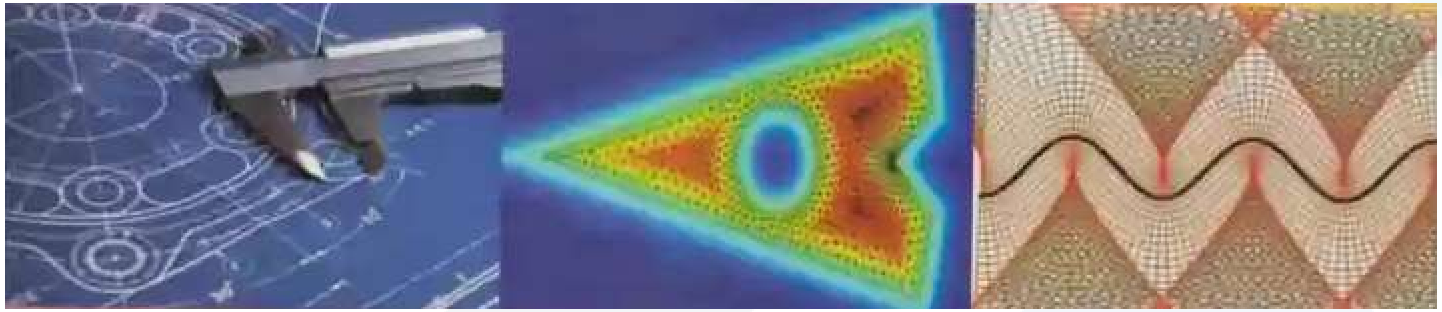
CEDC301: Engineering Mathematics

Lecture Notes 3: Integration in the Complex Plan



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Chapter 2

Integration in the Complex Plan

1. Contour Integrals
2. Cauchy-Goursat Theorem
3. Independence of the Path
4. Cauchy's Integral Formulas

1. Contour Integrals

A Definition

Suppose C is a curve parameterized by $x = x(t)$, $y = y(t)$, $a \le t \le b$, and A and B are the points $(x(a), y(a))$ and $(x(b), y(b))$, respectively. We say that:

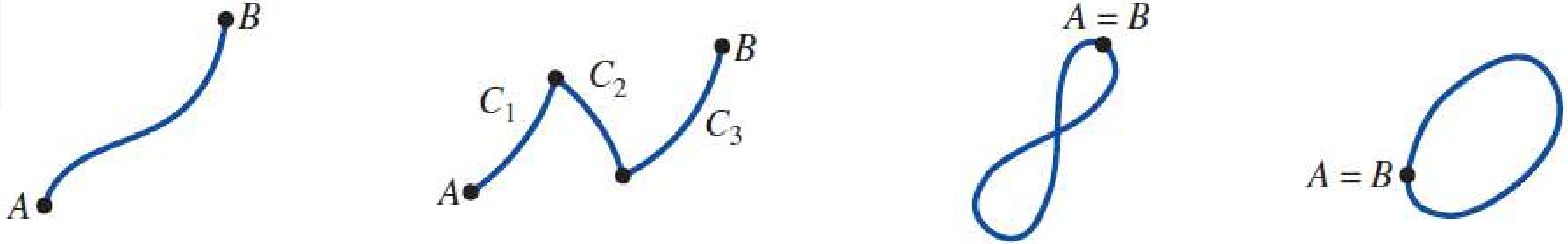
(i) C is a smooth curve if x' and y' are continuous on the closed interval $[a, b]$ and not simultaneously zero on the open interval (a, b).

(ii) C is piecewise smooth if it consists of a finite number of smooth curves $C_1, C_2, \dots, C_n$ joined end to end; that is, $C = C_1 \cup C_2 \cup \dots \cup C_n$.

(iii) C is a closed curve if $A = B$.

(iv) C is a simple closed curve if $A = B$ and the curve does not cross itself.

(v) If C is not a closed curve, then the positive direction on C is the direction corresponding to increasing values of t.



Smooth curve *Piecewise smooth curve* *Closed but not simple* *Simple closed curve*

A Definition

- Integral of a complex function $f(z)$ that is defined along a curve C in the complex plane. Let C be defined by the parametric equations $x = x(t)$, $y = y(t)$, $a \leq t \leq b$, where t is a real parameter.
- By using $x(t)$ and $y(t)$ as real and imaginary parts, we can also describe a curve C in the complex plane by means of a complex-valued function of a real variable t: $z(t) = x(t) + iy(t)$, $a \leq t \leq b$.

- For example, $x = \cos\ t$, $y = \sin\ t$, $0 \le t \le 2\pi$, describes a unit circle centered at the origin. This circle can also be described by $z(t) = \cos t + i \sin t$, or even more compactly by $z(t) = e^{it}$, $0 \le t \le 2\pi$.
- In complex variables, a piecewise-smooth curve C is also called a contour or path.
- An integral of $f(z)$ on C is denoted by $\int_C f(z)dz$ or $\oint_C f(z)dz$ if the contour C is closed; it is referred to as a contour integral or simply as a complex integral.

1. Let $f(z) = u(x,\ y) + iv(x,\ y)$ be defined at all points on a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \le t \le b$.

2. Divide C into n subarcs according to the partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a,\ b]$. The corresponding points on the curve C are:

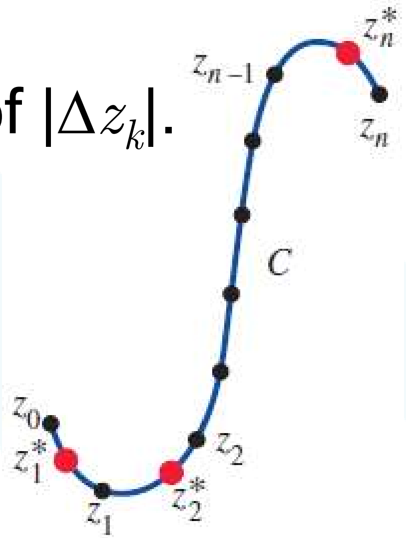
$z_0 = x_0 + iy_0 = x(t_0) + iy(t_0)$, $z_1 = x_1 + iy_1 = x(t_1) + iy(t_1)$, ..., $z_n = x_n + iy_n = x(t_n) + iy(t_n)$. Let $\Delta z_k = z_k - z_{k-1}$, $k = 1, 2, ..., n$.

3. Let $\|P\|$ be the norm of the partition, i.e., the maximum value of $|\Delta z_k|$.

4. Choose a sample point $z_k^* = x_k^* + iy_k^*$ on each subarc.

5. Form the sum:

$$\sum_{k=1}^{n} f(z_k^*)\Delta z_k$$



- **Definition:** Let f be defined at points of a smooth curve C defined by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$. The contour integral of f along C is

$$\int_C f(z)\,dz = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(z_k^*)\Delta z_k$$

The limit exists if f is continuous at all points on C and C is either smooth or piecewise smooth.

- **Theorem 1 (Evaluation of a Contour Integral):** If f is continuous on a smooth curve C given by $z(t) = x(t) + iy(t)$, $a \leq t \leq b$, then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

- **Example 1:** Evaluating a Contour Integral

Evaluate $\int_C \bar{z}\, dz$, where C is given by $x(t) = 3t$, $y(t) = t^2$, $-1 \leq t \leq 4$

$$\int_C \bar{z}\, dz = \int_{-1}^{4} (3t - it^2)(3 + 2it)dt$$

$$= \int_{-1}^{4} (2t^3 + 9t)dt + i\int_{-1}^{4} 3t^2 dt = 195 + 65i$$

- Example 2: Evaluating a Contour Integral

Evaluate $\oint_C \frac{1}{z}dz$, where C is the circle $x(t) = \cos\, t$, $y(t) = \sin\, t$, $0 \leq t \leq 2\pi$

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi} (e^{-it})ie^{it}dt = i\int_0^{2\pi} dt = 2\pi i$$

Properties

- Theorem 2 (Properties of Contour Integrals): Suppose f and g are continuous in a domain D and C, C_1 and C_2 are smooth curves lying entirely in D. Then

$$(i)\quad \int_C kf(z)dz = k\int_C f(z)dz,\ k \text{ a constant}$$

$$(ii)\ \int_C [f(z)+g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

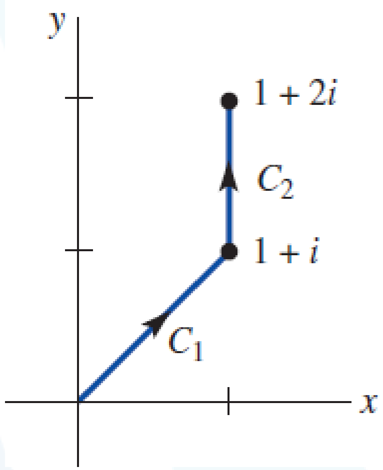
$$(iii)\ \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz,\ C = C_1 \cup C_2$$

$$(iv)\ \int_{-C} f(z)dz = -\int_C f(z)dz$$

where $-C$ denotes the curve having the opposite orientation of C

- Note: Theorem 2 also hold when C is a piecewise-smooth curve in D.
- Example 3: Evaluating a Contour Integral

Evaluate $\int_C (x^2 + iy^2)\, dz$, where C is the contour shown below



$$\int_C (x^2 + iy^2)\, dz = \int_{C_1} (x^2 + iy^2)\, dz + \int_{C_2} (x^2 + iy^2)\, dz$$

The curve C_1 is defined by $x(t) = y(t) = t$, $0 \le t \le 1$

The curve C_2 is defined by $x(t) = 1$, $y(t) = t$, $1 \le t \le 2$

$$\int_{C_1} (x^2 + iy^2)\, dz = \int_0^1 (t^2 + it^2)(1+i)\, dt = (1+i)^2 \int_0^1 t^2 dt = \frac{2}{3} i$$

$$\int_{C_2} (x^2 + iy^2)\, dz = \int_1^2 (1 + it^2) i dt = -\frac{7}{3} + i$$

$$\int_{C} (x^2 + iy^2)\, dz = \frac{2}{3} i - \frac{7}{3} + i = -\frac{7}{3} + \frac{5}{3} i$$

- Theorem 3 (A Bounding Theorem): If f is continuous on a smooth curve C and if $|f(z)| < M$ for all z on C, then $\left|\int_C f(z) dz\right| \leq ML$, where L is the length of C.

- Example 4: A Bound for a Contour Integral

  Find an upper bound for the absolute value of $\oint_C \frac{e^z}{z+1} dz$, where C is the circle $|z| = 4$.

The length s of the circle of radius 4 is 8π. $|z+1| \geq |z| - 1 = 4 - 1 = 3$,

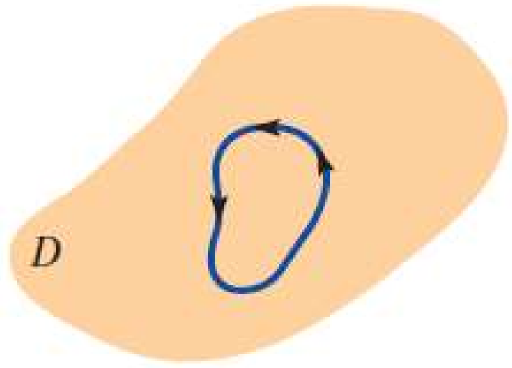
$$\left|\frac{e^z}{z+1}\right| \leq \frac{|e^z|}{|z|-1} = \frac{|e^z|}{3} = \frac{e^x}{3} \leq \frac{e^4}{3} \Rightarrow \left|\frac{e^z}{z+1}\right| \leq \frac{e^4}{3} \Rightarrow \left|\oint_C \frac{e^z}{z+1}dz\right| \leq \frac{8\pi e^4}{3}$$

2. Cauchy-Goursat Theorem

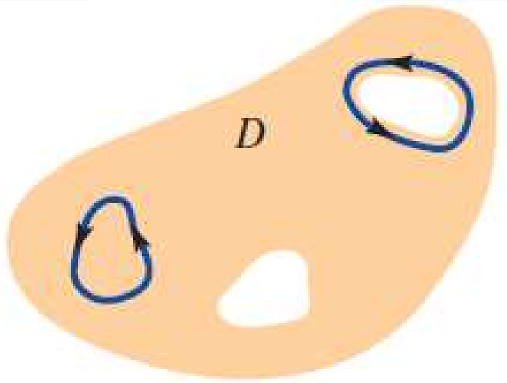
Simply and Multiply Connected Domains

- A domain D is said to be simply connected if every simple closed contour C lying entirely in D can be shrunk to a point without leaving D.
- In other words, in a simply connected domain, every simple closed contour C lying entirely within it encloses only points of the domain D.
- A simply connected domain has no "holes" in it.

- The entire complex plane is an example of a simply connected domain.
- A domain that is not simply connected is called a multiply connected domain; that is, a multiply connected domain has “holes” in it.
- We call a domain with one “hole” doubly connected, a domain with two “holes” triply connected, and so on.



Simply connected domain



Multiply connected domain

Cauchy's Theorem

Suppose that a function f is analytic in a simply connected domain D and that f is continuous in D. Then for every simple closed contour C in D, $\oint_C f(z)dz = 0$

- Theorem 4 (Cauchy-Goursat Theorem): Suppose a function f is analytic in a simply connected domain D. Then for every simple closed contour C in D,

$$\oint_C f(z)dz = 0$$

- Example 5: The functions z^n with n a positive integer, $\sin z$, $\cos z$, e^z, $\sinh z$, and $\cosh z$ are analytic (they are entire functions), so for any closed contour C in the complex plane,

$$\oint_C z^n dz = \oint_C \sin z dz = \oint_C \cos z dz = \oint_C e^z dz = \oint_C \sinh z dz = \oint_C \cosh z dz = 0$$

- Example 6: Applying the Cauchy-Goursat Theorem

Evaluate $\oint_C \frac{1}{z^2}\,dz$, where C is the ellipse $(x-2)^2 + \frac{(y-5)^2}{4} = 1$

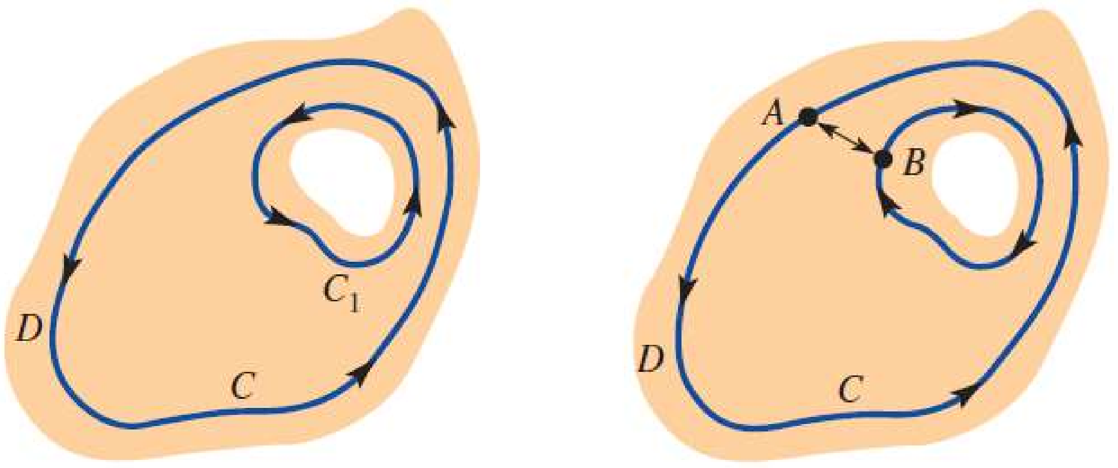
The rational function $f(z) = 1/z^2$ is analytic everywhere except at $z = 0$. But $z = 0$ is not a point interior to or on the contour C. Thus,

$$\oint_C \frac{1}{z^2}\,dz = 0$$

Cauchy-Goursat Theorem for Multiply Connected Domains

suppose D is a doubly connected domain and C and C_1 are simple closed contours such that C_1 surrounds the “hole” in the domain and is interior to C. Suppose, also, that f is analytic on each contour and at each point interior to C but exterior to C_1.

When we introduce the cut AB the region bounded by the curves is simply connected.



The integral from A to B has the opposite value of the integral from B to A, so

$$\oint_C f(z)dz + \int_{AB} f(z)dz + \int_{-AB} f(z)dz + \oint_{C_1} f(z)dz = 0 \Rightarrow \oint_C f(z)dz = \oint_{C_1} f(z)dz$$

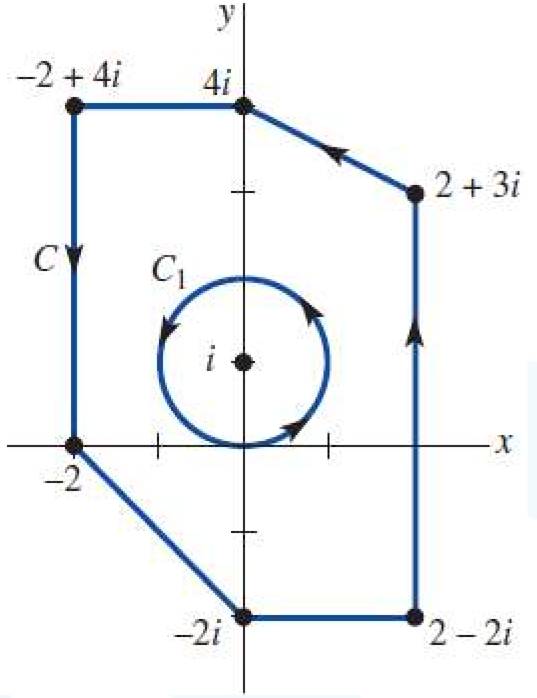
This result is sometimes called the principle of deformation of contours.

- Example 7: Applying Deformation of Contours

  Evaluate $\oint_C \frac{1}{z-i}\,dz$, where C is the outer contour shown

  We choose the more convenient circular contour C_1. By taking $r = 1$, we are guaranteed that C_1 lies within C. C_1 is the circle $|z - i| = 1$, which can be parameterized by $x = \cos t$, $y = 1 + \sin t$, or by $z = i + e^{it}$, $0 \le t \le 2\pi$.

$$\oint_C \frac{1}{z-i}\,dz = \oint_{C_1} \frac{1}{z-i}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i$$



- If z_0 is any constant complex number interior to any simple closed contour C, then

$$\oint_C \frac{dz}{(z-z_0)^n} = \begin{cases} 2\pi i, & n = 1 \\ 0, & n \text{ an integer} \neq 1 \end{cases}$$

- Example 8: Applying Deformation of Contours

Evaluate $\oint_C \frac{5z+7}{z^2+2z-3}\,dz$, where C is the circle $|z-2|=2$

Since the denominator factors as $z^2+2z-3=(z-1)(z+3)$, the integrand fails to be analytic at $z=1$ and $z=-3$. Only $z=1$ lies within the contour C, which is a circle centered at $z=2$ of radius $r=2$.

$$\frac{5z+7}{z^2+2z-3}=\frac{3}{z-1}+\frac{2}{z+3}\Rightarrow \oint_C \frac{5z+7}{z^2+2z-3}\,dz=3\oint_C \frac{dz}{z-1}+2\oint_C \frac{dz}{z+3}$$

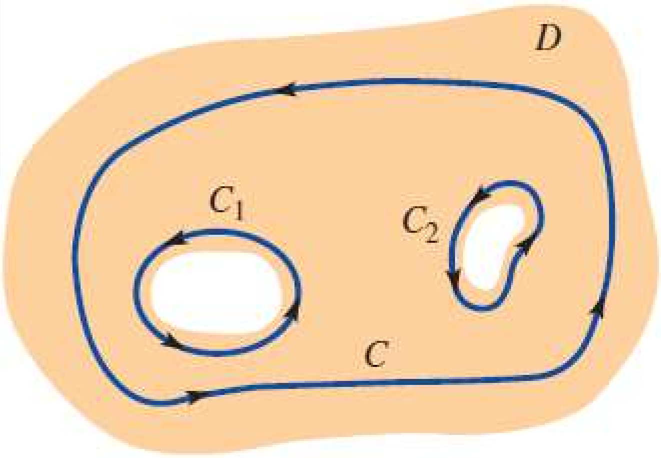
$$\oint_C \frac{5z+7}{z^2+2z-3}\,dz=3(2\pi i)+2(0)=6\pi i$$

- Theorem 5 (Cauchy-Goursat Theorem for Multiply Connected Domains): Suppose C, C_1, ..., C_n are simple closed curves with a positive orientation such that C_1, C_2, ..., C_n are interior to C but the regions interior to each C_k, $k =$ 1, 2, ..., n, have no points in common. If f is analytic on each contour and at each point interior to C but exterior to all the C_k, $k =$ 1, 2, ..., n, then

$$\oint_C f(z)dz = \sum_{k=1}^{n} \oint_{C_k} f(z)dz$$

For example: triply connected domain D,

$$\oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$$



- Note: Cauchy-Goursat theorem is valid for any closed contour C in a simply connected domain D.

- Example 9: Applying Cauchy-Goursat Theorem for triply Connected domain

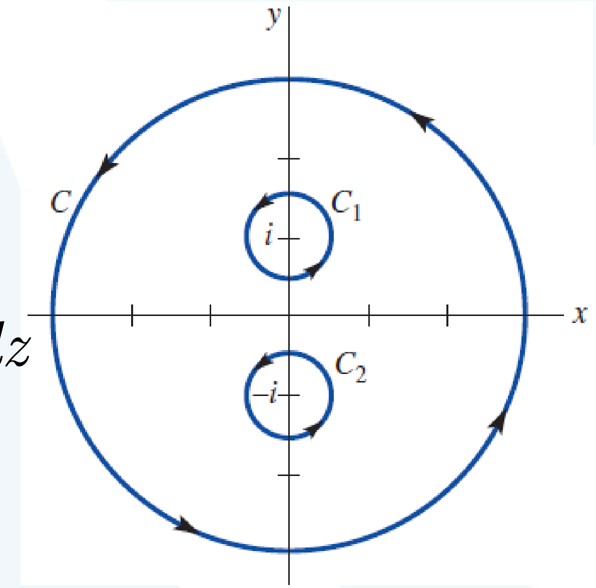
Evaluate $\oint_C \frac{dz}{z^2+1}$, where C is the circle $|z| = 3$

$z^2 + 1 = (z-i)(z+i)$, the integrand fails to be analytic at $z = i$ and $z = -i$. Both of these points lie within the contour C.

$$\frac{1}{z^2+1} = \frac{1/2i}{z-i} - \frac{1/2i}{z+i} \Rightarrow \oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_C \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\oint_{C_1} \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz + \frac{1}{2i}\oint_{C_2} \left[\frac{1}{z-i} - \frac{1}{z+i}\right] dz$$

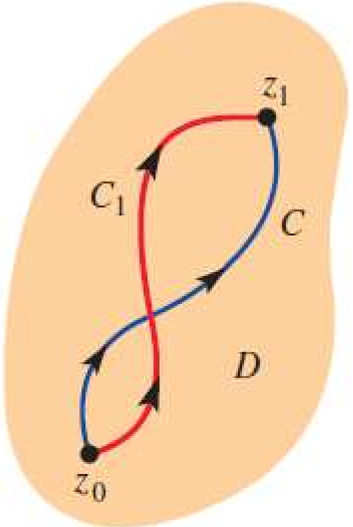
$$\oint_C \frac{dz}{z^2+1} = \frac{1}{2i}\left[(2\pi i) - (0)\right] + \frac{1}{2i}\left[(0) - (2\pi i)\right] = \pi - \pi = 0$$



3. Independence of the Path

- **Definition:** Let z_0 and z_1 be points in a domain D. A contour integral $\int_C f(z)dz$ is said to be **independent of the path** if its value is the same for all contours C in D with an initial point z_0 and a terminal point z_1.

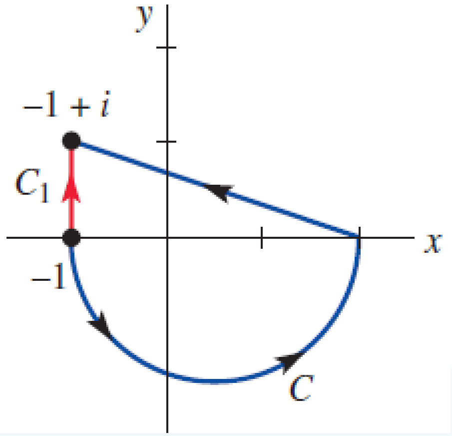
  Suppose, that C and C_1 are two contours in a simply connected domain D, both with initial point z_0 and terminal point $z1$. Note that C and $-C_1$ form a closed contour. Thus, if f is analytic in D, it follows from the Cauchy-Goursat theorem that



$$\oint_C f(z)dz + \oint_{-C_1} f(z)dz = 0 \Rightarrow \oint_C f(z)dz = \oint_{C_1} f(z)dz$$

- **Theorem 6 (Analyticity Implies Path Independence):** If f is an analytic function in a simply connected domain D, then $\int_C f(z)dz$ is independent of the path C.

- **Example 10:** Choosing a Different Path



Evaluate $\int_C 2z\,dz$, where C is the contour with initial point $z = -1$ and terminal point $z = 1 + i$ shown below

The function $f(z) = 2z$ is entire, we can replace the path C C_1 joining $z = -1$ and $z = -1 + i$. In particular, by choosing C_1 to be the straight line segment $x = -1$, $y = t$, $0 \leq t \leq 1$. $z = -1 + it$

$$\int_C 2z\,dz = \int_0^1 2(-1+it)i\,dt = -2i\int_0^1 dt - 2\int_0^1 t\,dt = -1 - 2i$$

- **Definition:** Suppose f is continuous in a domain D. If there exists a function F such that $F'(z) = f(z)$ for each z in D, then F is called an antiderivative of f.

For example, the function $F(z) = -\cos z$ is an antiderivative of $f(z) = \sin z$.

Antiderivative, or **indefinite integral**, of a function $f(z)$ is written

$$\int f(z)dz = F(z) + C$$

where $F'(z) = f(z)$ and C is some complex constant.

- **Theorem 7 (Fundamental Theorem for Contour Integrals):** Suppose f is continuous in a domain D and F is an antiderivative of f in D. Then for any contour C in D with initial point z_0 and terminal point z_1,

$$\int_C f(z)dz = F(z_1) - F(z_0)$$

- **Example 11:** Using an Antiderivative

$$\int_C 2z dz = \int_{-1}^{-1+i} 2z dz = z^2 \Big]_{-1}^{-1+i} = -1 - 2i$$

- Example 12: Using an Antiderivative

  Evaluate $\int_C \cos z dz$, where C is any contour with initial point $z = 0$ and terminal point $z = 2 + i$.

$$\int_C \cos zdz = \int_0^{2+i} \cos zdz = \sin z\Big]_0^{2+i} = \sin(2+i)$$

- If a continuous function f has an antiderivative F in D, then $\int_C f(z)dz$ is independent of the path.
- If f is continuous and $\int_C f(z)dz$ is independent of the path in a domain D, then f has an antiderivative everywhere in D.
- Theorem 8 (Existence of an Antiderivative): If f is analytic in a simply connected domain D, then f has an antiderivative in D; that is, there exists a function F such that $F'(z) = f(z)$ for all z in D.

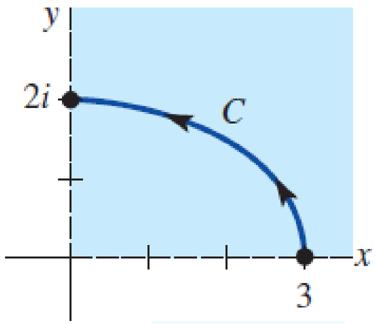
- Note: under some circumstances $\text{Log } z$ is an antiderivative of $1/z$. For example, suppose D is the entire complex plane without the origin. The function $1/z$ is analytic in this multiply connected domain.

- If C is any simple closed contour containing origin, $\oint_C (1/z)dz = 2\pi i \neq 0$. In this case, $\text{Log } z$ is not an antiderivative of $1/z$ in D, since $\text{Log } z$ is not analytic in D.

- Example 13: Using the Logarithmic Function

  Evaluate $\int_C \frac{dz}{z}$, where C is the contour shown below

  Suppose that D is the simply connected domain defined by $x = Re(z) > 0$, $y = Im(z) > 0$. In this case, $\text{Log } z$ is an antiderivative of $1/z$, since both these functions are analytic in D.



$$\int_C \frac{dz}{z} = \int_3^{2i} \frac{1}{z}\,dz = \operatorname{Log} z\Big]_3^{2i} = \operatorname{Log} 2i - \operatorname{Log} 2 = \operatorname{Ln} 2 + \frac{\pi}{2}i - \operatorname{Ln} 3 = \operatorname{Ln} \frac{2}{3} + \frac{\pi}{2}i$$

4. Cauchy's Integral Formulas

- The value of an analytic function f at any point z_0 in a simply connected domain can be represented by a contour integral.
- An analytic function f in a simply connected domain possesses derivatives of all orders.
- **Theorem 9 (Cauchy's Integral Formula):** Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz$$

- Example 14: Using Cauchy's Integral Formula

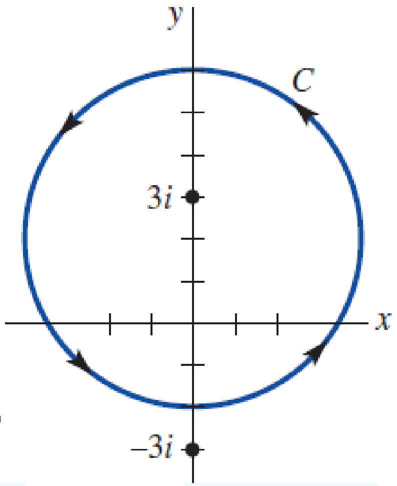
  Evaluate $\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz$, where C is the circle $|z| = 2$

  $f(z) = z^2 - 4z + 4$ and $z_0 = -i$ as a point within the circle C. f is analytic at all points within and on the contour C.

$$\oint_C \frac{z^2 - 4z + 4}{z + i}\, dz = 2\pi i f(-i) = 2\pi i(3 + 4i) = 2\pi(-4 + 3i)$$

- Example 15: Using Cauchy's Integral Formula

  Evaluate $\oint_C \frac{z}{z^2 + 9}\, dz$, where C is the circle $|z - 2i| = 4$



$$\frac{z}{z^2 + 9} = \frac{z/(z + 3i)}{z - 3i}$$

$z_0 = 3i$ is the only point within the circle C.

$f(z) = z/(z - 3i)$. This function is analytic at all points within and on the contour C.

$$\oint_C \frac{z}{z^2+9}\,dz = 2\pi i f(3i) = 2\pi i \frac{3i}{6i} = \pi i$$

- Theorem 10 (Cauchy's Integral Formula for Derivatives): Let f be analytic in a simply connected domain D, and let C be a simple closed contour lying entirely within D. If z_0 is any point within C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

- Example 16: Using Cauchy's Integral Formula for Derivatives

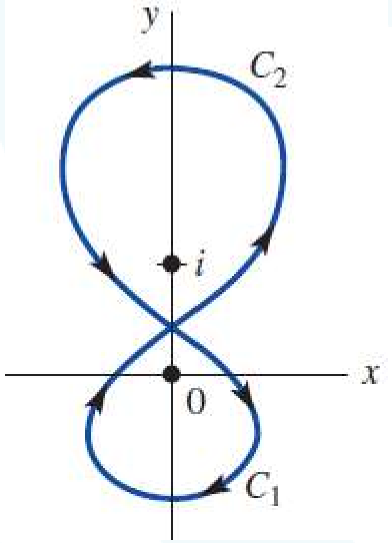
Evaluate $\oint_C \frac{z+1}{z^4+4z^3}\,dz$, where C is the circle $|z| = 1$

The integrand is not analytic at $z = 0$ and $z = -4$, but only $z = 0$ lies within the closed contour.

$$\frac{z+1}{z^4+4z^3} = \frac{(z+1)/(z+4)}{z^3} \Rightarrow \oint_C \frac{z+1}{z^4+4z^3}\,dz = \frac{2\pi i}{2!} f''(0) = \frac{3\pi}{32} i$$

- Example 17: Using Cauchy's Integral Formula for Derivatives

Evaluate $\oint_C \frac{z^3+3}{z(z-i)^2}\,dz$, where C is the contour shown below



C is not a simple closed contour, we can think of it as the union of two simple closed contours C_1 and C_2

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = \oint_{C_1} \frac{z^3+3}{z(z-i)^2}\,dz + \oint_{C_2} \frac{z^3+3}{z(z-i)^2}\,dz$$

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = -\oint_{C_1} \frac{\dfrac{z^3+3}{(z-i)^2}}{z}\,dz + \oint_{C_2} \frac{\dfrac{z^3+3}{z}}{(z-i)^2}\,dz = -I_1 + I_2$$

$$I_1 = \oint_{C_1} \frac{\dfrac{z^3+3}{(z-i)^2}}{z}\,dz = 2\pi i f(0) = -6\pi i$$

$$I_2 = \oint_{C_2} \frac{\dfrac{z^3+3}{z}}{(z-i)^2}\,dz = \frac{2\pi i}{1!} f'(i) = 2\pi i(3+2i) = 2\pi(-2+3i)$$

$$\oint_C \frac{z^3+3}{z(z-i)^2}\,dz = -I_1 + I_2 = 6\pi i + 2\pi(-2+3i) = 4\pi(-1+3i)$$