Exercises 6: Inner Product Spaces

CEDC102 : Linear Algebra and Matrix Theory

Manara University

2023-2024

Find (a) $\|\boldsymbol{u}\|$, (b) $\|\boldsymbol{v}\|$, and (c) $\|\boldsymbol{u}+\boldsymbol{v}\|$

1. $\boldsymbol{u} = (3,\ 1,\ 3),\ \boldsymbol{v} = (0,\ -1,\ 1)$

$(a)\ \|\boldsymbol{u}\| = \sqrt{3^2+1^2+3^2} = \sqrt{19}$ $\qquad (b)\ \|\boldsymbol{v}\| = \sqrt{0^2+(-1)^2+1^2} = \sqrt{2}$

$$(c)\ \|\boldsymbol{u}+\boldsymbol{v}\| = \|(3-0, 1-(-1), 3-1)\| = \|(3,2,2)\| = \sqrt{3^2+2^2+2^2} = \sqrt{17}$$

Find a unit vector (a) in the direction of $\boldsymbol{u}$ and (b) in the direction opposite that of $\boldsymbol{u}$

1. $\boldsymbol{u} = (3,\ 2,\ -5),\ \boldsymbol{v} = (0,\ -1,\ 1)$

(a) A unit vector $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$ is given by

$$\boldsymbol{v} = \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} = \frac{1}{\sqrt{3^2+2^2+(-5)^2}}(3,2,-5) = \frac{1}{\sqrt{38}}(3,2,-5) = \left(\frac{3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{-5}{\sqrt{38}}\right)$$

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(b) A unit vector in the direction opposite that of $\boldsymbol{u}$ is given by

$$-\boldsymbol{v} = -\left(\frac{3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{-5}{\sqrt{38}}\right) = \left(\frac{-3}{\sqrt{38}}, \frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}}\right)$$

Find the vector $\boldsymbol{v}$ with the given length and the same direction as $\boldsymbol{u}$

① $\|\boldsymbol{v}\| = 5,\ \boldsymbol{u} = (\sqrt{5}, 5, 0)$

First find a unit vector in the direction of $\boldsymbol{u}$

$$\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} = \frac{1}{\sqrt{5+25}}(\sqrt{5}, 5, 0) = \frac{1}{\sqrt{30}}(\sqrt{5}, 5, 0) = \left(\frac{1}{\sqrt{6}}, \frac{\sqrt{5}}{\sqrt{6}}, 0\right)$$

Then $\boldsymbol{v}$ is five times this vector $\boldsymbol{v} = 5\left(\frac{1}{\sqrt{6}}, \frac{\sqrt{5}}{\sqrt{6}}, 0\right) = \left(\frac{5}{\sqrt{6}}, \frac{5\sqrt{5}}{\sqrt{6}}, 0\right)$

For what values of c is $\|c(1,2,3)\| = 1$?

$$\|c(1,2,3)\| = |c|\|(1,2,3)\| = 1 \Rightarrow |c| = \frac{1}{\|(1,2,3)\|} = \frac{1}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}}$$

$$\Rightarrow c = \pm\frac{1}{\sqrt{14}}$$

Find (a) $\boldsymbol{u}\cdot\boldsymbol{v}$, (b) $\boldsymbol{v}\cdot\boldsymbol{v}$, (c) $\|\boldsymbol{v}\|^2$, (d) d($\boldsymbol{u},\boldsymbol{v}$), (e) $(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}$, and (f) $\boldsymbol{u}\cdot(5\boldsymbol{v})$

(1) $\boldsymbol{u} = (3, 4),\ \boldsymbol{v} = (2, -3)$

(a) $\boldsymbol{u}.\boldsymbol{v} = 3(2) + 4(-3) = -6$

(b) $\boldsymbol{v}.\boldsymbol{v} = 2(2) + (-3)(-3) = 13$

(c) $\|\boldsymbol{u}\|^2 = \boldsymbol{u}.\boldsymbol{u} = 3^2 + 4^2 = 25$

(d) $d(\boldsymbol{u},\boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\| = \|(3-2, 4-(-3))\| = \|(1,7)\| = \sqrt{1^2+7^2} = \sqrt{50} = 5\sqrt{2}$

(e) $(\boldsymbol{u}.\boldsymbol{v})\boldsymbol{v} = -6\boldsymbol{v} = (-12, 18)$

(f) $\boldsymbol{u}(5\boldsymbol{v}) = 5\boldsymbol{u}.\boldsymbol{v} = 5(-6) = -30$

② $\boldsymbol{u} = (2, -2, 1),\ \boldsymbol{v} = (2, -1, -6)$

(a) $\boldsymbol{u}.\boldsymbol{v} = 2(2) + (-2)(-1) + 1(-6) = 0$

(b) $\boldsymbol{v}.\boldsymbol{v} = 2(2) + (-1)(-1) + (-6)(-6) = 41$

(c) $\|\boldsymbol{u}\|^2 = \boldsymbol{u}.\boldsymbol{u} = 2^2 + (-2)^2 + 1^2 = 9$

(d) $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \|(2-2, -2-(-1), 1-(-6))\| = \|(0, -1, 7)\|$

$$= \sqrt{0^2 + (-1)^2 + 7^2} = \sqrt{50}$$

(e) $(\boldsymbol{u}.\boldsymbol{v})\boldsymbol{v} = 0\boldsymbol{v} = \boldsymbol{0} = (0, 0, 0)$

(f) $\boldsymbol{u}(5\boldsymbol{v}) = 5\boldsymbol{u}.\boldsymbol{v} = 0$

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Find $(3\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-3\boldsymbol{v})$ when $\boldsymbol{u}\cdot\boldsymbol{u}=8$, $\boldsymbol{u}\cdot\boldsymbol{v}=7$, and $\boldsymbol{v}\cdot\boldsymbol{v}=6$

$$(3\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-3\boldsymbol{v}) = 3\boldsymbol{u}.\boldsymbol{u}-9\boldsymbol{u}.\boldsymbol{v}-\boldsymbol{v}.\boldsymbol{u}+3\boldsymbol{v}.\boldsymbol{v}=3\boldsymbol{u}.\boldsymbol{u}-10\boldsymbol{u}.\boldsymbol{v}+3\boldsymbol{v}.\boldsymbol{v}$$

$$= 3(8) - 10(7) + 3(6) = -28$$

Find the angle θ between the vectors

① $\boldsymbol{u}=(1,-1),\ \boldsymbol{v}=(0,1)$

$$\cos\theta=\frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}=\frac{1(0)+(-1)(1)}{\sqrt{1^2+(-1)^2}\sqrt{0^2+(-1)^2}}=\frac{-1}{\sqrt{2}}\Rightarrow\theta=\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)=\frac{3\pi}{4}\text{ rad }(135^\circ)$$

② $\boldsymbol{u}=(\cos\frac{\pi}{6},\sin\frac{\pi}{6}),\ \boldsymbol{v}=(\cos\frac{3\pi}{4},\sin\frac{3\pi}{4})$

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$$\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{\cos\frac{\pi}{6}\cos\frac{3\pi}{4} + \sin\frac{\pi}{6}\sin\frac{3\pi}{4}}{\sqrt{\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}}\sqrt{\cos^2\frac{3\pi}{4} + \sin^2\frac{3\pi}{4}}} = \frac{\cos\left(\frac{\pi}{6} - \frac{3\pi}{4}\right)}{1\times 1}$$

$$= \cos\left(-\frac{7\pi}{12}\right) = \cos\left(\frac{7\pi}{12}\right) \Rightarrow \theta = \frac{7\pi}{12} \text{ radians } (105^\circ)$$

③ $\boldsymbol{u} = (1, 1, 1), \ \boldsymbol{v} = (2, 1, -1)$

$$\cos\theta = \frac{\boldsymbol{u}.\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{1(2) + 1(1) + 1(-1)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{2^2 + 1^2 + (-1)^2}} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\Rightarrow \theta = \cos^{-1}(\sqrt{2}/3) \approx 1.08 \text{ radians } (62^\circ)$$

Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, parallel, or neither

① $\boldsymbol{u} = (1, -1, 1),\ \boldsymbol{v} = (2, 1, -1)$

$\boldsymbol{u}.\boldsymbol{v} = 1(2) + (-1)(1) + 1(-1) = 0$, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

② $\boldsymbol{u} = (1, -1, 1),\ \boldsymbol{v} = (2, 1, 1)$

$\boldsymbol{u}.\boldsymbol{v} = 1(2)+(-1)(1)+1(1)=2\neq 0$, $\Rightarrow$ $\boldsymbol{u}$ and $\boldsymbol{v}$ are not orthogonal. Moreover, because one is not a scalar multiple of the other, they are not parallel

③ $\boldsymbol{u} = (1, -2, 4),\ \boldsymbol{v} = (-3, 6, -12)$

$\boldsymbol{u} = -3\boldsymbol{v}$, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel

④ $\boldsymbol{u} = (\cos\theta, \sin\theta, -1),\ \boldsymbol{v} = (\sin\theta, -\cos\theta, 0)$

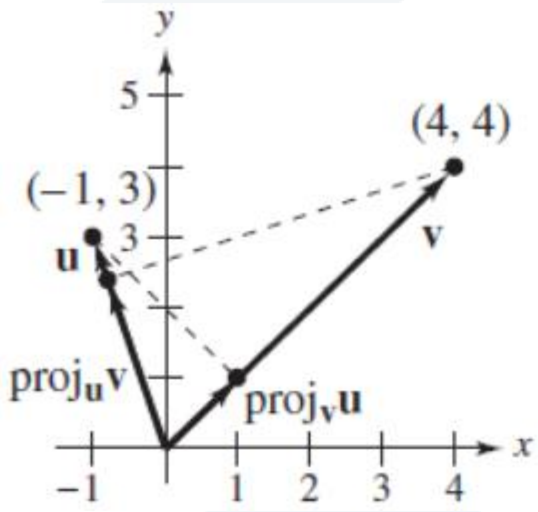
$\boldsymbol{u}.\boldsymbol{v} = \cos\theta\,(\sin\theta) + (\sin\theta)(-\cos\theta) + (-1)(0) = 0$, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal

Find (a) $\text{proj}_{\boldsymbol{v}}\boldsymbol{u}$ and (b) $\text{proj}_{\boldsymbol{u}}\boldsymbol{v}$

1. $\boldsymbol{u} = (-1, 3), \ \boldsymbol{v} = (4, 4)$

$$(a)\ \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}\boldsymbol{v} = \frac{-1(4)+3(4)}{4^2+4^2}(4,4) = \frac{1}{4}(4,4) = (1,1)$$

$$(b)\ \text{proj}_{\boldsymbol{u}}\boldsymbol{v} = \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u} = \frac{4(-1)+4(3)}{(-1)^2+3^2}(-1,3) = \frac{4}{5}(-1,3) = (-\frac{4}{5},\frac{12}{5})$$



② $\boldsymbol{u} = (1, 3, -2), \ \boldsymbol{v} = (0, -1, 1)$

$$(a)\ \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\boldsymbol{v}\cdot\boldsymbol{v}}\boldsymbol{v} = \frac{1(0)+3(-1)-2(1)}{0^2+(-1)^2+1^2}(0,-1,1) = \frac{-5}{2}(0,-1,1) = (0, \frac{5}{2}, \frac{-5}{2})$$

$$(b)\ \text{proj}_{\boldsymbol{u}}\boldsymbol{v} = \frac{\boldsymbol{v}\cdot\boldsymbol{u}}{\boldsymbol{u}\cdot\boldsymbol{u}}\boldsymbol{u} = \frac{0(1)+(-1)(3)+1(-2)}{1^2+3^2+(-2)^2}(1,3,-2)$$

$$= \frac{-5}{14}(1,3,-2) = (\frac{-5}{14}, \frac{-15}{14}, \frac{5}{7})$$

Find (a) $\|\boldsymbol{u}\|$, (b) $\|\boldsymbol{v}\|$, and (c) $\|\boldsymbol{u}+\boldsymbol{v}\|$

1. $\boldsymbol{u} = (2, 1, -1),\ \boldsymbol{v} = (0, -1, 1)$
2. $\boldsymbol{u} = (2, -1),\ \boldsymbol{v} = (1, 1)$

Find a unit vector (a) in the direction of $\boldsymbol{u}$ and (b) in the direction opposite that of $\boldsymbol{u}$

1. $\boldsymbol{u} = (-1, 3, 4)$
2. $\boldsymbol{u} = (2, -2)$

Find the vector $\boldsymbol{v}$ with the given length and the same direction as $\boldsymbol{u}$

1. $\|\boldsymbol{v}\| = 4,\ \boldsymbol{u} = (1, 2, -1)$
2. $\|\boldsymbol{v}\| = 2,\ \boldsymbol{u} = (3, 4)$

Find (a) $\boldsymbol{u}\cdot\boldsymbol{v}$, (b) $\boldsymbol{v}\cdot\boldsymbol{v}$, (c) $\|\boldsymbol{v}\|^2$, (d) $d(\boldsymbol{u},\boldsymbol{v})$, (e) $(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}$, and (f) $\boldsymbol{u}\cdot(5\boldsymbol{v})$

1. $\boldsymbol{u} = (-1, 2),\ \boldsymbol{v} = (2, -2)$
2. $\boldsymbol{u} = (0, -2, 1),\ \boldsymbol{v} = (2, 1, -2)$

For what values of c is $\|c(-2,2,1)\| = 6$?

Find $(u+v)\cdot(2u-v)$ when $u\cdot u = 4$, $u\cdot v = -5$, and $v\cdot v = 10$

Find the angle θ between the vectors

1. $\boldsymbol{u} = (3, 1)$, $\boldsymbol{v} = (-2, 4)$
2. $\boldsymbol{u} = (2, 3, 1)$, $\boldsymbol{v} = (-3, 2, 0)$
3. $\boldsymbol{u} = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$, $\boldsymbol{v} = (\cos\frac{\pi}{4}, \sin\frac{\pi}{4})$

Determine whether $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal, parallel, or neither

1. $\boldsymbol{u} = (0, 3, -4)$, $\boldsymbol{v} = (1, -8, 6)$
2. $\boldsymbol{u} = (1, -1, 2)$, $\boldsymbol{v} = (-3, 3, -6)$
3. $\boldsymbol{u} = (1, -1)$, $\boldsymbol{v} = (0, -1)$

Find (a) $\text{proj}_{\boldsymbol{v}}\boldsymbol{u}$ and (b) $\text{proj}_{\boldsymbol{u}}\boldsymbol{v}$

1. $\boldsymbol{u} = (1, 2)$, $\boldsymbol{v} = (2, 1)$
2. $\boldsymbol{u} = (5, -3, 1)$, $\boldsymbol{v} = (1, -1, 0)$

(a) Determine whether the set of vectors in $\boldsymbol{R^n}$ is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for $\boldsymbol{R^n}$

① $\boldsymbol{u} = (2, -4),\ \boldsymbol{v} = (2, 1)$

(a) $(2,-4)\cdot(2,\ 1) = 2(2)+(-4)(1) = 0$, the set is orthogonal

(b) $\|(2,-4)\| = \sqrt{2^2+(-4)^2} = \sqrt{20} \neq 1$, the set is not orthonormal

(c) 2 independents vectors, the set is a basis for R^2

② $\{\boldsymbol{u} = (3/5,\ 4/5),\ \boldsymbol{v} = (-4/5,\ 3/5)\}$

(a) $(\frac{3}{5},\frac{4}{5})\cdot(-\frac{4}{5},\ \frac{3}{5}) = \frac{3}{5}(-\frac{4}{5})+\frac{4}{5}(\frac{3}{5}) = 0$, the set is orthogonal

(b) $\|(\frac{3}{5},\frac{4}{5})\| = \sqrt{(\frac{3}{5})^2+(\frac{4}{5})^2} = 1,\quad \|(-\frac{4}{5},\frac{3}{5})\| = \sqrt{(-\frac{4}{5})^2+(\frac{3}{5})^2} = 1$, the set is orthonormal

(c) 2 independents vectors, the set is a basis for R^2

$$\left\|\left(\frac{3}{5},\frac{4}{5}\right)\right\| = \sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2} = 1, \quad \left\|\left(-\frac{4}{5},\frac{3}{5}\right)\right\| = \sqrt{\left(-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2} = 1$$

3 $$\left\{ \boldsymbol{u} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \boldsymbol{v} = \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right), \boldsymbol{w} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) \right\}$$

(a) The set is orthogonal because

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) = -\frac{\sqrt{12}}{12} + 0 + \frac{\sqrt{12}}{12} = 0$$

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) = \frac{\sqrt{6}}{6} + 0 - \frac{\sqrt{6}}{6} = 0$$

$$\left(-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)\cdot\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)=-\frac{\sqrt{18}}{18}+\frac{\sqrt{18}}{9}-\frac{\sqrt{18}}{18}=0$$

(b) The set is orthonormal because

$$\left\|\left(\frac{\sqrt{2}}{2},0,\frac{\sqrt{2}}{2}\right)\right\|=\sqrt{\frac{2}{4}+0+\frac{2}{4}}=1,\quad \left\|\left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},-\frac{\sqrt{3}}{3}\right)\right\|=\sqrt{\frac{3}{9}+\frac{3}{9}+\frac{3}{9}}=1$$

$$\left\|\left(-\frac{\sqrt{6}}{6},\frac{\sqrt{6}}{3},\frac{\sqrt{6}}{6}\right)\right\|=\sqrt{\frac{6}{36}+\frac{6}{9}+\frac{6}{36}}=1$$

(c) 3 independents vectors, the set is a basis for R^3

4 $\left\{ \boldsymbol{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \boldsymbol{v} = (0,1,0) \right\}$

(a) The set is orthogonal because $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot (0,1,0) = 0$

(b) The set is orthonormal because

$$\left\| \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1, \quad \|(0,1,0)\| = \sqrt{0+1+0} = 1$$

(c) Only 2 independents vectors, the set is not a basis for R^3

(a) Show that the set of vectors in $\boldsymbol{R}^n$ is orthogonal, and (b) normalize the set to produce an orthonormal set

① $\boldsymbol{u} = (-1, 4),\ \boldsymbol{v} = (8, 2)$

(a) $(-1, 4)\cdot(8,\ 2) = -1(8) + 4(2) = 0$, the set is orthogonal

(b) $\|(-1, 4)\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} \neq 1$, the set is not orthonormal. Normalizing

$$\boldsymbol{u}_1 = \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \frac{1}{\sqrt{17}}(-1, 4) = \left(-\frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17}\right)$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \frac{1}{2\sqrt{17}}(8, 2) = \left(\frac{4\sqrt{17}}{17}, \frac{\sqrt{17}}{17}\right)$$

② $\left\{ \boldsymbol{v}_1 = \left(\sqrt{3}, \sqrt{3}, \sqrt{3}\right), \boldsymbol{v}_2 = \left(-\sqrt{2}, 0, \sqrt{2}\right) \right\}$

(a) $(\sqrt{3}, \sqrt{3}, \sqrt{3}) \cdot (-\sqrt{2}, 0, \sqrt{2}) = -\sqrt{6} + 0 + \sqrt{6} = 0$, the set is orthogonal

(b) $\left\| (\sqrt{3}, \sqrt{3}, \sqrt{3}) \right\| = \sqrt{\sqrt{3}^2 + \sqrt{3}^2 + \sqrt{3}^2} = \sqrt{9} = 3 \neq 1$

the set is not orthonormal. Normalizing

$$\boldsymbol{u}_1 = \frac{1}{3}(\sqrt{3}, \sqrt{3}, \sqrt{3}) = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right)$$

$$\boldsymbol{u}_2 = \frac{1}{2}(-\sqrt{2}, 0, \sqrt{2}) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right)$$

Find the coordinate matrix of $\boldsymbol{w}$ relative to the orthonormal basis $\boldsymbol{B}$ in $\boldsymbol{R}^n$

① $\boldsymbol{w} = (-3, 4),\ B = \left\{ \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right), \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) \right\}$

$$(-3, 4) \cdot \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right) = \frac{-3\sqrt{5}}{5} + \frac{8\sqrt{5}}{5} = \frac{5\sqrt{5}}{5} = \sqrt{5}$$

$$(-3, 4) \cdot \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) = \frac{6\sqrt{5}}{5} + \frac{4\sqrt{5}}{5} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$$

So, $[\boldsymbol{w}]_B = \begin{bmatrix} \sqrt{5} \\ 2\sqrt{5} \end{bmatrix}$

2 $\mathbf{w}=(2,-2,1),\ B=\left\{\left(\frac{\sqrt{10}}{10},0,\frac{3\sqrt{10}}{10}\right),(0,1,0),\left(-\frac{3\sqrt{10}}{10},0,\frac{\sqrt{10}}{10}\right)\right\}$

$$(2,-2,1)\cdot\left(\frac{\sqrt{10}}{10},0,\frac{3\sqrt{10}}{10}\right)=\frac{2\sqrt{10}}{10}+0+\frac{3\sqrt{10}}{10}=\frac{5\sqrt{10}}{10}=\frac{\sqrt{10}}{2}$$

$$(2,-2,1)\cdot(0,1,0)=0-2+0=-2$$

$$(2,-2,1)\cdot\left(-\frac{3\sqrt{10}}{10},0,\frac{\sqrt{10}}{10}\right)=-\frac{6\sqrt{5}}{5}+0+\frac{\sqrt{10}}{10}=-\frac{5\sqrt{5}}{10}=-\frac{\sqrt{10}}{2}$$

So, $[\mathbf{w}]_B=\begin{bmatrix}\frac{\sqrt{10}}{2}\\ -2\\ -\frac{\sqrt{10}}{2}\end{bmatrix}$

Apply the Gram-Schmidt orthonormalization process to transform the given basis for $\boldsymbol{R}^n$ into an orthonormal basis

1. $\mathbf{v}_1 = (3, 4),\ \mathbf{v}_2 = (1, 0)$

$$\mathbf{w}_1 = \mathbf{v}_1 = (3, 4)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (1,0) - \frac{1(3)+0(4)}{3^2+4^2}(3,4) = \left(\frac{16}{25}, -\frac{12}{25}\right)$$

Normalize the vectors

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{3^2+4^2}}(3,4) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{\left(\frac{16}{25}\right)^2 + \left(-\frac{12}{25}\right)^2}}\left(\frac{16}{25}, -\frac{12}{25}\right) = \left(\frac{4}{5}, -\frac{3}{5}\right)$$

② $B = \{ v_1 = (1, -2, 2),\ v_2 = (2, 2, 1),\ v_3 = (2, -1, -2)\}$

Because $v_i \cdot v_j = 0$ for $i \neq j$, the given vectors are orthogonal. Normalize the vectors

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2}}(1, -2, 2) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2^2 + 2^2 + 1^2}}(2, 2, 1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2}}(2, -1, -2) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

③ $B = \{ v_1 = (4, -3, 0),\ v_2 = (1, 2, 0),\ v_3 = (0, 0, 4)\}$

Orthogonalize each vector in B vectors

$$w_1 = v_1 = (4, -3, 0)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = (1,2,0) - \tfrac{2}{25}(4,-3,0) = \left(\tfrac{33}{25}, \tfrac{44}{25}, 0\right)$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = (0,0,4) - 0(4,-3,0) - 0\left(\tfrac{33}{25}, \tfrac{44}{25}, 0\right) = (0,0,4)$$

Normalize the vectors

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{4^2 + (-3)^2 + 0^2}}(4,-3,0) = \left(\frac{4}{5}, -\frac{3}{5}, 0\right)$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{(\frac{33}{25})^2 + (\frac{44}{25})^2 + 0^2}}(\tfrac{33}{25}, \tfrac{44}{25}, 0) = \left(\frac{3}{5}, \frac{4}{5}, 0\right)$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{0^2 + 0^2 + 4^2}}(0,0,4) = (0,0,1)$$

(a) Determine whether the set of vectors in $\boldsymbol{R}^n$ is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for $\boldsymbol{R}^n$

1. $\{\boldsymbol{u} = (-3, 5),\ \boldsymbol{v} = (4, 0)\}$
2. $\{\boldsymbol{u} = (2, 1),\ \boldsymbol{v} = (1/3, -2/3)\}$
3. $\left\{\boldsymbol{u} = \left(\frac{\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6}\right), \boldsymbol{v} = \left(0, \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right), \boldsymbol{w} = \left(\frac{\sqrt{5}}{5}, 0, \frac{1}{2}\right)\right\}$
4. $\{(2, 5, -3), (4, 2, 6)\}$

(a) Show that the set of vectors in $\boldsymbol{R}^n$ is orthogonal, and (b) normalize the set to produce an orthonormal set

1. $\{\boldsymbol{v}_1 = (-1, 3),\ \boldsymbol{v}_2 = (12, 4)\}$
2. $\{\boldsymbol{v}_1 = (1, 3, 1),\ \boldsymbol{v}_2 = (3, 0, -3)\}$

Find the coordinate matrix of $\boldsymbol{w}$ relative to the orthonormal basis $\boldsymbol{B}$ in $\boldsymbol{R}^n$

1. $\boldsymbol{w} = (1, 2),\ B = \left\{\left(-\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right), \left(\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13}\right)\right\}$
2. $\boldsymbol{w} = (1, -3, 2),\ B = \left\{\left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10}\right), (0, 1, 0), \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10}\right)\right\}$

Apply the Gram-Schmidt orthonormalization process to transform the given basis for R^n into an orthonormal basis

1. $B = \{\mathbf{v}_1 = (4, -3),\ \mathbf{v}_2 = (3, 2)\}$

2. $B = \{\mathbf{v}_1 = (2, 1, -2),\ \mathbf{v}_2 = (1, 2, 2),\ \mathbf{v}_3 = (2, -2, 1)\}$

3. $B = \{\mathbf{v}_1 = (1, 0, 0),\ \mathbf{v}_2 = (1, 1, 1),\ \mathbf{v}_3 = (1, 1, -1)\}$