stiffness” approach for springs

1.2 Derivation of the Stiffness Matrix for a Spring Element:

Step 3 Derive the Element Stiffness Matrix and Equations

We now derive the spring element stiffness matrix. By the sign convention for nodal forces and equilibrium, (see Figures 2 and 3) we have

$$f_{1x} = -T \qquad f_{2x} = T \tag{1.2.5}$$

Using Eqs. (1.2.4) and (1.2.5), we have

$$\begin{aligned} T &= -f_{1x} = k(u_2 - u_1) \\ T &= f_{2x} = k(u_2 - u_1) \end{aligned} \tag{1.2.6}$$

Rewriting Eqs. (1.2.6), we obtain

$$\begin{aligned} f_{1x} &= k(u_1 - u_2) \\ f_{2x} &= k(u_2 - u_1) \end{aligned} \tag{1.2.7}$$

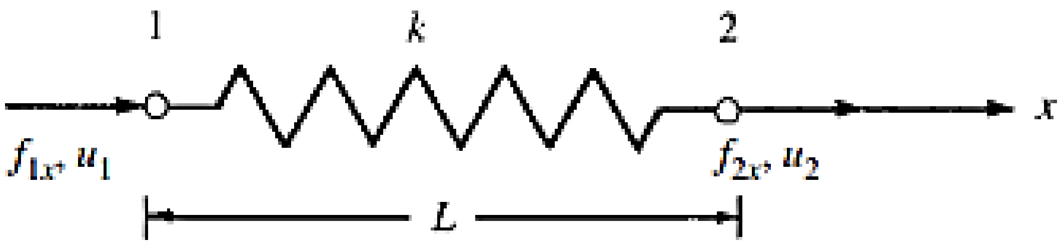


Figure 2 Linear spring element with positive nodal displacement and force conventions

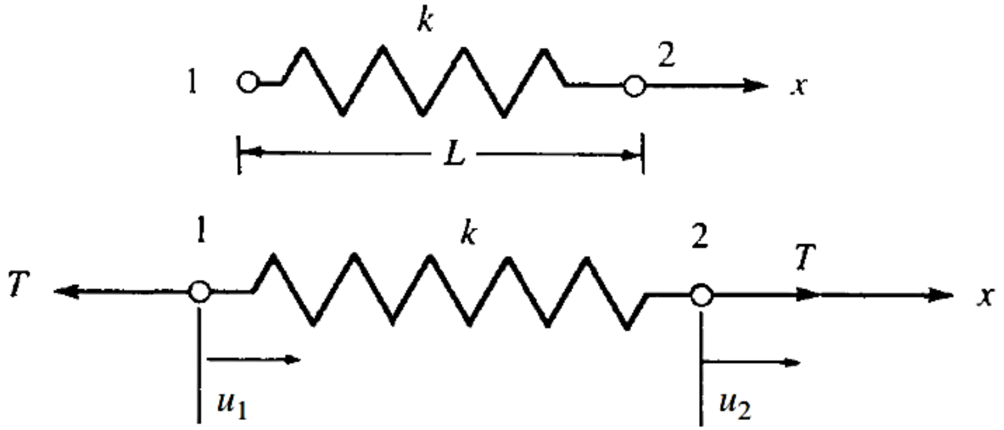


Figure 3 Linear spring subjected to tensile forces

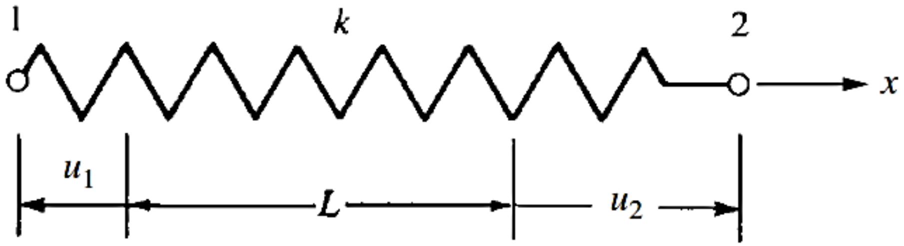


Figure 4 Deformed spring

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1.2 Derivation of the Stiffness Matrix for a Spring Element:

Step 3 Derive the Element Stiffness Matrix and Equations

Now expressing Eqs (1.2.7) in a single matrix equation yields

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad (1.2.8)$$

This relationship holds for the spring along the x axis. From our basic definition of a stiffness matrix and application of Eq. (1.2.1) to Eq. (1.2.8), we obtain

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad (1.2.9)$$

"Direct Stiffness" approach for springs

1.2 Derivation of the Stiffness Matrix for a Spring Element:

Step 3 Derive the Element Stiffness Matrix and Equations

$$[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad (1.2.9)$$

as the stiffness matrix for a linear spring element. Here [k] is called the local stiffness matrix for the element. We observe from Eq. (1.2.9) that [k] has the following properties:

1. It is symmetric (that is, kij 5 k ji for i ? j).
2. It is square (the number of rows equals the number of columns in [k]) as it relates the same number of nodal forces to nodal displacements.
3. It is singular. That is, the determinant of [k] is equal to zero, so [k] cannot be inverted.

"Direct Stiffness" approach for springs

1.2 Derivation of the Stiffness Matrix for a Spring Element:

Step 4 Assemble the Element Equations to Obtain the Global Equations and Introduce Boundary Conditions

The global stiffness matrix and global force matrix are assembled using nodal force equilibrium equations, force/deformation and compatibility equations from Section 1.3, and the direct stiffness method described in Section 1.4. This step applies for structures composed of more than one element such that

$$[K] = \sum_{e=1}^{N} [k^{(e)}] \quad \text{and} \quad \{F\} = \sum_{e=1}^{N} \{f^{(e)}\} \tag{1.2.10}$$

where [k (e)] and { f (e)} are now element stiffness and force matrices expressed in a global reference frame. This concept becomes relevant for instance when considering truss structures in following chapters. (Throughout this text, the $\sum$ sign used in this context does not imply a simple summation of element matrices but rather denotes that these element matrices must be assembled properly according to the direct stiffness method described in Section 1.4.)

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1.2 Derivation of the Stiffness Matrix for a Spring Element:

Step 5 Solve for the Nodal Displacements

The displacements are then determined by imposing boundary conditions, such as support conditions, and solving a system of equations simultaneously as

$$\{F\} = [K]\{d\} \tag{1.2.11}$$

Step 6 Solve for the Element Forces

Finally, the element forces are determined by back-substitution, applied to each element, into equations similar to Eqs. (1.2.7).

"Direct Stiffness" approach for springs

1.3 Example of a Spring Assemblage

Structures such as trusses, building frames, and bridges comprise basic structural components connected together to form the overall structures. To analyze these structures, we must deter- mine the total structure stiffness matrix for an interconnected system of elements. Before considering the truss and frame, we will determine the total structure stiffness matrix for a spring assemblage by using the force/displacement matrix relationships derived in Section 1.2 for the spring element, along with fundamental concepts of nodal equilibrium and compatibility. Step 5 will then have been illustrated.

"Direct Stiffness" approach for springs

1.3 Example of a Spring Assemblage

We will consider the specific example of the two-spring assemblage shown in Figure 5. This example is general enough to illustrate the direct equilibrium approach for obtaining the total stiffness matrix of the spring assemblage. Here we f$_{ix}$ node 1 and apply axial forces for F_{3x} at node 3 and F_{2x} at node 2. The stiffnesses of spring elements 1 and 2 are k_1 and k_2 , respectively. The nodes of the assemblage have been numbered 1, 3, and 2 for further generalization because sequential numbering between elements generally does not occur in large problems.

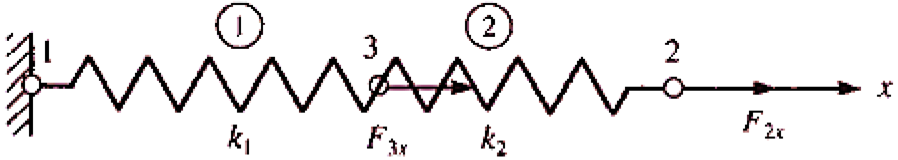

Figure .5 Two-spring assemblage

"Direct Stiffness" approach for springs

1.3 Example of a Spring Assemblage

The x axis is the global axis of the assemblage. The local x axis of each element coincides with the global axis of the assemblage. For element 1, using Eq. (1.2.8), we have

$$\begin{Bmatrix} f_{1x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_3^{(1)} \end{Bmatrix} \tag{1.3.1}$$

and for element 2, we have

$$\begin{Bmatrix} f_{3x}^{(2)} \\ f_{2x}^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_3^{(2)} \\ u_2^{(2)} \end{Bmatrix} \tag{1.3.2}$$

Furthermore, elements 1 and 2 must remain connected at common node 3 throughout the displacement. This is called the continuity or compatibility requirement. The compatibility requirement yields

$$u_3^{(1)} = u_3^{(2)} = u_3 \tag{1.3.3}$$

"Direct Stiffness" approach for springs

1.3 Example of a Spring Assemblage

Where, throughout this text, the superscripts in parentheses above u refers to the element number to which they are related. Recall that the subscript to the right identifies the node of displacement and that u_3 is the node 3 displacement of the total or global spring assemblage. Free-body diagrams of each element and node (using the established sign conventions for element nodal forces in Figure 2) are shown in Figure 6. Based on the free-body diagrams of each node shown in Figure 6 and the fact that external forces must equal internal forces at each node, we can write nodal equilibrium equations at nodes 3, 2, and 1 as

$$F_{3x} = f_{3x}^{(1)} + f_{3x}^{(2)} \tag{1.3.4}$$

$$F_{2x} = f_{2x}^{(2)} \tag{1.3.5}$$

$$F_{1x} = f_{1x}^{(1)} \tag{1.3.6}$$

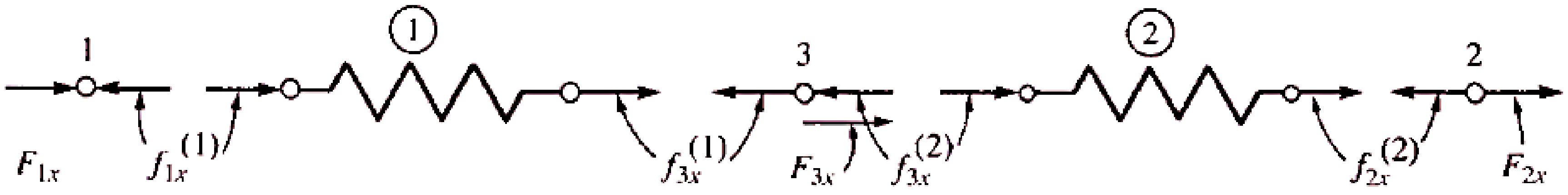

Figure 6 Nodal forces consistent with element force sign convention

whereF1x results from the external applied reaction at the fixed support

"Direct Stiffness" approach for springs

1.3 Example of a Spring Assemblage

Here Newton's third law, of equal but opposite forces, is applied in moving from a node to an element associated with the node. Using Eqs. (1.3.1) through (1.3.3) in Eqs. (1.3.4) through (1.3.6), we obtain

$$\begin{aligned} F_{3x} &= (-k_1 u_1 + k_1 u_3) + (k_2 u_3 - k_2 u_2) \\ F_{2x} &= -k_2 u_3 + k_2 u_2 \\ F_{1x} &= k_1 u_1 - k_1 u_3 \end{aligned} \tag{1.3.7}$$

In matrix form, Eqs. (1.3.7) are expressed by

$$\begin{Bmatrix} F_{3x} \\ F_{2x} \\ F_{1x} \end{Bmatrix} = \begin{bmatrix} k_1 + k_2 & -k_2 & -k_1 \\ -k_2 & k_2 & 0 \\ -k_1 & 0 & k_1 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_2 \\ u_1 \end{Bmatrix} \tag{1.3.8}$$

Rearranging Eq. (1.3.8) in numerically increasing order of the nodal degrees of freedom, we have

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{1.3.9}$$

"Direct Stiffness" approach for springs

1.3 Example of a Spring Assemblage

Equation (1.3.9) is now written as the single matrix equation

$$\{F\} = [K]\{d\} \tag{1.3.7}$$

where $\{F\} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$ is called the *global nodal force matrix*, $\{d\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$ is called the *global nodal displacement matrix*, and

$$[K] = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix}$$

is called the total or global or system stiffness matrix.

In summary, to establish the stiffness equations and stiffness matrix, Eqs. (1.3.9) and (1.3.11), for a spring assemblage, we have used force/deformation relationships Eqs. (1.3.1) and (1.3.2), compatibility relationship Eq. (1.3.3), and nodal force equilibrium Eqs. (1.3.4) through (1.3.6).

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1.4 Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)

We will now consider a more convenient method for constructing the total stiffness matrix. This method is based on proper superposition of the individual element stiffness matrices making up a structure. Referring to the two-spring assemblage of Section 1.3, the element stiffness matrices are given in Eqs. (1.3.1) and (1.3.2) as

$$[k^{(1)}] = \begin{array}{cc} & \begin{array}{cc} u_1 & u_3 \end{array} \\ & \left[\begin{array}{cc} k_1 & -k_1 \\ -k_1 & k_1 \end{array}\right] \begin{array}{c} u_1 \\ u_3 \end{array} \end{array} \qquad [k^{(2)}] = \begin{array}{cc} & \begin{array}{cc} u_3 & u_2 \end{array} \\ & \left[\begin{array}{cc} k_2 & -k_2 \\ -k_2 & k_2 \end{array}\right] \begin{array}{c} u_3 \\ u_2 \end{array} \end{array} \tag{1.4.1}$$

Here the u_i s written above the columns and next to the rows in the [k]s indicate the degrees of freedom associated with each element row and column.

"Direct Stiffness" approach for springs

1.4 Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)

The two element stiffness matrices, Eqs. (1.4.1), are not associated with the same degrees of freedom; that is, element 1 is associated with axial displacements at nodes 1 and 3, whereas element 2 is associated with axial displacements at nodes 2 and 3. Therefore, the element stiffness matrices cannot be added together (superimposed) in their present form. To superimpose the element matrices, we must expand them to the order (size) of the total structure (spring assemblage) stiffness matrix so that each element stiffness matrix is associated with all the degrees of freedom of the structure. To expand each element stiffness matrix to the order of the total stiffness matrix, we simply add rows and columns of zeros for those displacements not associated with that particular element.

For element 1, we rewrite the stiffness matrix in expanded form so that Eq. (1.3.1) becomes

"Direct Stiffness" approach for springs

1.4 Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)

For element 1, we rewrite the stiffness matrix in expanded form so that Eq. (1.3.1) becomes

$$k_1 \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix} \qquad (1.4.2)$$

where, from Eq. (1.4.2), we see that u(1), and f(1) are not associated with [k (1)]. Similarly, for element 2, we have

$$k_2 \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} \qquad (1.4.3)$$

Now, considering force equilibr

$$\begin{Bmatrix} f_{1x}^{(1)} \\ 0 \\ f_{3x}^{(1)} \end{Bmatrix} + \begin{Bmatrix} 0 \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} \qquad (1.4.4)$$

"Direct Stiffness" approach for springs

1.4 Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)

For element 1, we rewrite the stiffness matrix in expanded form so that Eq. (1.3.1) becomes

$$k_1 \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix} \quad (1.4.2)$$

where, from Eq. (1.4.2), we see that u(1), and f(1) are not associated with [k (1)]. Similarly, for element 2, we have

$$k_2 \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} \quad (1.4.3)$$

Now, considering force equilibr

$$\begin{Bmatrix} f_{1x}^{(1)} \\ 0 \\ f_{3x}^{(1)} \end{Bmatrix} + \begin{Bmatrix} 0 \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} \quad (1.4.4)$$

“Direct Stiffness” approach for springs

1.5 Boundary Conditions

We must specify boundary (or support) conditions for structure models such as the spring assemblage of Figure 5, or [K] will be singular; that is, the determinant of [K] will be zero, and its inverse will not exist. This means the structural system is unstable. Without our specifying adequate kinematic constraints or support conditions, the structure will be free to move as a rigid body and not resist any applied loads. In general, the number of boundary conditions necessary to make [K] nonsingular is equal to the number of possible rigid body modes.

Boundary conditions relevant for spring assemblages are associated with nodal displacements. These conditions are of two types. Homogeneous boundary conditions—the more common—occur at locations that are completely prevented from movement; nonhomogeneous boundary conditions occur where finite nonzero values of displacement are specified, such as the settlement of a support.

"Direct Stiffness" approach for springs

1.5 Boundary Conditions

To illustrate the two general displacement types of boundary conditions, let us consider Eq. (1.4.6), derived for the spring assemblage of Figure 5. which has a single rigid body mode in the direction of motion along the spring assemblage.

Homogeneous Boundary Conditions: We first consider the case of homogeneous boundary conditions. Hence all boundary conditions are such that the displacements are zero at certain nodes. Here we have $u_1=0$ because node1 is fixed. Therefore, Eq. (1.4.6) can be written as

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} \tag{1.5.1}$$

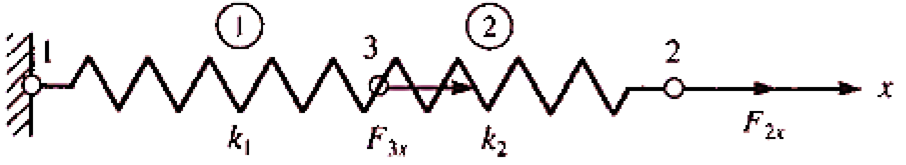

Figure 5 Two-spring assemblage

"Direct Stiffness" approach for springs

1.5 Boundary Conditions

Homogeneous Boundary Conditions: Equation (1.5.1), written in expanded form, becomes,

$$\begin{aligned} k_1(0) + (0)u_2 - k_1u_3 &= F_{1x} \\ 0(0) + k_2u_2 - k_2u_3 &= F_{2x} \\ -k_1(0) - k_2u_2 + (k_1 + k_2)u_3 &= F_{3x} \end{aligned} \qquad (1.5.2)$$

where F_{1x} is the unknown reaction and F_{2x} and F3x are known applied loads. Writing the second and third of Eqs. (1.5.2) in matrix form, we have

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} \qquad (1.5.3)$$

We have now effectively partitioned off the first column and row of [K] and the first row of {d} and {F} to arrive at Eq. (1.5.3)

"Direct Stiffness" approach for springs

1.5 Boundary Conditions

Homogeneous Boundary Conditions: For homogeneous boundary conditions, Eq. (1.5.3) could have been obtained directly by deleting the row and column of Eq. (1.5.1) corresponding to the zero-displacement degrees of freedom. Here row 1 and column 1 are deleted because one is really multiplying column 1 of [K] by $u_1 = 0$. However, F_{1x} is not necessarily zero and can be determined once u_2 and u_3 are solved for. After solving Eq. (1.5.3) for u2 and u3, we have

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}^{-1} \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{k_2} + \dfrac{1}{k_1} & \dfrac{1}{k_1} \\ \dfrac{1}{k_1} & \dfrac{1}{k_1} \end{bmatrix} \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} \tag{1.5.4}$$

Now that u2 and u3 are known from Eq. (1.5.4), we substitute them in the first of Eqs. (1.5.2) to obtain the reaction F_{1x} as

$$F_{1x} = -k_1 u_3 \tag{1.5.5}$$

We can express the unknown nodal force at node 1 (also called the reaction) in terms of the applied nodal forces F_{2x} and F_{3x} by using Eq. (1.5.4) for u3 substituted into Eq. (1.5.5). The result is

$$F_{1x} = -F_{2x} - F_{3x} \tag{1.5.6}$$

"Direct Stiffness" approach for springs

1.5 Boundary Conditions

Homogeneous Boundary Conditions:

Therefore, for all homogeneous boundary conditions, we can delete the rows and columns corresponding to the zero-displacement degrees of freedom from the original set of equations and then solve for the unknown displacements. This procedure is useful for hand calculations.