Lecture 1-B: Systems of Linear Equations - Matrices

CEDC102: Linear Algebra

Manara University

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- Introduction to Systems of Linear Equations
- Gaussian Elimination
- Operations with Matrices
- Properties of Matrix Operations
- Inverse matrices
- Elementary Matrices
- LU factorization

- Inverse matrix:

Consider $A \in M_{n\times n}(K)$

If there exists a matrix $B \in M_{n\times n}(K)$ such that $AB = BA = I_n$,

Then (1) A is invertible (or nonsingular)

(2) B is the inverse of A

- Note:

A matrix that does not have an inverse is called noninvertible (or singular).

- Theorem (The inverse of a matrix is unique):

If B and C are both inverses of the matrix A, then $B = C$.

Consequently, the inverse of a matrix is unique.

- Notes:

(1) The inverse of A is denoted by A^{-1}

(2) $AA^{-1} = A^{-1}A = I$

- Find the inverse of a matrix by Gauss-Jordan Elimination:

$$\begin{bmatrix} A & | & I \end{bmatrix} \longrightarrow \begin{bmatrix} I & | & A^{-1} \end{bmatrix}$$ Gauss-Jordan Elimination

- Ex : (Find the inverse of the matrix)

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

Sol:

$$AX = I$$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x_{11} + 4x_{21} & x_{12} + 4x_{22} \\ -x_{11} - 3x_{21} & -x_{12} - 3x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{array}{rcrcl} x_{11} & + & 4x_{21} & = & 1 \\ -x_{11} & - & 3x_{21} & = & 0 \end{array}\right\} \quad (1)$$

$$\left.\begin{array}{rcrcl} x_{12} & + & 4x_{22} & = & 0 \\ -x_{12} & - & 3x_{22} & = & 1 \end{array}\right\} \quad (2)$$

$$(1) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 1 \\ -1 & -3 & 0 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -3 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{11} = -3,\ x_{21} = 1$$

$$(2) \Rightarrow \left[\begin{array}{rr:r} 1 & 4 & 0 \\ -1 & -3 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{rr:r} 1 & 0 & -4 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow x_{12} = -4,\ x_{22} = 1$$

Thus

$$X = A^{-1} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \quad (AX = I = AA^{-1})$$

- Note:

$$\left[\begin{array}{cc:cc} 1 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{r_{12}^{(1)},\ r_{21}^{(-4)}} \left[\begin{array}{cc:cc} 1 & 0 & -3 & -4 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\quad A \qquad\quad I \qquad\qquad\qquad\qquad\qquad I \qquad\quad A^{-1}$$

If A can't be row reduced to I, then A is singular.

- Ex : (Find the inverse of the following matrix)

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{bmatrix}$$

Sol: $\left[A \ \vdots \ I\right] = \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow{r_{12}^{(-1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ -6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_{13}^{(6)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -4 & 3 & 6 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_{23}^{(4)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 4 & 1 \end{array}\right] \xrightarrow{r_{3}^{(-1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

$$\xrightarrow{r_{32}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] \xrightarrow{r_{21}^{(1)}} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right] = \left[\begin{array}{c:c} I & A^{-1} \end{array}\right]$$

So the matrix A is invertible, and its inverse is $A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}$

- Check: $AA^{-1} = A^{-1}A = I$

- **Power of a square matrix:**

$$(1)\ A^0 = I$$

$$(2)\ A^k = \underbrace{AA\cdots A}_{k \text{ factors}} \qquad (k > 0)$$

$$(3)\ A^r \cdot A^s = A^{r+s} \qquad r, s\text{: integers}$$

$$(4)\ (A^r)^s = A^{rs}$$

$$(5)\ D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

- **Theorem (Properties of inverse matrices):**

If A is an invertible matrix, k is a positive integer, and c is a scalar not equal to zero, then

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(2) A^k is invertible and $(A^k)^{-1} = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{k \text{ factors}} = (A^{-1})^k = A^{-k}$

(3) cA is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1},\ c \neq 0$

(4) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

- Theorem (The inverse of a product):

If A and B are invertible matrices of size n, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

- Note:

$$(A_1A_2A_3\cdots A_n)^{-1} = A_n^{-1}\cdots A_3^{-1}A_2^{-1}A_1^{-1}$$

- Theorem (Cancellation properties)

If C is an invertible matrix, then the following properties hold:

(1) If $AC = BC$, then $A = B$ (Right cancellation property)

(2) If $CA = CB$, then $A = B$ (Left cancellation property)

- Note: If C is not invertible, then cancellation is not valid.

- Theorem (Systems of equations with unique solutions):

If A is an invertible matrix, then the system of linear equations $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution given by $\boldsymbol{x} = A^{-1}\boldsymbol{b}$

- Note:

For square systems (those having the same number of equations as variables), Previous Theorem can be used to determine whether the system has a unique solution.

- Note: $A\boldsymbol{x} = \boldsymbol{b}$ (A is an invertible matrix)

$$\left[A \mid \boldsymbol{b}\right] \xrightarrow{A^{-1}} \left[A^{-1}A \mid A^{-1}\boldsymbol{b}\right] = \left[I \mid A^{-1}\boldsymbol{b}\right]$$

- Ex : Use an inverse matrix to solve each system

$$(a)\quad \begin{aligned} 2x + 3y + z &= -1 \\ 3x + 3y + z &= 1 \\ 2x + 4y + z &= -2 \end{aligned} \qquad\qquad (b)\quad \begin{aligned} 2x + 3y + z &= 0 \\ 3x + 3y + z &= 0 \\ 2x + 4y + z &= 0 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$(a)\quad \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$(b)\quad \boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Elementary Matrices

- Row elementary matrix:

An $n \times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- Three row elementary matrices:

(1) $R_{ij} = r_{ij}(I)$ — Interchange two rows .

(2) $R_i^{(k)} = r_i^{(k)}(I) \qquad (k \neq 0)$ — Multiply a row by a nonzero constant.

(3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ — Add a multiple of a row to another row.

- Note:

Only do a single elementary row operation.

Note: Permutation matrix

Row Exchange Matrix $\boldsymbol{P_{ij}}$ is the identity matrix with rows i and j reversed. When this "permutation matrix" $\boldsymbol{P_{ij}}$ multiplies a matrix, it exchanges rows i and j.

To exchange equations 1 and 3 multiply by

$$\boldsymbol{P_{ij}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- Ex : (Elementary matrices and non elementary matrices)

$(a)\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}$ Yes $(r_2^{(3)}(I_3))$

$(b)\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$ No (not square)

$(c)\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ No (Row multiplication must be by a nonzero constant)

$(d)\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ Yes $(r_{23}(I_3))$

$(e)\begin{bmatrix}1&0\\2&1\end{bmatrix}$ Yes $(r_{12}^{(2)}(I_2))$

$(f)\begin{bmatrix}1&0&0\\0&2&0\\0&0&-1\end{bmatrix}$ No (Use two elementary row operations)

- Theorem (Representing elementary row operations):

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m\times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$

- Notes:

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

- Ex : (Elementary matrices and elementary row operation)

$$(a)\ \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix}\ (r_{12}(A)=R_{12}A)$$

$$(b)\ \begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}\ (r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\ \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}\ (r_{12}^{(2)}(A)=R_{12}^{(2)}A)$$

- Ex : (Using elementary matrices)

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Sol:

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} \qquad E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} \qquad E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$B = E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- Row-equivalent:

Matrix B is row-equivalent to A if there exists a finite number of elementary matrices such that

$$B = E_k E_{k-1} \cdots E_2 E_1 A$$

- Theorem : (Elementary matrices are invertible)

If E is an elementary matrix, then E^{-1} exists and is an elementary matrix.

- Notes:

(1) $(R_{ij})^{-1} = R_{ij}$

(2) $(R_i^{(k)})^{-1} = R_i^{(\frac{1}{k})}$

(3) $(R_{ij}^{(k)})^{-1} = R_{ij}^{(-k)}$

- Ex :

Elementary Matrix | **Inverse Matrix**

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$

$$(R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$

(Elementary Matrix)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)}$$

$$(R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)}$$

(Elementary Matrix)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})}$$

$$(R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)}$$

(Elementary Matrix)

- Theorem (A property of invertible matrices):

A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

- Ex 5: Find a sequence of elementary matrices whose product is

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}$$

Sol:

$$A=\begin{bmatrix}-1 & -2\\ 3 & 8\end{bmatrix}\xrightarrow{r_1^{(-1)}}\begin{bmatrix}1 & 2\\ 3 & 8\end{bmatrix}\xrightarrow{r_{12}^{(-3)}}\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix}$$

$$\xrightarrow{r_2^{(1/2)}}\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\xrightarrow{r_{21}^{(-2)}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I$$

Therefore $R_{21}^{(-2)}R_2^{(\frac{1}{2})}R_{12}^{(-3)}R_1^{(-1)}A=I$

Thus $A = (R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$

$$= R_1^{(-1)} R_{12}^{(3)} R_2^{(2)} R_{21}^{(2)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Note:

If A is invertible then

$$E_k \cdots E_3 E_2 E_1 A = I$$

$$A^{-1} = E_k \cdots E_3 E_2 E_1$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

- Theorem (Equivalent conditions):

If A is an $n \times n$ matrix, then the following statements are equivalent.

(1) A is invertible.

(2) $A\mathbf{x} = \boldsymbol{b}$ has a unique solution for every $n \times 1$ column matrix $\boldsymbol{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(4) A is row-equivalent to I_n.

(5) A can be written as the product of elementary matrices.

- $LU-$factorization:

If the $n\times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an $LU-$factorization of A

- Note:

If a square matrix A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is easy to find an $LU-$factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU \quad (L = E_1^{-1} E_2^{-1} \cdots E_k^{-1})$$

- Ex :(LU −factorization)

$$(a)\ A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \qquad (b)\ A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \xrightarrow{r_{12}^{(-1)}} \begin{bmatrix}1 & 2\\ 0 & -2\end{bmatrix}=U$$

$$\Rightarrow R_{12}^{(-1)}A=U$$

$$\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}\xrightarrow{r_{13}^{(-2)}}\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 0 & -4 & 2\end{bmatrix}\xrightarrow{r_{23}^{(4)}}\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 0 & 0 & 14\end{bmatrix}=U$$

$$\Rightarrow R_{23}^{(4)}R_{13}^{(-2)}A=U$$

$$\Rightarrow A=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}U=LU$$

$$\Rightarrow L=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}=R_{13}^{(2)}R_{23}^{(-4)}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & -4 & 1\end{bmatrix}$$

- Solving $\boldsymbol{Ax} = \boldsymbol{b}$ with an LU–factorization of A

  $A\boldsymbol{x} = \boldsymbol{b}$ If $A = LU$, then $LU\boldsymbol{x} = \boldsymbol{b}$

  Let $\boldsymbol{y} = U\boldsymbol{x}$, then $L\boldsymbol{y} = \boldsymbol{b}$

- Two steps:

  (1) Write $\boldsymbol{y} = U\boldsymbol{x}$, and solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$

  (2) Solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$

- Ex : (Solving a linear system using LU–factorization)

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & & & = & -5 \\ & & x_2 & + & 3x_3 & = & -1 \\ 2x_1 & - & 10x_2 & + & 2x_3 & = & -20 \end{array}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $\boldsymbol{y} = U\boldsymbol{x}$, and solve for $L\boldsymbol{y} = \boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{array}{l} y_1 = -5 \\ y_2 = -1 \\ y_3 = -20 - 2y_1 + 4y_2 = -14 \end{array}$$

(2) Solve the following system $U\boldsymbol{x} = \boldsymbol{y}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So $x_3 = -1$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

The Cost of Elimination

A very practical question is cost—or computing time. We can solve 1000 equations on a PC. What if $n = 100,000$? (*Is A dense or sparse*?) Large systems come up all the time in scientific computing, where a three-dimensional problem can easily lead to a million unknowns. We can let the calculation run overnight, but we can't leave it for 100 years.

The first stage of elimination produces zeros below the first pivot in column 1. To find each new entry below the pivot row requires one multiplication and one subtraction. *We will count this first stage as n^2 multiplications and n^2 subtractions*. It is actually less, $n^2 - n$, because row 1 does not change.

The next stage clears out the second column below the second pivot. The working matrix is now of size $n-1$. Estimate this stage by $(n-1)^2$ multiplications and subtractions. The matrices are getting smaller as elimination goes forward. The rough count to reach U is the sum of squares $n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$.

There is an exact formula $\frac{1}{3}n\left(n + \frac{1}{2}\right)(n+1)$ for this sum of squares. When n is large, the $\frac{1}{2}$ and the 1 are not important. *The number that matters is* $\frac{1}{3}n^3$. The sum of squares is like the integral of x^2 ! The integral from 0 to n is $\frac{1}{3}n^3$: