Problem sets 3,4 : Vector spaces

CEDC102: Linear Algebra and Matrix Theory

Manara University

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Problem 1. Give examples of matrices A for which the number of solutions to Ax=b is

(a) 0 or 1, depending on b.
(b) ∞, regardless of b
(c) 0 or ∞, depending on b
(d) 1, regardless of b.

Problem 2. Consider

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 5 & -3 \\ 1 & 2 & 7 & -7 \end{bmatrix}$$

- Find the rank(A)

- Find the basis of $N(A)$.

- Give a permutation matrix P such that doing Gaussian elimination on AP (which re-orders the _______ of A) leads to the first two columns being the pivot columns

- How do the nullspace $N(AP)$ and column space $C(AP)$ relate to the null and column spaces of A, respectively?

- How we can get basis for N(AP) from the basis of $N(A)$?

Problem 3 Consider the equation $ABx = b$, given by:

$$\underbrace{\begin{pmatrix} 2 & 1 & 2 \\ & -1 & 2 \\ & & 1 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} -1 & 1 & 2 & 1 \\ 3 & -2 & -5 & -1 \\ 0 & 1 & 1 & 2 \end{pmatrix}}_{B} x = \underbrace{\begin{pmatrix} 17 \\ 19 \\ 7 \end{pmatrix}}_{b}$$

1. Give **bases for the nullspaces** $N(A), N(B)$, and $N(AB)$. (Without doing the multiblication)

2. Give the **complete solution** x to $ABx = b$, and show how you can do this without ever multiplying A and B together to explicitly form the matrix AB.

Problem 4

1. Compute the reduced row echelon form of the matrix:

$$\begin{bmatrix} 0 & -1 & 2 & 1 \\ 1 & 2 & -3 & 0 \\ 1 & 3 & -5 & -1 \end{bmatrix}$$

2. Use the result of part (1) to find the full set of solutions to the equation

$$A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

Problem 5 justify your answer

1) If X is an invertible square matrix, what can you say about $C(X)$ and $N(X)$?

2) If $Y = \left[\frac{A}{B}\right]$ is a block matrix, what is $N(Y)$ in terms of $N(A)$ and $N(B)$?

3) If $Z = [A|B]$ is a block matrix, what is $C(Z)$ in terms of $C(A)$ and $C(B)$?

Problem 6

Consider a matrix A such that the general solution to the equation:

$$A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \qquad \text{is} \qquad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

for arbitrary numbers λ and μ.

1. How many rows and columns does A have?
2. Based on the information in the equation above, what is the second column of A?
3. Find the entire matrix A.

Problem 7

Find a basis for the vector space spanned by the vectors:

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 6 \\ -3 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -2 \\ 3 \\ -5 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -5 \\ -8 \\ 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ -1 \\ 9 \\ -4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 3 \\ 8 \\ -5 \end{bmatrix}$$

Explain your method.

Problem 8

1. Suppose that A is a 3×3 matrix. What relation is there between the nullspace of A and the nullspace of A^2? How about the nullspace of A^3?

2. The set of polynomials of degree at most four in the variable x is a vector space. What is the nullspace of $\frac{d^2}{dx^2}$? What is the nullspace of $\left(\frac{d^2}{dx^2}\right)^2$?

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Problem 9 TRUE or FALSE?

(a) Every upper-triangular matrix is in reduced row echelon form?

(b) Every lower-triangular matrix is in reduced row echelon form?

(c) Every permutation matrix is in reduced row echelon form?

(d) The following matrix is in reduced row echelon form?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(e) The reduced row echelon form of A is unique?

(f) The full solution set of $Ax = b$, where A is $m \times n$ and $b \in \mathbb{R}^m$, is always a vector subspace of $\mathbb{R}^n$?

(g) The difference $a = x_1 - x_2$, between any two solutions x_1 and x_2 to $Ax = b$, is a vector that belongs to the null space $N(A)$?