Lecture 2-A: Vector spaces

CEDC102: Linear Algebra

Manara University

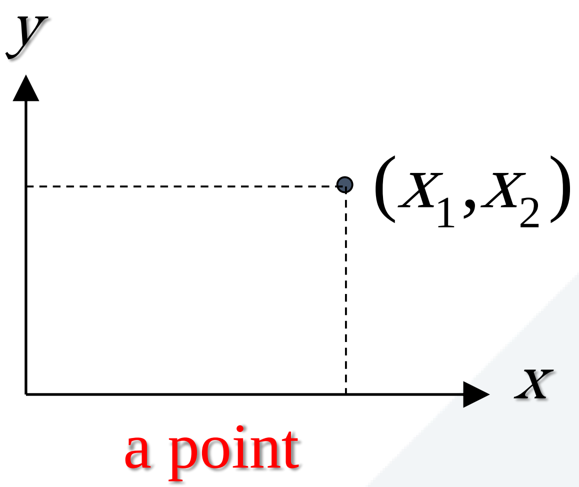
2023-2024

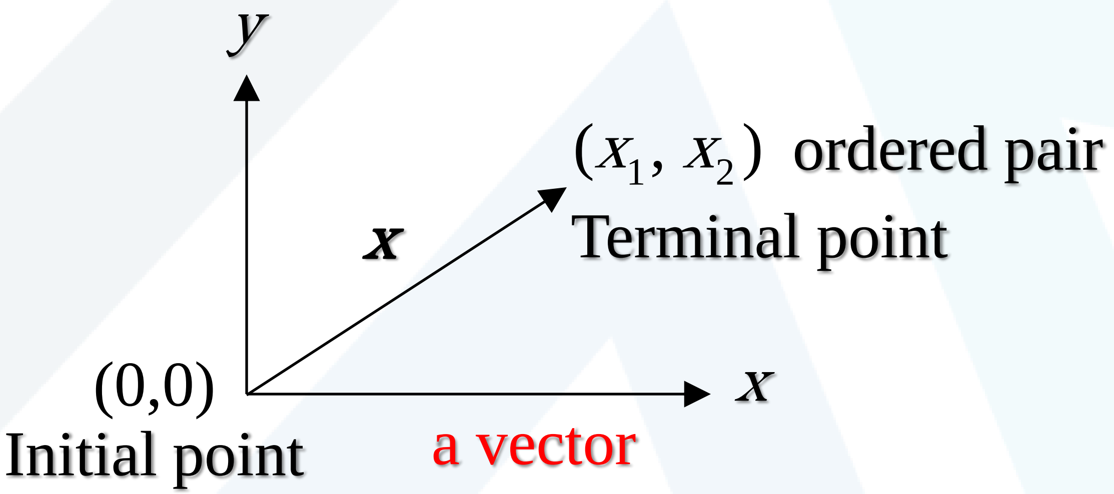
- Vectors in $\mathbb{R}^n$
- Vector Spaces
- Subspaces of Vector Spaces
- Spanning Sets and Linear Independence
- Basis and Dimension
- Rank and Nullity of a Matrix

Vectors in $\mathbb{R}^n$

- **Vectors in the plane:**

a vector $\boldsymbol{x}$ in the plane is represented by a directed line segment with its initial point at the origin and its terminal point at (x_1, x_2).



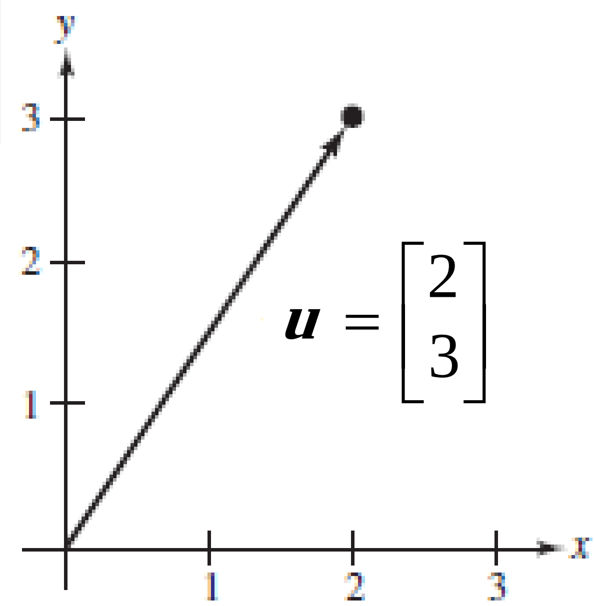


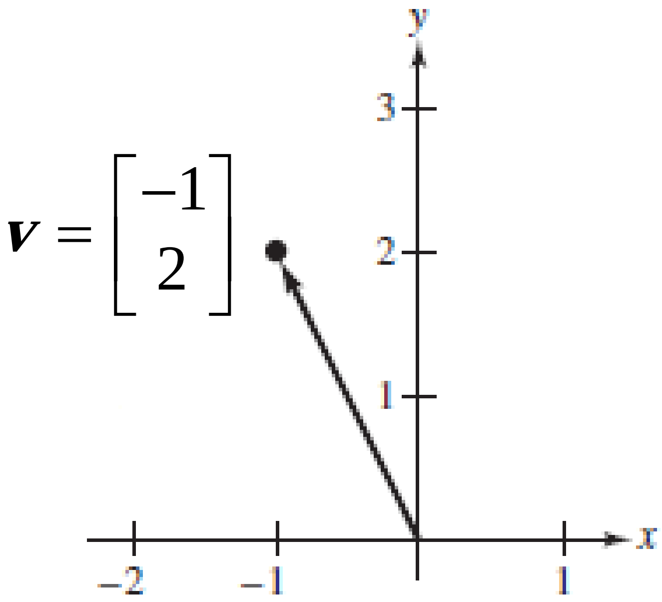
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

x_1 = first component of $\boldsymbol{x}$

x_2 = second component of $\boldsymbol{x}$

- Ex :

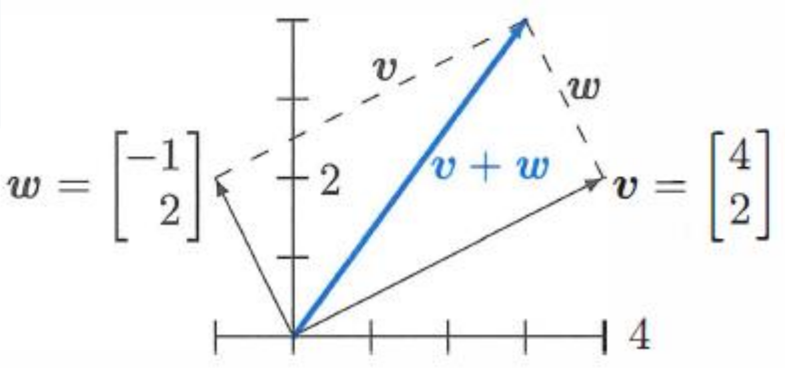


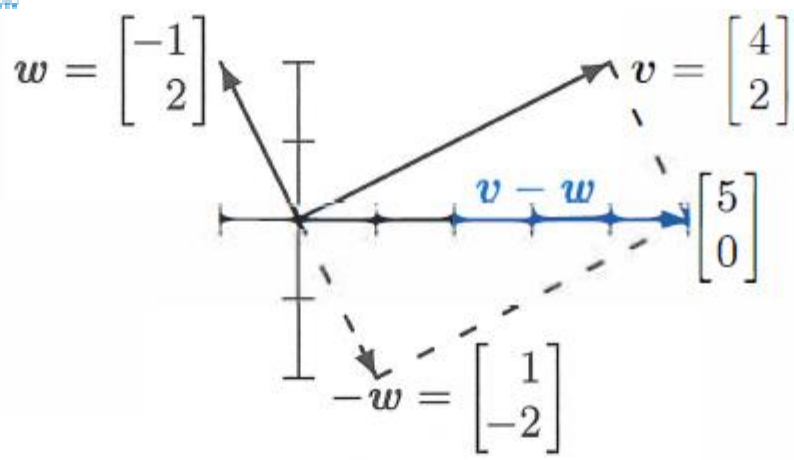


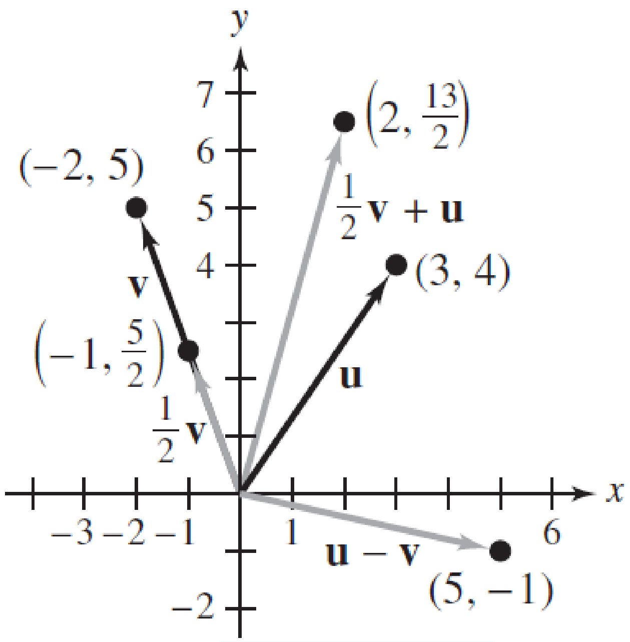
Vector Addition $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$

Scalar Multiplication $c\boldsymbol{v} = c\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \qquad -\boldsymbol{v} = (-1)\boldsymbol{v} \Rightarrow \boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

- Ex :


$w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
v
w
$v + w$
2
4
$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$


$w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$
$v - w$
$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$
$-w = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$


y
x
$(-2, 5)$
$\left(2, \frac{13}{2}\right)$
$\frac{1}{2}\mathbf{v} + \mathbf{u}$
$(3, 4)$
$\mathbf{v}$
$\left(-1, \frac{5}{2}\right)$
$\mathbf{u}$
$\frac{1}{2}\mathbf{v}$
$\mathbf{u} - \mathbf{v}$
$(5, -1)$
−3 −2 −1
1
6
7
6
5
4
−2

- n-space: $\mathbb{R}^n$

$\mathbb{R}^1$ = 1-space = set of all real number (x_1, x_2)

$\mathbb{R}^2$ = 2-space = set of all ordered pair of real numbers (x_1, x_2, x_3)

$\mathbb{R}^3$ = 3-space = set of all ordered triple of real numbers $(x_1, x_2, \cdots, x_n)$

$\vdots$

$\mathbb{R}^n$ = n-space = set of all ordered n-tuple of real numbers

- Notes: An n-tuple $(x_1, x_2, \ldots, x_n)$ can be viewed as

(1) a point in $\mathbb{R}^n$ with the x_i's as its coordinates.

(2) a vector $\boldsymbol{x}$ in $\mathbb{R}^n$ with the x_i's as its components.

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

a vector $\boldsymbol{x}$ in $\mathbb{R}^n$ will be represented also as $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$

$$\boldsymbol{u} = (u_1, u_2, \cdots, u_n), \quad \boldsymbol{v} = (v_1, v_2, \cdots, v_n) \quad \text{(two vectors in } \mathbb{R}^n\text{)}$$

- Equal:

$\boldsymbol{u} = \boldsymbol{v}$ if and only if $u_1 = v_1,\ u_2 = v_2,\ \cdots,\ u_n = v_n$

- Vector addition (the sum of $\boldsymbol{u}$ and $\boldsymbol{v}$):

$$\boldsymbol{u} + \boldsymbol{v} = (u_1 + v_1,\ u_2 + v_2,\ \cdots,\ u_n + v_n)$$

- Scalar multiplication (the scalar multiple of $\boldsymbol{u}$ by c):

$$c\boldsymbol{u} = (cu_1, cu_2, \cdots, cu_n)$$

- Notes:

The sum of two vectors and the scalar multiple of a vector in $\mathbb{R}^n$ are called the standard operations in $\mathbb{R}^n$.

- Negative:

$$-\boldsymbol{u} = (-u_1, -u_2, \cdots, -u_n)$$

- Difference:

$$\boldsymbol{u} - \boldsymbol{v} = (u_1 - v_1,\ u_2 - v_2,\ \cdots,\ u_n - v_n)$$

- Zero vector:

$$\mathbf{0} = (0,\ 0,\ \cdots,\ 0)$$

- Notes:

(1) The zero vector $\mathbf{0}$ in $\mathbb{R}^n$ is called the additive identity in $\mathbb{R}^n$.

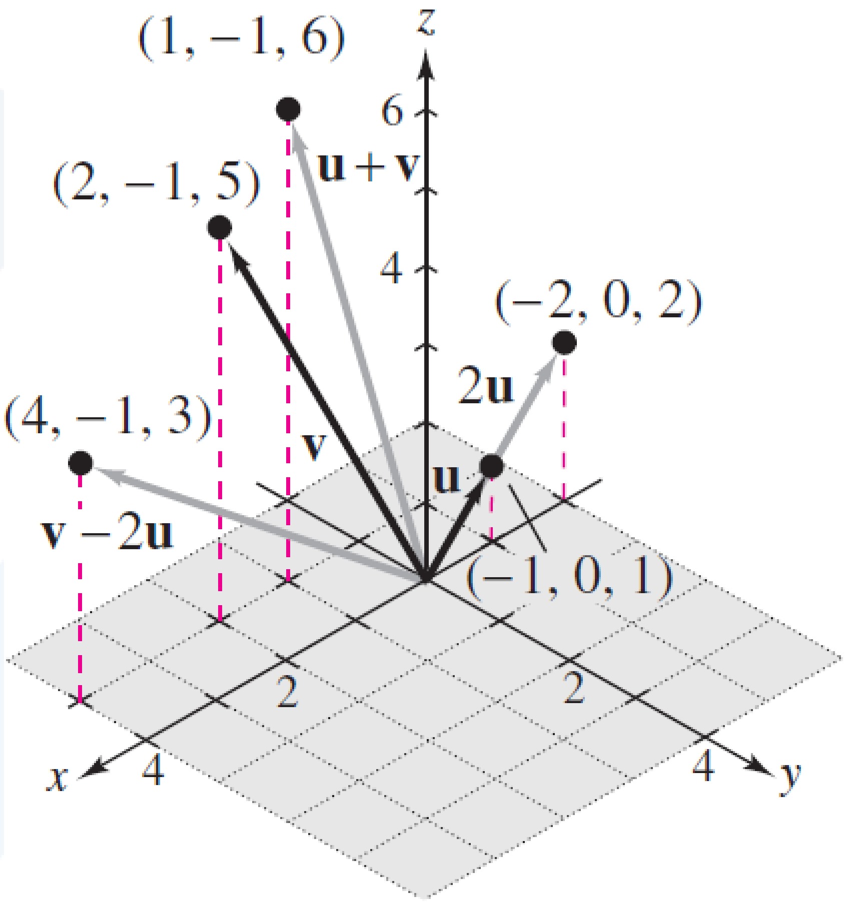
(2) The vector $-\boldsymbol{v}$ is called the additive inverse of $\boldsymbol{v}$.

- Ex :

Let $\boldsymbol{u} = (-1, 0, 1)$ and $\boldsymbol{v} = (2, -1, 5)$ in $\mathbb{R}^3$.

Perform each vector operation:

$(a)\ \boldsymbol{u} + \boldsymbol{v}$ $\quad (b)\ 2\boldsymbol{u}$ $\quad (c)\ \boldsymbol{v} - 2\boldsymbol{u}$



Sol:

$(a)\ \boldsymbol{u} + \boldsymbol{v} = (-1, 0, 1) + (2, -1, 5) \quad = (1, -1, 6)$

$(b)\ 2\boldsymbol{u} = 2\,(-1, 0, 1) = (-2, 0, 2)$

$(c)\ \boldsymbol{v} - 2\boldsymbol{u} = (2, -1, 5) - (-2, 0, 2) \quad = (4, -1, 3)$

- Theorem: (Properties of vector addition and scalar multiplication)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c and d be scalars

(1) $\mathbf{u}+\mathbf{v}$ is a vector in $\mathbb{R}^n$	Closure under addition
(2) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	Commutative property of addition
(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	Associative property of addition
(4) $\mathbf{u}+\mathbf{0}=\mathbf{u}$	Additive identity property
(5) $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$	Additive inverse property
(6) $c\mathbf{u}$ is a vector in R^n	Closure under scalar multiplication
(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$	Distributive property
(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$	Distributive property
(9) $c(d\mathbf{u})=(cd)\mathbf{u}$	Associative property of multiplication
(10) $1(\mathbf{u})=\mathbf{u}$	Multiplicative identity property

- Ex : (Vector operations in $\mathbb{R}^4$)

Let $\boldsymbol{u}=(2,-1,5,0)$, $\boldsymbol{v}=(4,3,1,-1)$ and $\boldsymbol{w}=(-6,2,0,3)$ be vectors in $\mathbb{R}^4$. Solve $\boldsymbol{x}$ for each of the following:

(*a*) $\boldsymbol{x}=2\boldsymbol{u}-(\boldsymbol{v}+3\boldsymbol{w})$

(*b*) $3(\boldsymbol{x}+\boldsymbol{w})=2\boldsymbol{u}-\boldsymbol{v}+\boldsymbol{x}$

Sol: (*a*) $\boldsymbol{x}=2\boldsymbol{u}-(\boldsymbol{v}+3\boldsymbol{w})=2\boldsymbol{u}-\boldsymbol{v}-3\boldsymbol{w}$

$$=(4,-2,10,0)-(4,3,1,-1)-(-18,6,0,9)=(18,-11,9,-8)$$

(*b*) $3(\boldsymbol{x}+\boldsymbol{w})=2\boldsymbol{u}-\boldsymbol{v}+\boldsymbol{x}\Rightarrow 3\boldsymbol{x}+3\boldsymbol{w}=2\boldsymbol{u}-\boldsymbol{v}+\boldsymbol{x}$

$3\boldsymbol{x}-\boldsymbol{x}=2\boldsymbol{u}-\boldsymbol{v}-3\boldsymbol{w}\Rightarrow 2\boldsymbol{x}=2\boldsymbol{u}-\boldsymbol{v}-3\boldsymbol{w}\Rightarrow \boldsymbol{x}=\boldsymbol{u}-\frac{1}{2}\boldsymbol{v}-\frac{3}{2}\boldsymbol{w}$

$\boldsymbol{x}=(2,1,5,0)+(-2,-\frac{3}{2},-\frac{1}{2},\frac{1}{2})+(9,-3,0,-\frac{9}{2})=(9,-\frac{11}{2},\frac{9}{2},-4)$

- Theorem : (Properties of additive identity and additive inverse)

Let $\boldsymbol{v}$ be a vector in $\mathbb{R}^n$, and c be a scalars. Then the properties below are true:

(1) The additive identity is unique. That is, if $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}$, then $\boldsymbol{u}=\mathbf{0}$

(2) The additive inverse of $\boldsymbol{v}$ is unique. That is, if $\boldsymbol{v}+\boldsymbol{u}=\mathbf{0}$, then $\boldsymbol{u}=-\boldsymbol{v}$

(3) $0\boldsymbol{v}=\mathbf{0}$

(4) $c\mathbf{0}=\mathbf{0}$

(5) If $c\boldsymbol{v}=\mathbf{0}$, then $c=0$ or $\boldsymbol{v}=\mathbf{0}$

(6) $-(-\boldsymbol{v})=\boldsymbol{v}$

- Linear combination:

The vector $\boldsymbol{x}$ is called a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ if it can be expressed in the form $\boldsymbol{x} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n}$ $\quad c_1, c_2, \ldots, c_n$: scalars

- Ex 5: Given $\boldsymbol{x} = (-1, -2, -2)$, $\boldsymbol{u} = (0, 1, 4)$, $\boldsymbol{v} = (-1, 1, 2)$, and $\boldsymbol{w} = (3, 1, 2)$ in $\mathbb{R}^3$, find a, b, and c such that $\boldsymbol{x} = a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w}$.

Sol:

$$\begin{array}{rcrcrcr} & & -b & + & 3c & = & -1 \\ a & + & b & + & c & = & -2 \\ 4a & + & 2b & + & 2c & = & -2 \end{array}$$

$$\Rightarrow a = 1, \;\; b = -2, \;\; c = -1$$

Thus $\boldsymbol{x} = \boldsymbol{u} - 2\boldsymbol{v} - \boldsymbol{w}$

Vector Spaces

- Vector spaces:

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in V and every scalar c and d, then V is called a vector space.

Addition:

(1) $\boldsymbol{u}+\boldsymbol{v}$ is in V — Closure under addition

(2) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$ — Commutative property

(3) $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$ — Associative property

(4) V has a zero vector $\mathbf{0}$: for every $\boldsymbol{u}$ in V, $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$ — Additive identity

(5) For every $\boldsymbol{u}$ in V, there is a vector in V denoted by $-\boldsymbol{u}$: $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$ — Scalar identity

Scalar multiplication:

(6) $c\boldsymbol{u}$ is a vector in V Closure under scalar multiplication

(7) $c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u}+c\boldsymbol{v}$ Distributive property

(8) $(c+d)\boldsymbol{u} = c\boldsymbol{u}+d\boldsymbol{u}$ Distributive property

(9) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ Associative property

(10) $1(\boldsymbol{u}) = \boldsymbol{u}$ Scalar identity

- Notes:

(1) A vector space $(V, +, .)$ consists of four entities:

a nonempty set V of vectors, a set of scalars, and two operations (+, .)

(2) $V = \{\mathbf{0}\}$ zero vector space

(3) $K = R$: Real Vector Space $K = C$: Complex Vector Space

- Examples of vector spaces:

(1) n-tuple space: $V = R^n$

$$(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) \quad \text{vector addition}$$

$$k(u_1, u_2, \cdots, u_n) = (ku_1, ku_2, \cdots, ku_n) \quad \text{scalar multiplication}$$

(2) Matrix space: $V = M_{m \times n}$ (the set of all $m \times n$ matrices with real values)

Ex: ($m = n = 2$)

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix} \quad \text{vector addition}$$

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \quad \text{scalar multiplication}$$

(3) Infinite Sequences space: $V = R^{\infty}$ (set of all infinite sequences of real numbers)

$$(u_n)_{n\in N} + (v_n)_{n\in N} = (u_n + v_n)_{n\in N} \qquad c(u_n)_{n\in N} = (cu_n)_{n\in N}$$

(4) polynomial space: $V = P_{\infty}$ (the set of all real polynomials)

$$(p+q)(x) = p(x) + q(x) \qquad (cp)(x) = cp(x)$$

(5) n-th degree polynomial space: $V = P_n(x)$

(the set of all real polynomials of degree n or less)

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n$$

$$kp(x) = ka_0 + ka_1 x + \cdots + ka_n x^n$$

(6) Function space: $V = c(-\infty, \infty)$ (the set of all real functions)

$$(f+g)(x) = f(x) + g(x) \qquad (kf)(x) = kf(x)$$

- Theorem: (Properties of scalar multiplication)

  Let $\boldsymbol{v}$ any element of a vector space V, and let c be any scalars. Then the following properties are true:

  (1) $0\boldsymbol{v} = \mathbf{0}$

  (2) $c\mathbf{0} = \mathbf{0}$

  (3) If $c\boldsymbol{v} = \mathbf{0}$, then $c = 0$ or $\boldsymbol{v} = \mathbf{0}$

  (4) $(-1)\boldsymbol{v} = -\boldsymbol{v}$

- Note: To show that a set is not a vector space, you need only find one axiom that is not satisfied

- Ex : $V = R^2 =$ the set of all ordered pairs of real numbers

  vector addition: $(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

  scalar multiplication: $c(u_1, u_2) = (cu_1, 0)$ Verify that V is not a vector space.

Sol:

$$1(1, 1) = (1, 0) \neq (1, 1)$$

$\Rightarrow$ the set (together with the two given operations) is not a vector space

- Ex 2: Set of all real polynomials of degree n Is Not a vector space. Why?

- Complex Vector Spaces C^n:

A vector space in which scalars are allowed to be complex numbers is called a complex vector space

Vectors in C^n: If n is a positive integer, then a complex n-tuple is a sequence of n complex numbers $\mathbf{v} = (v_1, v_2, ..., v_n)$. The set of all complex n-tuples is called complex n-space and is denoted by C^n.

- Ex (C^3): $\boldsymbol{u} = (1, -i+1, -2)$, $\boldsymbol{v} = (i, 3, 2i)$, $\boldsymbol{u} + \boldsymbol{v} = (1+i, -i+4, -2+2i)$
- Note: The complex vector space C^n is a generalization of the real vector space R^n
- Vector Conjugate

$$\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \Rightarrow \overline{\boldsymbol{v}} = (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_n})$$

- Ex : $\boldsymbol{u} = (3+i, -2i, 5) \Rightarrow \overline{\boldsymbol{u}} = (3-i, 2i, 5)$
- Properties of vector conjugate

$$\boldsymbol{u}, \boldsymbol{v} \in C^n$$

(1) $\overline{\overline{\boldsymbol{u}}} = \boldsymbol{u}$ (2) $\overline{c\boldsymbol{u}} = \overline{c}\,\overline{\boldsymbol{u}}, \quad c \in C$

(3) $\overline{\boldsymbol{u} \pm \boldsymbol{v}} = \overline{\boldsymbol{u}} \pm \overline{\boldsymbol{v}}$

Subspaces of Vector Spaces

- **Subspace:**

$(V, +, .)$: a vector space

$W \neq \emptyset$
$W \subseteq V$: a nonempty subset

$(W, +, .)$: a vector space (under the operations of addition and scalar multiplication defined in V)

$\Rightarrow W$ is a subspace of V

- **Trivial subspace:** Every vector space V has at least two subspaces

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Theorem: (Test for a subspace)

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold:

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

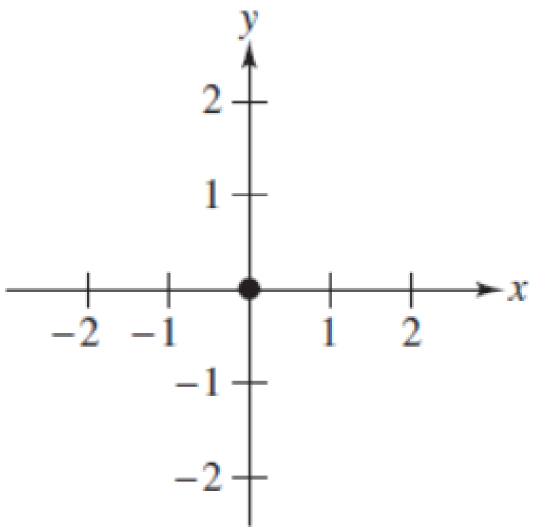
- Notes:

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, c and d are any scalars, then $c\mathbf{u}+d\mathbf{v}$ is in W.

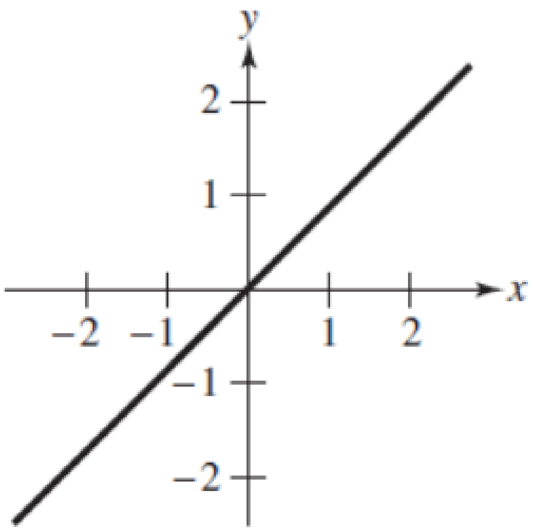
$\Rightarrow$ W is a subspace of V

(2) If W is a subspace of a vector space V, then W contains the zero vector $\mathbf{0}$ of V

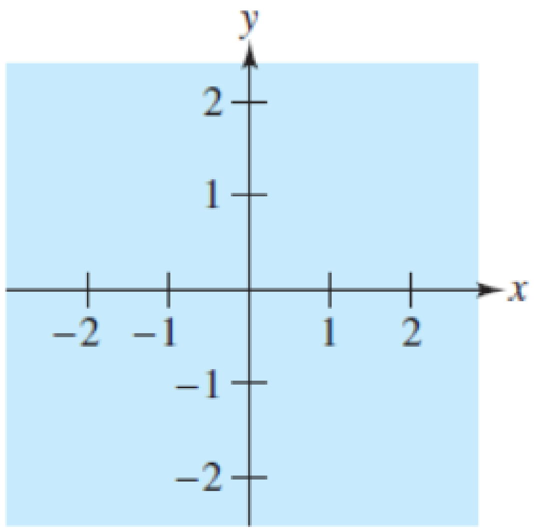
- Ex : Subspace of $\mathbb{R}^2$



$W = \{(0, 0)\}$



W = all points on a line passing through the origin



$W = R^2$

(1) $\{\mathbf{0}\}$ $\qquad \mathbf{0} = (0, 0)$

(2) Lines through the origin

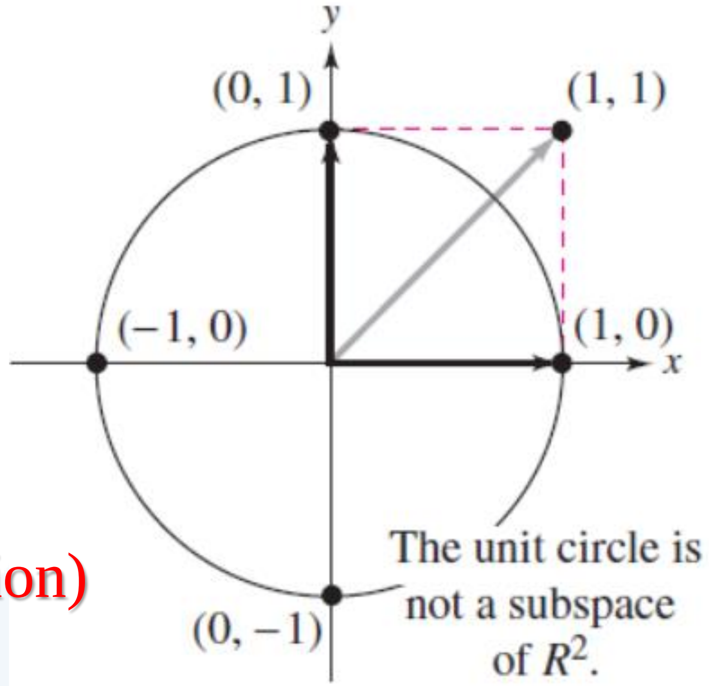
(3) R^2

- Ex : (A Subset of $\mathbb{R}^2$ That Is Not a Subspace)

  Show that the subset of R^2 consisting of all points on $x^2 + y^2 = 1$ is not a subspace

  Sol:

  points (1, 0) and (0, 1) are in the subset, but their sum $(1, 0) + (0, 1) = (1, 1)$ is not. (not closed under addition)



- Ex : Subspace of R^3

  (1) $\{\mathbf{0}\}$ $\quad\quad \mathbf{0} = (0, 0, 0)$

  (2) Lines through the origin

  (3) Planes through the origin

  (4) R^3

- Ex : (The set of first-quadrant vectors is not a subspace of $\mathbb{R}^2$)

Show that $W = \{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2.

Sol:

Let $\boldsymbol{u} = (1, 1) \in W$

$(-1)\boldsymbol{u} = (-1)(1, 1) = (-1, -1) \notin W$ (not closed under scalar multiplication)

$\Rightarrow W$ is not a subspace of R^2

- Ex 5: (Determining subspaces of R^3)

Which of the following subsets is a subspace of R^3?

$(a)\ W = \{(x_1, x_2, 1) \mid x_1, x_2 \in R\}$ No ($\mathbf{0} = (0, 0, 0) \notin W$)

$(b)\ W = \{(x_1, x_1 + x_3, x_3) \mid x_1, x_3 \in R\}$ Yes

- Ex : (A subspace of $M_{2\times2}$)

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times2}$, with the standard operations of matrix addition and scalar multiplication

- Ex : (The set of singular matrices is not a subspace of $M_{2\times2}$)

Let W be the set of singular matrices of order 2. Show that W is not a subspace of $M_{2\times2}$ with the standard operations

- Ex : (Determining subspaces of R^2)

Which of the following two subsets is a subspace of R^2?

(a) The set of points on the line given by $x + 2y = 0$. Yes

(b) The set of points on the line given by $x + 2y = 1$. No

- Ex : (Subspaces of Functions)

Let W_5 be the vector space of all functions defined on [0, 1]

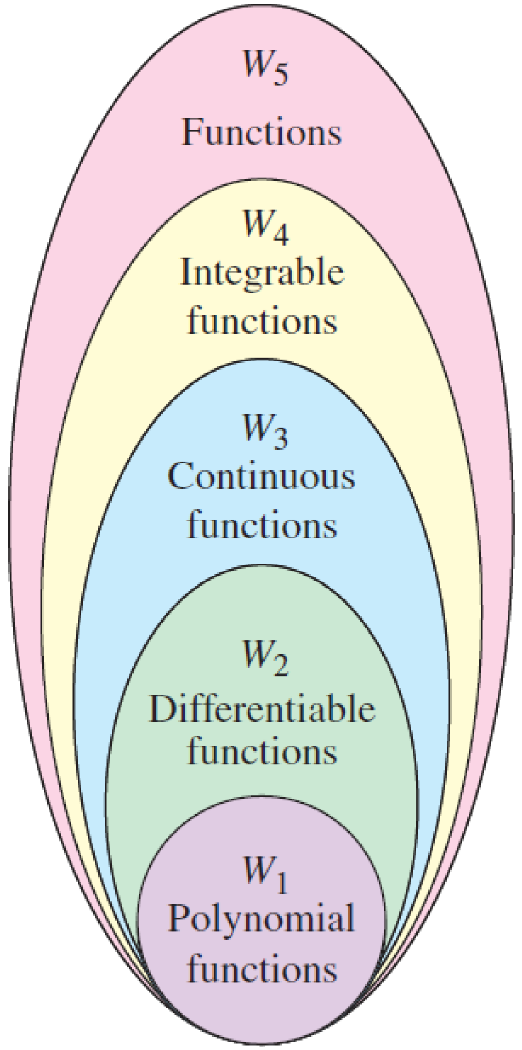
W_1 = set of all polynomial defined on [0, 1]

W_2 = set of all functions differentiable on [0, 1]

W_3 = set of all functions continuous on [0, 1]

W_4 = set of all functions integrable on [0, 1]

Show that $W_1 \subset W_2 \subset W_3 \subset W_4 \subset W_5$ and that W_i is a subspace of W_j for $i \leq j$



- Ex 10: P_n is a subspace of P_∞

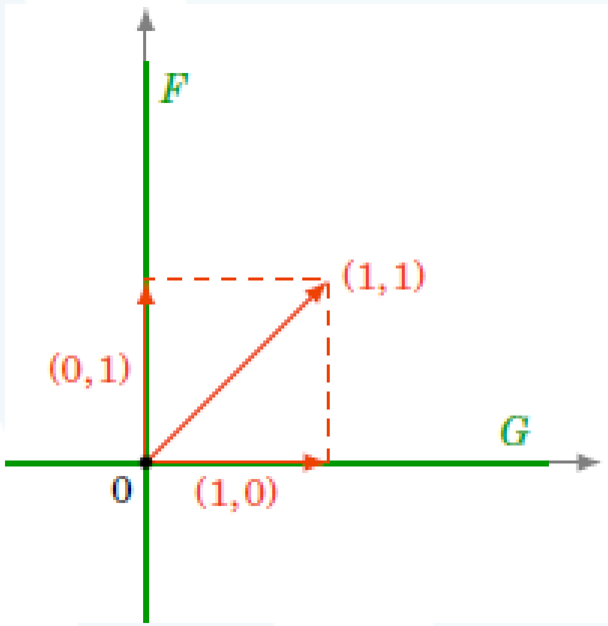
- Theorem (The intersection of two subspaces is a subspace)

  If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.

- Note:

  The union of F and G (denoted by $F \cup G$) is not necessarily a subspace of V

- Ex 12: Let $V = R^2$



$$F = \{(x, y) \in R^2 \mid x = 0\}, G = \{(x, y) \in R^2 \mid y = 0\}$$

$$F \cap G = \{\mathbf{0}\}$$

$$(0, 1)\,(\in F) + (1, 0)\,(\in G) = (1, 1) \notin F \cup G$$

- Theorem : (The sum of two subspaces is a subspace)

If F and G are both subspaces of a vector space V, then the sum of F and G (denoted by $F + G$), consisting of all the elements $\boldsymbol{u} + \boldsymbol{v} | \boldsymbol{u} \in F, \boldsymbol{v} \in G$. It is also a subspace of V.

- Ex 13: Let $V = R^2$

$$F = \{(x,y) \in R^2 | x = 0\}, G = \{(x,y) \in R^2 | y = 0\}$$

$$F + G = R^2$$

- Ex 14: Let $V = R^3$

$$F = \{(x,y,z) \in R^3 | y = z = 0\} \text{ and } G = \{(x,y,z) \in R^3 | x = z = 0\}$$

$$F + G = \{(x,y,z) \in R^3 | z = 0\}$$

The Column Space of A

The most important subspaces are tied directly to a matrix A.

To solve $Ax = b$.

If A is not invertible, the system is solvable for some b and not solvable for other b.

We want to describe the good right sides b-the vectors that can be written as A times some vector x

Those b' s form the "column space" of A

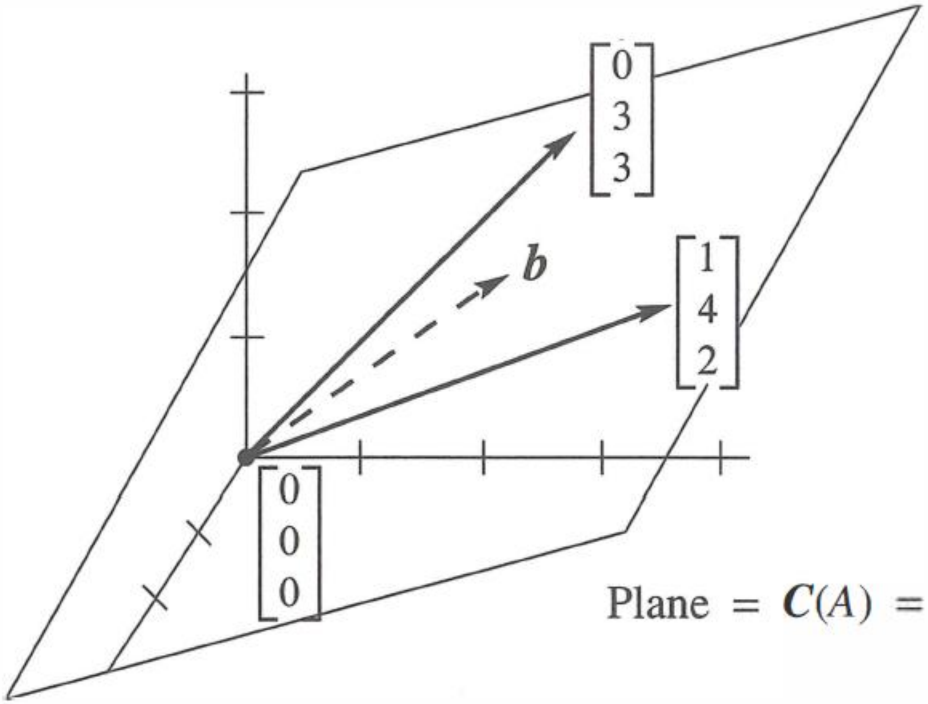
Remember: Ax is a combination of the columns of A.

Start with the columns of A and take all their linear combinations.
This produces the column space of A.

It is a vector subspace space of $\mathbb{R}^m$ made up of column vectors

DEFINITION The column space consists of all linear combinations of the columns . The combinations are all possible vectors Ax. They fill the column space C(A).

Note: The system Ax = b is solvable if and only if b is in the column space of A.



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\boldsymbol{b} = .4 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + .3 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$A\boldsymbol{x} = \boldsymbol{b} \text{ has } \boldsymbol{x} = \begin{bmatrix} .4 \\ .3 \end{bmatrix}$$

Plane $= \boldsymbol{C}(A) =$ all vectors $A\boldsymbol{x}$

The column space $\boldsymbol{C}(A)$ is a plane containing the two columns. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is on that plane. Then $\boldsymbol{b}$ is a combination of the columns.

Spanning Sets and Linear Independence

- **Linear combination:**

  A vector $\boldsymbol{v}$ in a vector space V is called a linear combination of the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ in V if $\boldsymbol{v}$ can be written in the form

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \qquad c_1, c_2, \ldots, c_k\text{: scalars}$$

- **Ex 1: (Finding a linear combination)**

$$\boldsymbol{v}_1 = (1,\ 2,\ 3), \qquad \boldsymbol{v}_2 = (0,\ 1,\ 2), \qquad \boldsymbol{v}_3 = (-1,\ 0,\ 1)$$

Prove (*a*) $\boldsymbol{w} = (1, 1, 1)$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$

(*b*) $\boldsymbol{w} = (1, -2, 2)$ is not a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$

Sol: $(a)\ \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$(1, 1, 1) = c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)$$

$$= (c_1 - c_3,\ 2c_1 + c_2,\ 2c_2 + c_3)$$

$$\Rightarrow \begin{array}{rcrcrl} c_1 & & & - & c_3 & = 1 \\ 2c_1 & + & c_2 & & & = 1 \\ 3c_1 & + & 2c_2 & + & c_3 & = 1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow c_1 = 1 + t,\ c_2 = -1 - 2t,\ c_3 = t$ (this system has infinitely many solutions)

$t = 1 \Rightarrow \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$

$$(b)\ \boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ this system has no solution $(0 \neq 7)$

$$\Rightarrow\ \boldsymbol{w} \neq c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3}$$

- The span of a set: span (S)

  If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \left\{c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_k\boldsymbol{v_k} \mid \forall c_i \in K\right\}$$

(the set of all linear combinations of the vectors in S)

- A spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called a spanning set of the vector space.

- Notes:

$$\text{span}(S) = V$$

$\Rightarrow$ S spans (generates) V or V is spanned (generated) by S

S is spanning set of V

- Notes:

(1) $\text{span}(\emptyset) = \{0\}$ (2) $S \subseteq \text{span}(S)$

(3) $S_1, S_2 \subseteq V$

$$S_1 \subseteq S_2 \Rightarrow \text{span}(S_1) \subseteq \text{span}(S_2)$$

- Ex 2: (Examples of Spanning Sets)

  The set $S = (1, 0, 0) + (0, 1, 0) + (0, 0, 1)$ spans R^3

  The set $S = \{1, x, x^2\}$ spans P_2

- Ex 3: (A spanning set for R^3)

  The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\boldsymbol{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3)$$

- Ex 4: (A spanning set for R^3)

  Show that the set $S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$ spans R^3

  Sol: We must determine whether an arbitrary vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in R^3 can be as a linear combination of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$.

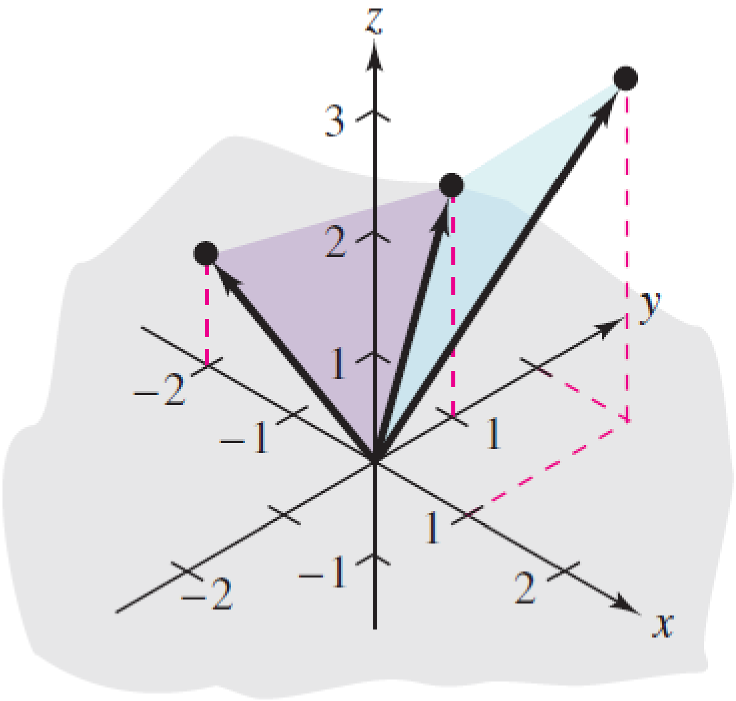
$$\boldsymbol{u} \in R^3 \Rightarrow \boldsymbol{u} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3} \Rightarrow \begin{array}{rcrcrcl} c_1 & & & - & 2c_3 & = & u_1 \\ 2c_1 & + & c_2 & & & = & u_2 \\ 3c_1 & + & 2c_2 & + & c_3 & = & u_3 \end{array}$$

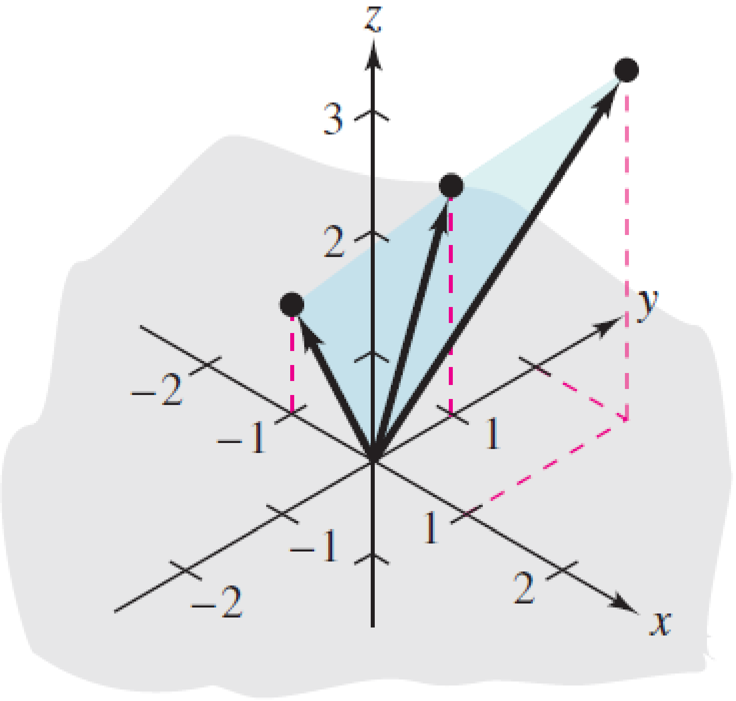
$$|A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{u} \Rightarrow \text{span}(S_1) = R^3$

- Ex 5: (A Set Does Not Span R^3)

  From Example 1: the set $S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$ does not span R^3 because $\boldsymbol{w} = (1, -2, 2)$ is in R^3 and cannot be expressed as a linear combination of the vectors in S_2.





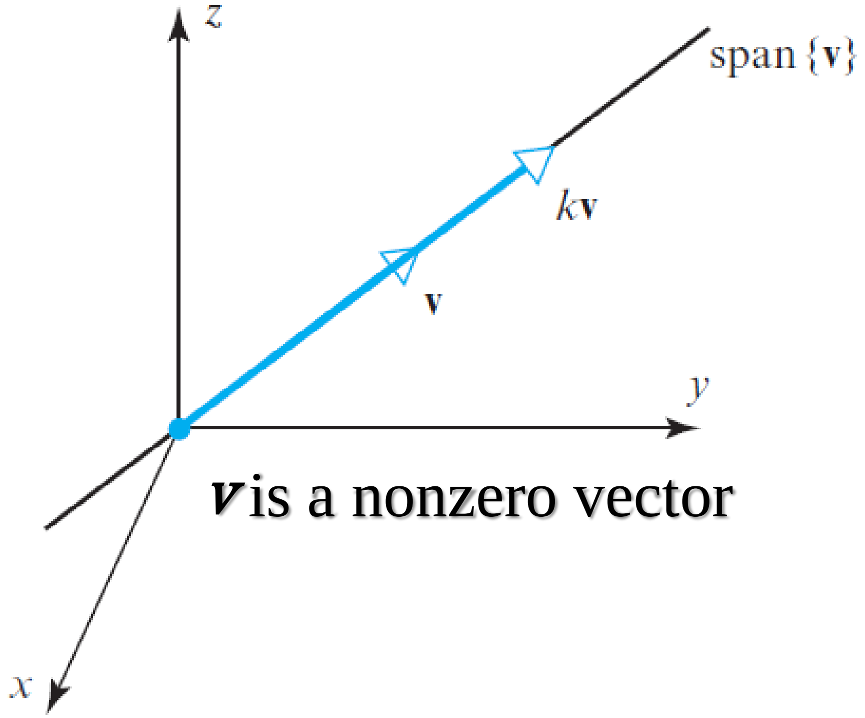
$S_1 = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$

The vectors in S_1 do not lie in a common plane

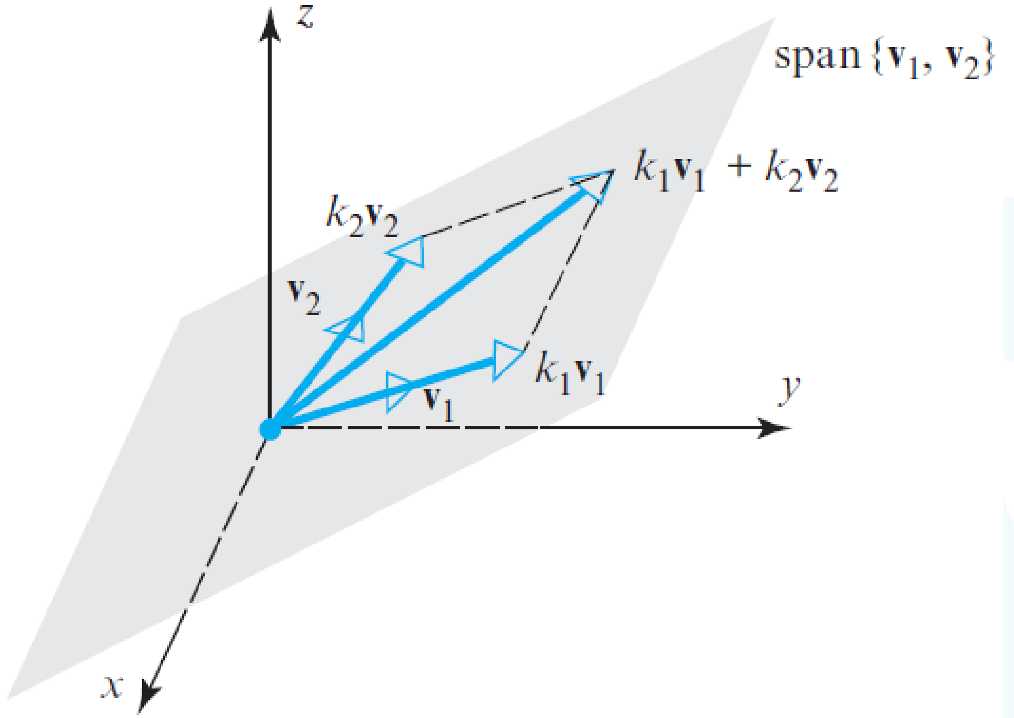
$S_2 = \{(1, 2, 3), (0, 1, 2), (-1, 0, 1)\}$

The vectors in S_2 lie in a common plane

- Ex 6: (A Geometric View of Spanning in $\mathbb{R}^3$)



$\boldsymbol{v}$ is a nonzero vector



span$\{\boldsymbol{v}\}$ is the line through the origin determined by $\boldsymbol{v}$

span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is the plane through the origin determined by $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$