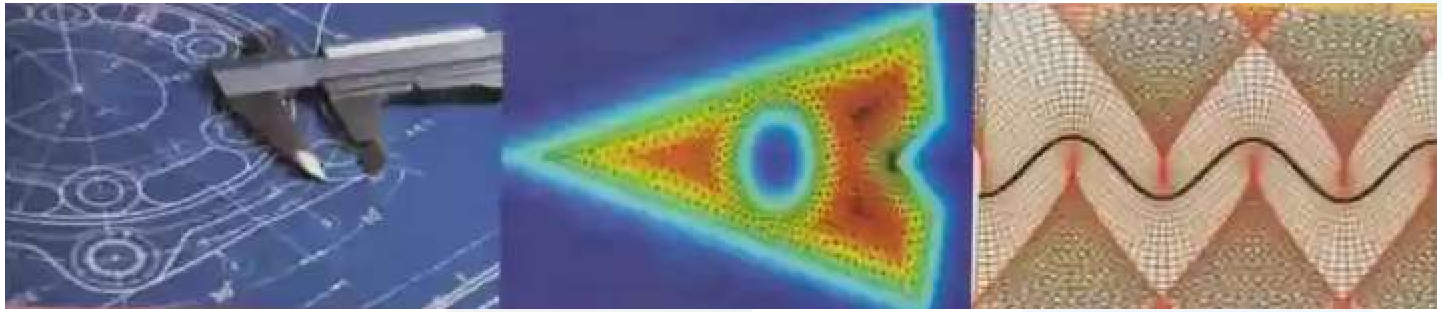
جَامعة المَنارة
MANARA UNIVERSITY

CEDC301: Engineering Mathematics

Lecture Notes 2: Functions of a Complex Variable: Part B



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Chapter 1

Functions of a Complex Variable

1. Complex Numbers
2. Powers and Roots
3. Sets in the Complex Plane
4. Functions of a Complex Variable
5. Cauchy-Riemann Equations
6. Exponential and Logarithmic Functions
7. Trigonometric and Hyperbolic Functions
8. Inverse Trigonometric and Hyperbolic Functions

6. Exponential and Logarithmic Functions

Exponential Function

We want the definition of the complex function $f(z) = e^z$, where $z = x + iy$, to reduce e^x for $y = 0$ and to possess the properties $f'(z) = f(z)$ and $f(z_1 + z_2) = f(z_1)f(z_2)$.

- **Definition:** The complex exponential function is defined as:

$$e^z = e^{x+iy} = e^x(\cos y + i\sin y)$$

- The real and imaginary parts of e^z are continuous and have continuous first partial derivatives at every point z of the complex plane. Moreover, the Cauchy-Riemann equations are satisfied at all points of the complex plane:

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

$f(z) = e^z$ is analytic for all z

f is an entire function

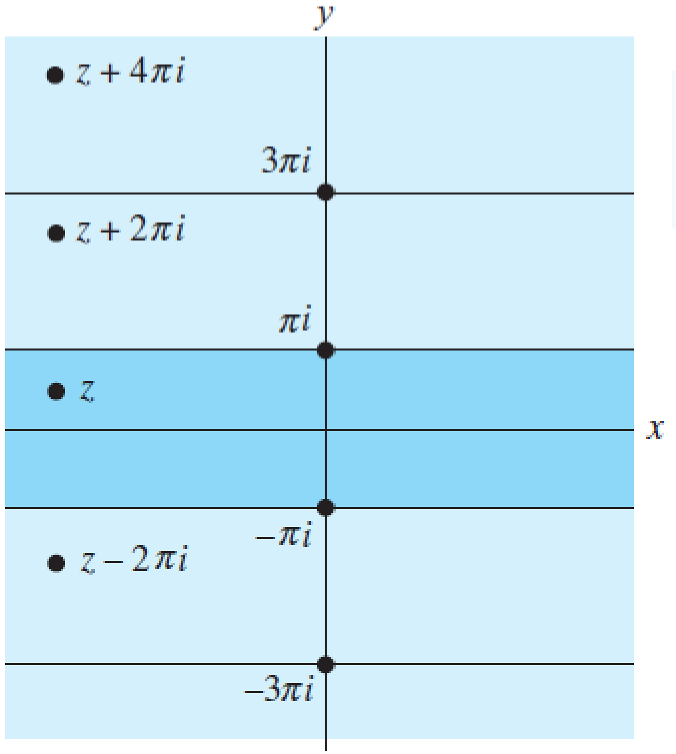
Properties

$$\frac{d}{dz}e^z = e^z, \quad e^0 = 1, \quad e^{z_1}e^{z_2} = e^{z_1+z_2}, \quad \frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}, \quad \overline{e^z} = e^{\bar{z}}$$

Periodicity

Unlike the real function e^x, the complex function $f(z) = e^z$ is periodic with the complex period $2\pi i$. $f(z+2\pi i) = f(z)$

If we divide the complex plane into horizontal strips defined by $(2n-1)\pi < y \leq (2n+1)\pi$, $n = 0, \pm 1, \pm 2, \ldots$, then, for any point z in the strip $-\pi < y \leq \pi$, the values $f(z)$, $f(z+2\pi i)$, $f(z-2\pi i)$, $f(z+4\pi i)$, and so on, are the same. The strip $-\pi < y \leq \pi$ is called the fundamental region for the exponential function $f(z) = e^z$.



Logarithmic Function

The logarithm of a complex number $z = x + iy$, $z \neq 0$, is defined as the inverse of the exponential function, $w = \log z$ if $z = e^w$.

- Definition: The multiple-valued function Logarithm of a Complex Number $z = x + iy$, $z \neq 0$, is defined as:

$$\log z = \ln|z| + i \arg z = \ln|z| + i(\operatorname{Arg} z + 2\pi n), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\log(-2) = \ln 2 + i(\pi + 2\pi n)$$
$$\log(i) = i\left(\frac{\pi}{2} + 2\pi n\right)$$
$$\log(-1-i) = \ln\sqrt{2} + i\left(\frac{5\pi}{4} + 2\pi n\right)$$

Principal Value

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \quad z \neq 0, \quad -\pi < \operatorname{Arg} z \leq \pi$$

$f(z) = \operatorname{Log} z$ is called the principal branch of $\log z$, or the principal logarithmic function.

$$\operatorname{Log}(-2) = \ln 2 + \pi i$$

$$\operatorname{Log}(i) = \frac{\pi}{2} i$$

$$\operatorname{Log}(-1-i) = \ln\sqrt{2} - \frac{3\pi}{4} i$$

Properties

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

$$\log \frac{z_1}{z_2} = \log z_1 - \log z_2$$

$$\log z^n = n\log z$$

- Note: The identities above are not necessarily satisfied by the principal value. For example, it is not true that $\mathrm{Log}(z_1 z_2) = \mathrm{Log}\ z_1 + \mathrm{Log}\ z_2$ for all complex numbers z_1 and z_2 (although it may be true for some complex numbers).

- Example 4: $\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}\ z_1 + \mathrm{Log}\ z_2$

  If $z_1 = i$ and $z_2 = -1 + i$, then

$$\mathrm{Log}(z_1 z_2) = \mathrm{Log}\,(-1 - i) = \ln\sqrt{2} - \frac{3\pi}{4} i$$

$$\mathrm{Log} z_1 + \mathrm{Log} z_2 = \frac{\pi}{2} i + \left(\ln\sqrt{2} + \frac{3\pi}{4} i\right) = \ln\sqrt{2} + \frac{5\pi}{4} i \neq \mathrm{Log}(z_1 z_2)$$

$\mathrm{Log}\ z$ as an Inverse Function

$$e^{\mathrm{Log}\ z} = z,\ z \neq 0$$

$$\mathrm{Log}\ e^{z} = z \text{ if } -\infty < x < \infty \text{ and } -\pi < y \leq \pi$$

- If the complex exponential function $f(z) = e^z$ is defined on the fundamental region $-\infty < x < \infty$, $-\pi < y \leq \pi$, then f is one-to-one and the inverse function of f is the principal value of the complex logarithm $f^{-1}(z) = \text{Log } z$.

  For example, for the point $z = 1 + 3/2\pi i$, which is not in the fundamental region, we have:

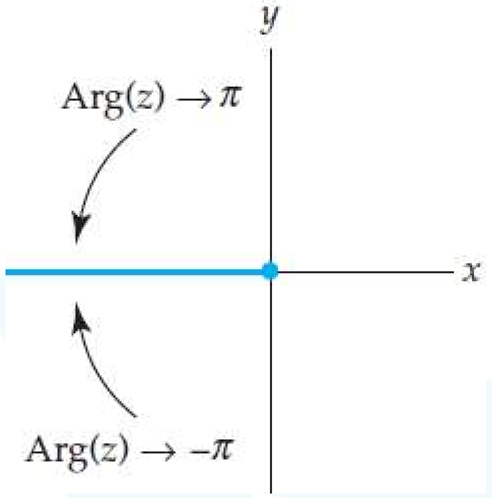
$$\text{Log } e^{1+3\pi i/2} = 1 - \pi i/2 \neq 1 + 3\pi i/2$$

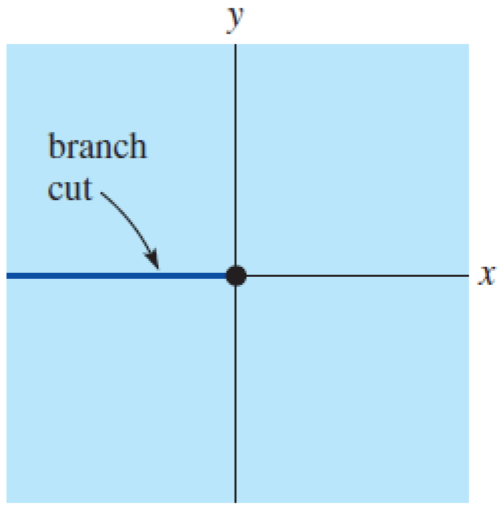
Analyticity

- The logarithmic function $f(z) = \text{Log } z$ is not continuous at $z = 0$ since $f(0)$ is not defined.
- The logarithmic function $f(z) = \text{Log } z$ is discontinuous at all points of the negative real axis.

- This is because the imaginary part of the function, $v = \operatorname{Arg} z$, is discontinuous only at these points.
- Suppose x_0 is a point on the negative real axis. As $z \to x_0$ from the upper half-plane, Arg $z \to \pi$, whereas if $z \to x_0$ from the lower half-plane, then Arg $z \to -\pi$.
- This means that $f(z) = \operatorname{Log} z$ is not analytic on the nonpositive real axis.
- However, $f(z) = \operatorname{Log} z$ is analytic throughout the domain D consisting of all the points in the complex plane except those on the nonpositive real axis.

$$|z| > 0,\ -\pi < \arg(z) < \pi$$





- It is convenient to think of D as the complex plane from which the nonpositive real axis has been cut out.
- Since $f(z) = \text{Log } z$ is the principal branch of $\log z$, the nonpositive real axis is referred to as a branch cut for the function.
- The Cauchy-Riemann equations are satisfied throughout this cut plane and that the derivative of $\text{Log } z$ is given by:

$$\frac{d}{dz}\text{Log } z = \frac{1}{z} \qquad \text{for all } z \text{ in } D$$

- Example 5: Derivatives of Logarithmic Functions

  Find the derivatives of the following functions in an appropriate domain:

  (a) $z \log z$ and (b) $\log(z+1)$

(a) $z \log z$ is differentiable at all points where both of the functions z and $\log z$ are differentiable. z is entire and $\log z$ is differentiable on the domain:

$$|z| > 0,\ -\pi < \arg\ z \leq \pi$$

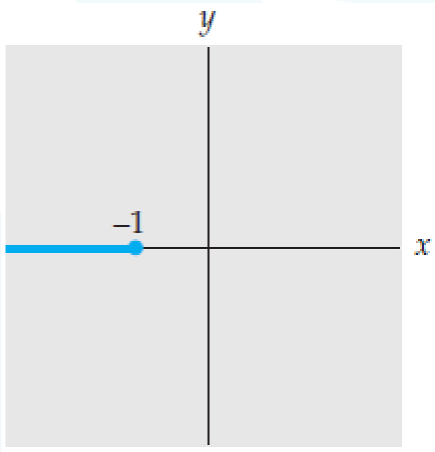
So $z \log z$ is differentiable on the domain defined by:

$$|z| > 0,\ -\pi < \arg\ z < \pi$$

$$\frac{d}{dz}[z\,\mathrm{Log}\ z] = z \cdot \frac{1}{z} + 1 \cdot \mathrm{Log}\ z = 1 + \mathrm{Log}\ z$$

(b)

$$\frac{d}{dz}\mathrm{Log}\,(z+1) = \frac{1}{z+1} \cdot 1 = \frac{1}{z+1}$$



Complex Powers

- If α is a complex number and $z = x + iy$, then z^α is defined by:

$$z^\alpha = e^{\alpha \log z}, \quad z \neq 0$$

- In general, z^α is multiple-valued since $\log z$ is multiple-valued. However, in the special case when $\alpha = n =$ 0, ±1, ± 2, ... z^α is single-valued.

- Note: If we use $\mathrm{Log}\, z$ in place of $\log z$, then z^α gives the principal value.

- Example 6: Complex Power

  Find the value of: (a) i^{2i} (b) $(1 + i)^i$

  (a) $i^{2i} = e^{2i[\ln 1 + i(\pi/2 + 2\pi n)]} = e^{-(1+4n)\pi}, \quad n = 0, \pm 1, \pm 2, \ldots$

  The principal value of i^{2i} for $n = 0$: $i^{2i} = e^{-\pi}$

(b) $(1+i)^i = e^{i[\frac{1}{2}\ln 2 + i(\pi/4 + 2\pi n)]}, \quad n = 0, \pm 1, \pm 2, \ldots$

The principal value of $(1+i)^i$ for $n = 0$: $(1+i)^i = e^{-\frac{\pi}{4} + i\frac{\ln 2}{2}}$

Complex powers satisfy the following properties

$$z^\alpha z^\beta = z^{\alpha+\beta}, \qquad \frac{z^\alpha}{z^\beta} = z^{\alpha-\beta}; \alpha, \beta \in C$$

$$(z^\alpha)^n = z^{n\alpha}; \alpha \in C, n \in Z$$

Analyticity

- The principal value of the complex power $z^\alpha = e^{\alpha \operatorname{Log} z}$ is differentiable and:

$$\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1}$$

- Example 7: Derivative of a Power Function

Find the derivative of the principal value z^i at the point $z = 1 + i$

$z = 1 + i$ is in the domain $|z| > 0$, $-\pi < \arg\ z \leq \pi$, $\frac{d}{dz} z^i = iz^{i-1}$

$$\left.\frac{d}{dz} z^i\right|_{z=1+i} = \left. iz^{i-1}\right|_{z=1+i} = i(1+i)^{i-1} = i(1+i)^i \frac{1}{1+i} = \frac{1+i}{2}(1+i)^i$$

the principal value of $(1 + i)^i$: $(1+i)^i = e^{-\pi/4 + i(\ln 2)/2}$

$$\left.\frac{d}{dz} z^i\right|_{z=1+i} = \frac{1+i}{2} e^{-\pi/4 + i(\ln 2)/2}$$

7. Trigonometric and Hyperbolic Functions

Trigonometric Functions

- **Definition:** For any complex number $z = x + iy$,

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

additional trigonometric functions

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{1}{\tan z},$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

$$\cos i = \frac{e^{-1} + e}{2}$$

Periodicity

- The complex exponential function e^z is periodic with a pure imaginary period of $2\pi i$.
- e^{iz} and e^{-iz} are periodic functions with real period 2π.
- So, the complex sine and cosine are periodic functions with a real period of 2π.

$$\sin(z+2\pi)=\sin z \text{ and } \cos(z+2\pi)=\cos z$$

- The complex tangent and cotangent are periodic with a real period of π.

$$\tan(z+\pi)=\tan z \text{ and } \cot(z+\pi)=\cot z$$

Analyticity

- Since the exponential functions e^{iz} and e^{-iz} are entire functions, it follows that $\sin z$ and $\cos z$ are entire functions.

- $\sin z = 0$ only for the real numbers $z = n\pi$, n an integer, and $\cos z = 0$ only for the real numbers $z = (2n + 1)\pi/2$, n an integer.
- Thus, $\tan z$ and $\sec z$ are analytic except at the points $z = (2n + 1)\pi/2$, and $\cot z$ and $\csc z$ are analytic except at the points $z = n\pi$.

Derivatives

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z$$

$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

Identities

$$\sin(-z) = -\sin z \qquad \cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin(2z) = 2\sin z \cos z \qquad \cos(2z) = \cos^2 z - \sin^2 z$$

Zeros

$$\sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \sin x \frac{e^{y} + e^{-y}}{2} + i\cos x \frac{e^{y} - e^{-y}}{2}$$

$$\sin z = \sin x \cosh y + i\cos x \sinh y$$

$$\cos z = \cos x \cosh y - i\sin x \sinh y$$

$$\cosh^2 y = 1 + \sinh^2 y \Rightarrow |\sin z|^2 = \sin^2 x + \sinh^2 y$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y = 0 \Rightarrow \begin{cases} \sin x = 0 \\ \sinh y = 0 \end{cases} \Rightarrow \begin{cases} x = n\pi \\ y = 0 \end{cases}$$

$$\sin z = 0 \Rightarrow z = n\pi,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

$$\cos z = 0 \Rightarrow z = (2n+1)\pi/2,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

- Note: $|\sin x| \le 1,\ |\cos x| \le 1$ do not hold for the complex sine and cosine.
- Example 8: Solving a Trigonometric Equation

Solve the equation $\cos z = 10$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = 10 \Rightarrow e^{2iz} - 20e^{iz} + 1 = 0 \Rightarrow e^{iz} = 10 \pm 3\sqrt{11}$$

$$iz = \ln(10 \pm 3\sqrt{11}) + 2\pi n i, \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

$$\ln(10 - 3\sqrt{11}) = -\ln(10 + 3\sqrt{11})$$

$$z = 2\pi n \pm i\ln(10 + 3\sqrt{11}), \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

Hyperbolic Functions

- **Definition:** For any complex number $z = x + iy$,

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{1}{\tanh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z}$$

Analyticity

- $\sinh z$ and $\cosh z$ are entire functions.
- $\tanh z$, $\coth z$, $\operatorname{sech} z$, and $\operatorname{csch} z$ are analytic except where the denominators are zero.

Derivatives

$$\frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

$\sinh(iz) = i\sin z$ and $\cosh(iz) = \cos z$.

$\sin z = -i\sinh(iz)$, $\cos z = \cosh(iz)$

$\sinh z = -i\sin(iz)$, $\cosh z = \cos(iz)$.

Zeros

$$\sinh z = \sinh x \cos y + i\cosh x \sin y$$

$$\cosh z = \cosh x \cos y + i\sinh x \sin y$$

$\sinh z = 0 \Rightarrow z = n\pi i,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$

$\cosh z = 0 \Rightarrow z = (2n+1)\pi i/2,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$

Periodicity

$\sin z$ and $\cos z$ are also periodic with the same real period 2π.

$\sinh z$ and $\cosh z$ have the imaginary period $2\pi i$.

8. Inverse Trigonometric and Hyperbolic Functions

Inverse Trigonometric Functions

The inverse multiple-valued sine function, $\sin^{-1} z$ or $\arcsin z$, is defined by:

$$w = \sin^{-1} z \text{ if } z = \sin w.$$

$$\frac{e^{iw} - e^{-iw}}{2i} = z \Rightarrow e^{2iw} - 2ize^{iw} - 1 = 0 \Rightarrow e^{iw} = iz + (1 - z^2)^{1/2}$$

$$\sin^{-1} z = -i\log[iz + (1 - z^2)^{1/2}]$$

$$\cos^{-1} z = -i\log[z + i(1 - z^2)^{1/2}]$$

$$\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$$

- Example 9: Values of an Inverse Sine

  Find all values of $\sin^{-1}\sqrt{5}$

  $$\sin^{-1}\sqrt{5} = -i\log[\sqrt{5}i + (1-5)^{1/2}] = -i\log[(\sqrt{5} \pm 2)i] \qquad \left((1-5)^{1/2} = \pm 2i\right)$$

  $$= -i[\ln(\sqrt{5} \pm 2) + (\pi/2 + 2\pi \mathrm{n})i],\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$

  $$\ln(\sqrt{5} - 2) = -\ln(\sqrt{5} + 2) \Rightarrow \sin^{-1}\sqrt{5} = \pi/2 + 2\pi \mathrm{n} \pm i\ln(\sqrt{5} + 2),\ n = 0, \pm 1, \pm 2, \ldots$$

  To obtain particular values of, $\sin^{-1} z$, we must choose a specific root of $1 - z^2$ and a specific branch of the logarithm. For example, if we choose $(-4)^{1/2} = 2i$ and the principal branch of the logarithm, then $\sin^{-1}\sqrt{5} = \pi/2 - i\ln(\sqrt{5} + 2)$

Derivatives

$$\frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}, \qquad \frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

Inverse Hyperbolic Functions

$$\sinh^{-1} z = \log[z + (z^2+1)^{1/2}] \qquad \frac{d}{dz}\sin^{-1} z = \frac{1}{(1-z^2)^{1/2}}$$

$$\cosh^{-1} z = \log[z + (z^2-1)^{1/2}] \qquad \frac{d}{dz}\cos^{-1} z = \frac{-1}{(1-z^2)^{1/2}}$$

$$\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z} \qquad \frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$$

- Example 10: Values of an Inverse Hyperbolic Cosine

Find all values of $\cosh^{-1}(-1)$

$$\cosh^{-1}(-1) = \log(-1) = \ln 1 + (\pi + 2\pi n)i = (2n+1)\pi i,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots$$