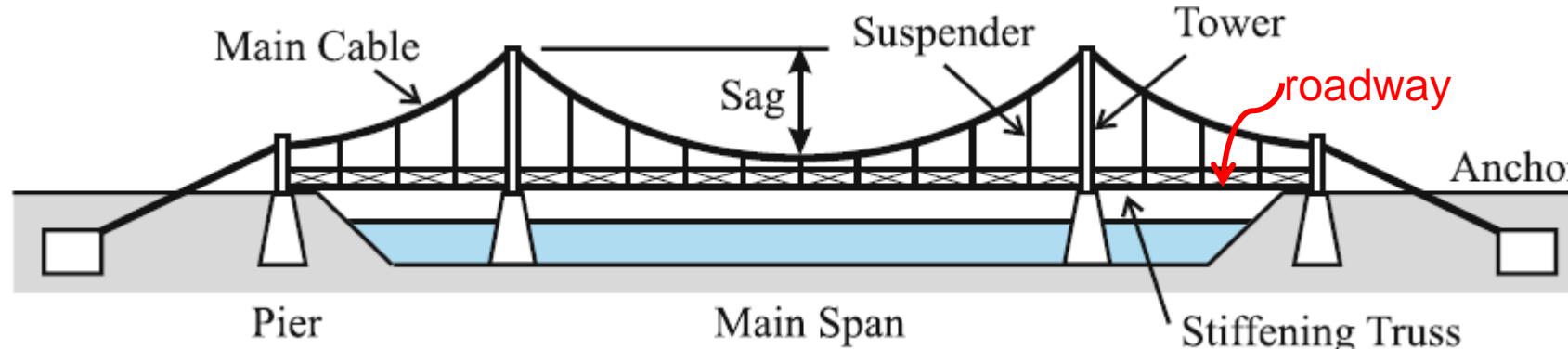
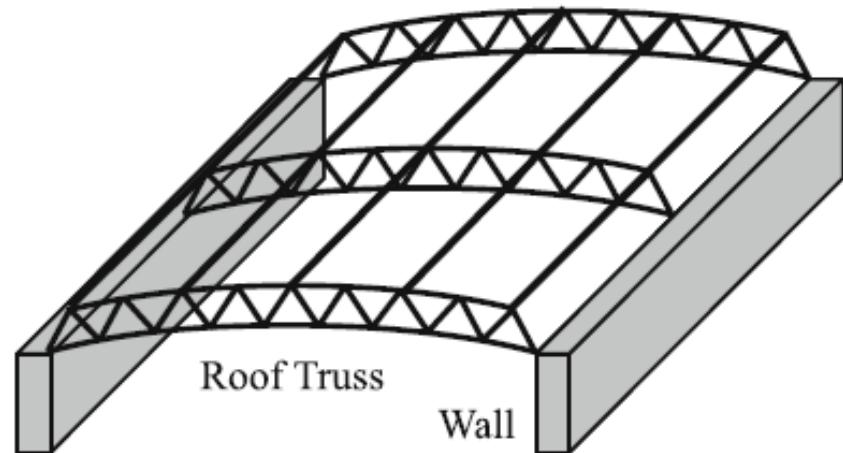


Axial Members and Pressure Vessels (Biaxial)

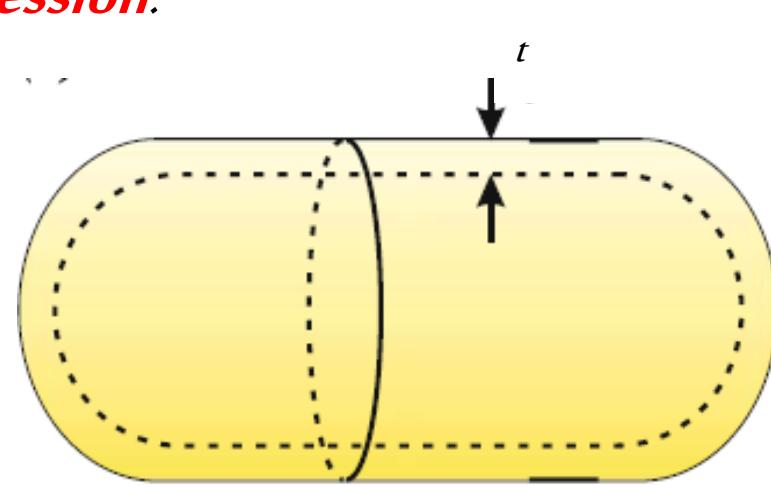
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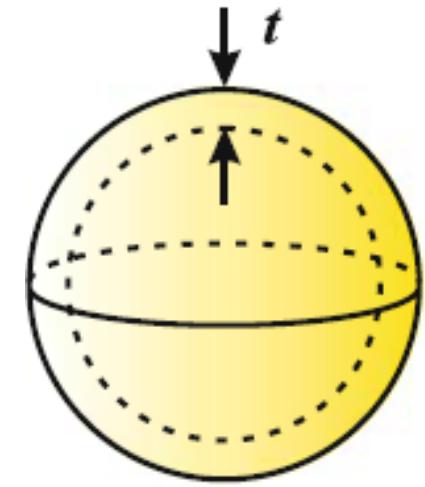
Main components of a suspension bridge. The main cables and suspenders are axial members in **tension**; the towers are primarily axial members in **compression**.



Trusses are made of straight members that are subjected to axial loads.



A cylindrical pressure vessel with hemispherical end caps.



A spherical pressure vessel.

The behaviour of Mechanical Systems under **mechanical and thermal actions**, is determined by

1. The *stresses, strains, & elongations* of each individual components.
2. The *stiffness (force–displacement)* of an assembly of components.

Two Methods of Analysis

Force Method

1. Apply *equilibrium conditions* to loads, reactions, → internal forces → stresses for a bar in tension: $\sigma = P/A$.
2. Calculate the strains from the stresses using *Hooke's Law*; e.g., $\varepsilon = \sigma/E$.
3. Integrate the strain in each component to find its elongation; e.g., $\Delta = \varepsilon L$.
4. Elongations of members must be *compatible*, then → displacement

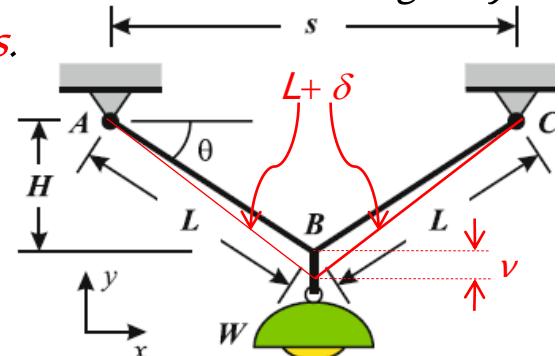
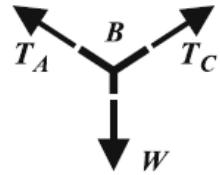
Displacement Method

1. Determine *elongation of each member* in terms of the *overall displacement*
2. Determine the *strain in each component* in terms of its *elongation*; e.g., $\varepsilon = \Delta/L$.
3. Calculate the stresses from *Hooke's Law*; e.g., $\sigma = E\varepsilon$.
4. Apply *equilibrium conditions* to determine the internal & external forces; e.g., $P = \sigma A$.

Types of Systems

Statically Determinate Systems

Reactions & internal forces can be calculated using only statics: *Equilibrium equations*.



2 Unknowns (T_A & T_C) with 2 Eq. Eqs.

$$\sum F_x = 0: -T_A \cos\theta + T_C \cos\theta = 0$$

$$\sum F_y = 0: T_A \sin\theta + T_C \sin\theta - W = 0$$

$$\Rightarrow T_A = T_C = T = W / 2\sin\theta$$

$$\Rightarrow \sigma_{AB} = \sigma_{CB} = W / 2A\sin\theta$$

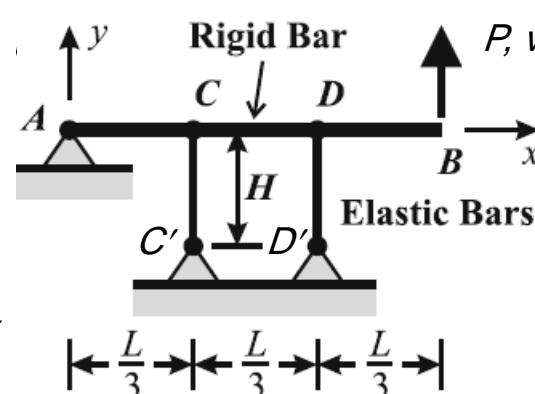
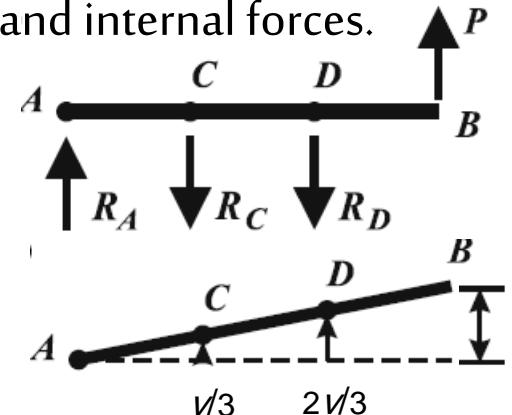
$$\Rightarrow \varepsilon_{AB} = \varepsilon_{CB} = W / 2EA\sin\theta$$

$$\Rightarrow \delta_{AB} = \delta_{CB} = \delta = WL / 2EA\sin\theta$$

$$\Rightarrow v = \delta / \sin\theta = WL / 2EA\sin^2\theta$$

Statically Indeterminate (Redundant) Systems

Statics alone (*Equilibrium equations*) is not enough to calculate reactions and internal forces.



3 Unknowns (R_A , R_C & R_D) with 2 Eq. Eqs. (impossible)

$$\sum F_y = 0: R_A - R_C - R_D + P = 0$$

$$\sum MA = 0: -R_C(L/3) - R_D(2L/3) + PL = 0$$

Additional *compatibility equation* is needed.

$$\delta_{C'C} = v/3 \Rightarrow R_C = EA\delta_{C'C}/H = EA v/3H$$

$$\delta_{D'D} = 2v/3 \Rightarrow R_D = EA\delta_{D'D}/H = 2EA v/3H \quad \boxed{R_D = 2R_C}$$

3Eqs solution gives: $R_A = 0.8P$, $R_C = 0.6P$, $R_D = 1.2P$

$$\Rightarrow \sigma_{C'C} = 0.6P/A \text{ & } \sigma_{D'D} = 1.2P/A.$$

$$\Rightarrow \delta_{C'C} = 0.6PH/EA, \sigma_{D'D} = 1.2PH/EA \text{ & } v = 1.8PH/EA$$

Strain-Elongation-Displacement of an axially loaded member.

The axial member IJ of length L , directed in x,y,z , by the unit vector \vec{e}_{IJ}

moves under loading to $I'J'$ by two **small displacement vectors** $\vec{\delta}_I$ & $\vec{\delta}_J$

getting a new length $L'=L+\Delta_{IJ}$, and a new direction, to determine Δ_{IJ} ,
we observe from the figure that: $\overrightarrow{IJ'} = \overrightarrow{IJ} + \overrightarrow{JJ'} - \overrightarrow{II'}$

using unit vector, the length, and two displacement vectors, the last equation is rewritten:

$$\overrightarrow{IJ'} = L\vec{e}_{IJ} + (\vec{\delta}_J - \vec{\delta}_I), \text{ squaring we get}$$

$$L'^2 = (L\vec{e}_{IJ})^2 + 2L(\vec{\delta}_J - \vec{\delta}_I) \cdot \vec{e}_{IJ} + (\vec{\delta}_J - \vec{\delta}_I)^2 = L^2 + 2L(\vec{\delta}_J - \vec{\delta}_I) \cdot \vec{e}_{IJ} + (\vec{\delta}_J - \vec{\delta}_I)^2$$

$$L'^2 - L^2 = 2L(\vec{\delta}_J - \vec{\delta}_I) \cdot \vec{e}_{IJ} + (\vec{\delta}_J - \vec{\delta}_I)^2 \Rightarrow (L'+L)(L'-L) = 2L(\vec{\delta}_J - \vec{\delta}_I) \cdot \vec{e}_{IJ} + (\vec{\delta}_J - \vec{\delta}_I)^2$$

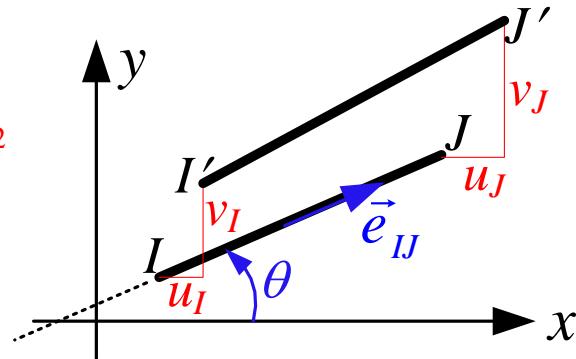
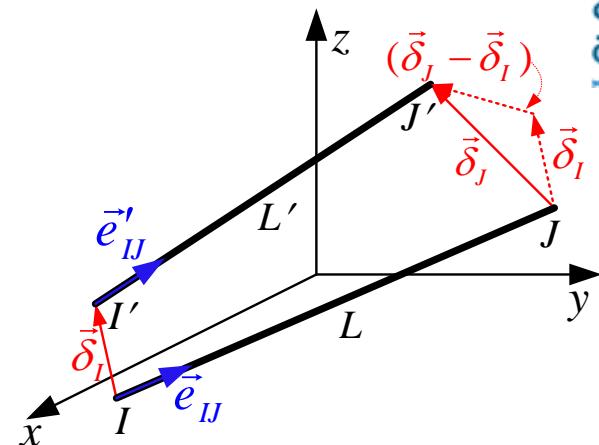
$$\Rightarrow (2L + \delta_{IJ})\delta_{IJ} = 2L(\vec{\delta}_J - \vec{\delta}_I) \cdot \vec{e}_{IJ} + (\vec{\delta}_J - \vec{\delta}_I)^2 \Rightarrow 2L\delta_{IJ} + \delta_{IJ}^2 = 2L(\vec{\delta}_J - \vec{\delta}_I) \cdot \vec{e}_{IJ} + (\vec{\delta}_J - \vec{\delta}_I)^2$$

Neglecting the two squared terms for small elongation & displacements, then:

$$\delta_{IJ} = (\vec{\delta}_J - \vec{\delta}_I) \cdot \vec{e}_{IJ}$$

In the 2D case where $\vec{\delta}_J = u_J \vec{e}_x + v_J \vec{e}_y$, $\vec{\delta}_I = u_I \vec{e}_x + v_I \vec{e}_y$ & $\vec{e}_{IJ} = \cos \theta \vec{e}_x + \sin \theta \vec{e}_y$

$$\boxed{\delta_{IJ} = (u_J - u_I) \cos \theta + (v_J - v_I) \sin \theta}$$

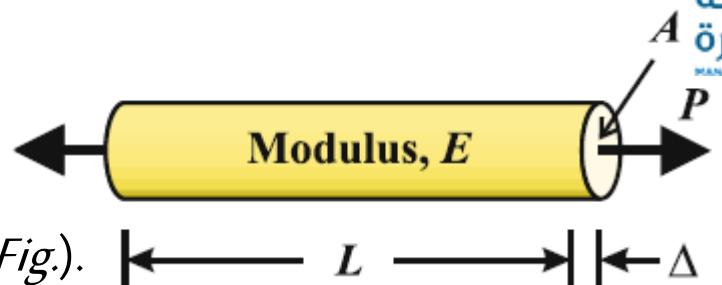


1. Axial Members – Force Method *Statically Determinate Systems*

Examples 1– 6, are statically determinate systems, solved using the force method.

Example 1 Uniform Bar in Tension

Given: Load P is applied at the ends of a bar of constant cross-section A and length L (Fig.).



The bar is made of an elastic material with modulus E .

Required: Determine the stress σ & strain ϵ in the bar, and its elongation Δ in terms of P , L , A , & E .

Notes: $(\sigma, \epsilon, \Delta)$ response to action or excitation P . (L, A , & E) sys. parameters

Solution: Following the steps of the *force method*:

Step 1. Equilibrium. The relationship between the applied force & the internal stress is: $P = \sigma A \Rightarrow \sigma = P/A$

Step 2. Elasticity (Stress–Strain relationship). From Hooke's Law, the strain is: $\epsilon = \sigma/E \Rightarrow \epsilon = P/EA$

Step 3. Strain–Elongation. The elongation as a function of strain is: $\Delta = \epsilon L \Rightarrow \Delta = PL/EA$

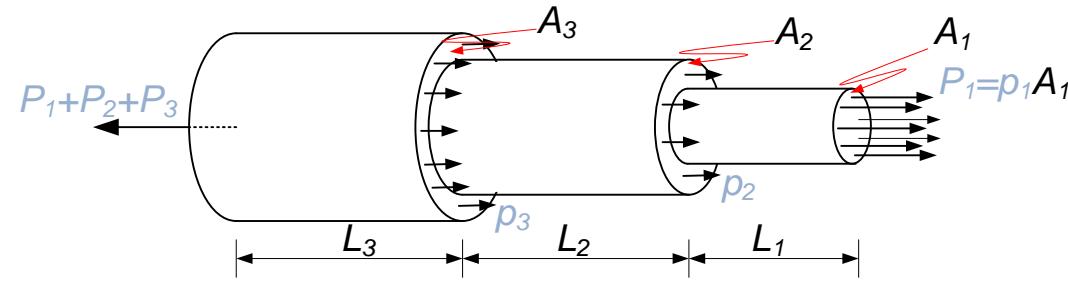
The force method directly gives the *flexibility* f of the bar, where $\Delta = fP$. Thus: $f = L/EA$

Then the *stiffness* K of the bar which is defined as: $P = K\Delta$, is given: $K = 1/f = EA/L$.

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The previous solution is valid when P , A , and E are all constant over the entire length of the bar L , resulting in a uniform strain.



To determine the elongation when P , A , and/or E change over the length of the bar, the bar must be broken up into shorter segments L_i , where P , A , and E are all constant. The total elongation is the sum of the elongations of each segment Δ_i :

$$\Delta = \sum \Delta_i = \sum \frac{P_i L_i}{E_i A_i}$$

Example 2. Rods Supporting Elevated Walkways

Given: A set of six stepped rods supports two elevated walkways overlooking a hotel atrium (Fig.).

The upper segment of each rod, AB , has area A_1 , length L_1 , and modulus E_1 .

The lower segment, BC , has properties A_2 , L_2 , and E_2 .

The load of the upper walkway is supported at joint B , the junction of the two bars.

The upper walkway applies a load to each rod of W_U at point B , and the lower walkway applies a load to each rod of W_L at point C .

Each rod is assumed to carry the same load (!!).

Required: Determine

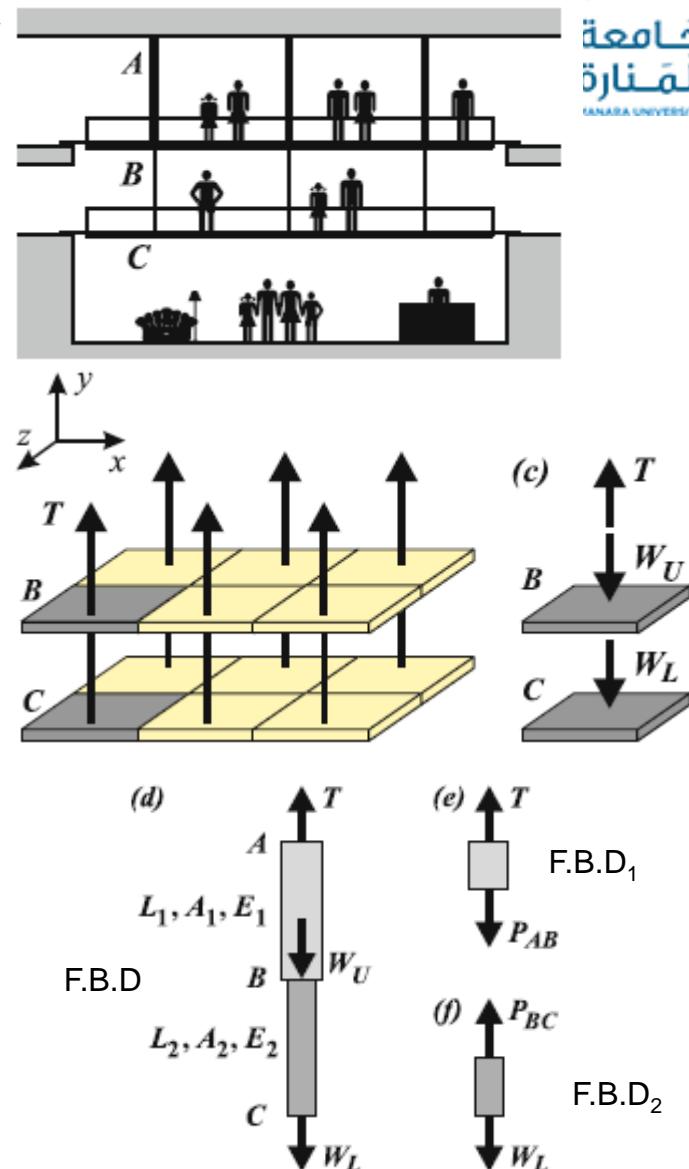
- (a) the internal stresses in each segment of the rod AB and BC .
- (b) the total elongation Δ of stepped rod AC .

Solution

Step 1. Equilibrium

Step 2. Elasticity (Stress–Strain relationship).

Step 3. Strain–elongation–displacement



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Continuously Varying Stress in Axial Members

The tapered column in Fig. supports a compressive load F . While the force through the column is constant, the cross-sectional area is not. Hence, stress and strain vary continuously over the length of the member.

Example 3. Tapered Column under Compressive Load

Given: A tapered concrete column has a square cross-section that varies from side $a = 125$ mm at the top to side $2a = 250$ mm at the bottom (Fig.). The total length of the column is $L = 1.20$ m, and it carries a compressive load of $F = 200$ kN. The modulus of concrete is $E = 30$ GPa. Neglect the weight of the concrete !. Since the force is constant, & area increases from top ($x=0$) to bottom ($x=L$), the axial compressive stress varies; it is maximum at the top and minimum at the bottom.

Required: Determine

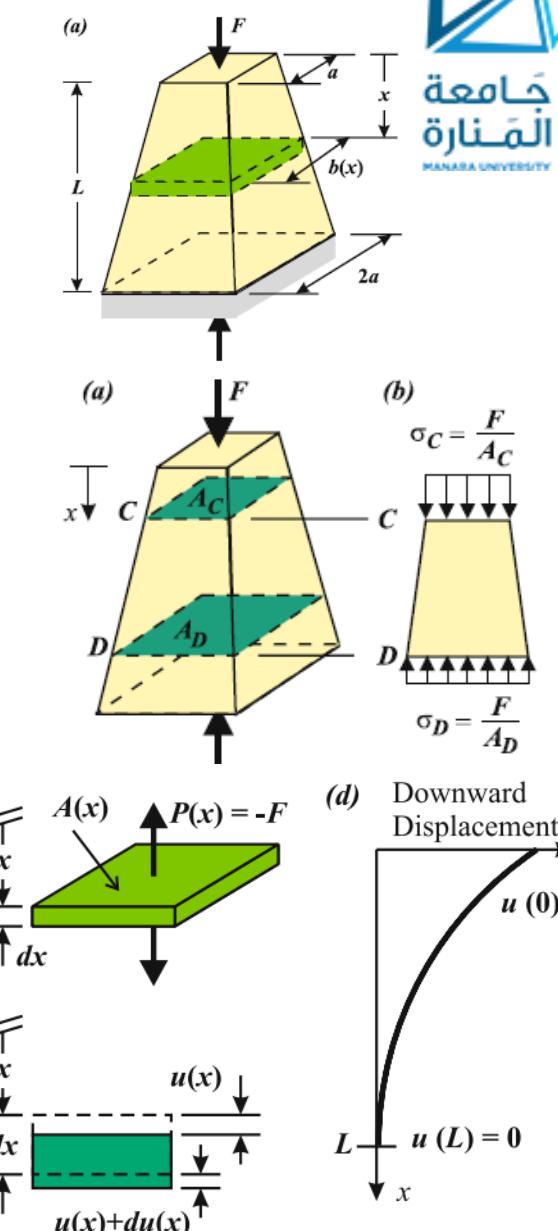
- (a) the variation of stress in the column $\sigma(x)$,
- (b) the variation of strain $\epsilon(x)$, and (c) the change in length of the column Δ .

Solution

Step 1. Equilibrium.

Step 2. Stress–Strain.

Step 3. Strain–Elongation–Displacement.



Example 4. Extraction of a Nail

Given: A nail is pulled out of a piece of wood (Fig.). The nail diameter is $D = 2R$, and its embedded length is L . Force T applied to extract the nail is resisted by an interfacial shear stress τ acting between the nail surface and the wood; τ is assumed to be constant (Fig.). The tip of the nail provides no resistance to pull-out. The nail begins to slide (pull-out) when the sliding stress τ_s is reached (τ_s is the stress to overcome the nail–wood friction).

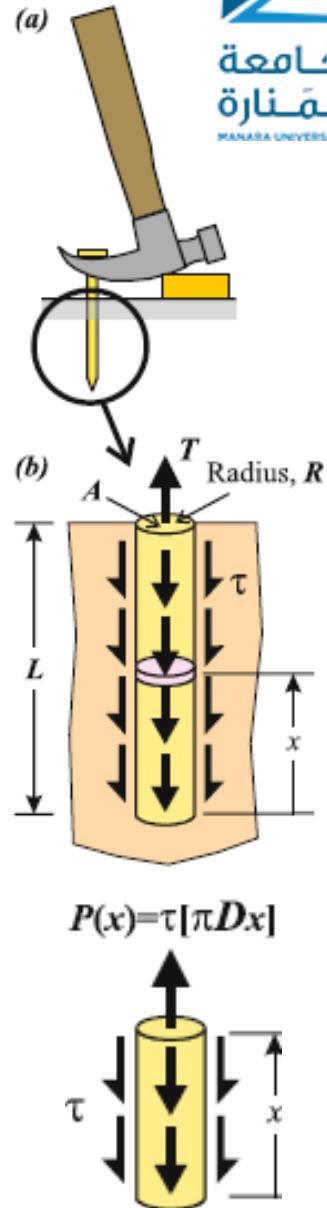
Required: Determine the change in length of the embedded part of the nail just before it starts to slide, in terms of τ_s , L , D , and E .

Solution:

Step 1. Equilibrium.

Step 2. Stress–Strain.

Step 3. Strain–Elongation–Displacement.



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Compatibility

Under load, the components of a system must deform and deflect in such a way that the system remains intact. In other words, the members of an assembly are geometrically constrained to deform together.

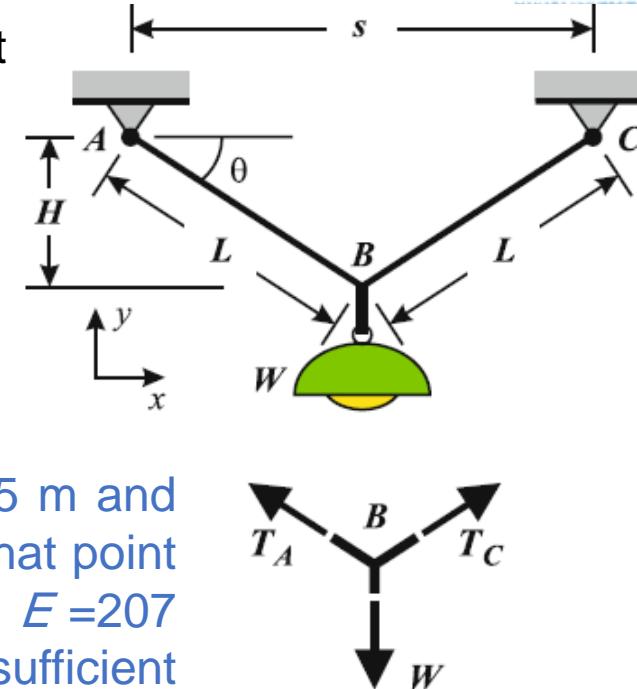
This condition is called *compatibility*. Applying the concept of compatibility is a key step when solving systems where several components are joined together, whether the systems are *statically determinate or statically indeterminate*.

Example 5. Hanging Lamp

Given: A lamp weighing $W=60$ N is supported by two wires, both of length $L = 1.5$ m and diameter $D = 2.5$ mm. The distance between the two cable mounts is $s = 2.4$ m so that point B is $H = 0.9$ m below horizontal line AC . The wires are made of steel with modulus $E = 207$ GPa and yield strength $S_y = 345$ MPa. Assume the wire below point B is rigid and of sufficient strength.

Required: Determine

- the downward displacement of the lamp v (i.e., of point B) due to its own weight,
- the stiffness of the wire assembly in the vertical direction, $K = W/v$,
- the factor of safety against yielding.

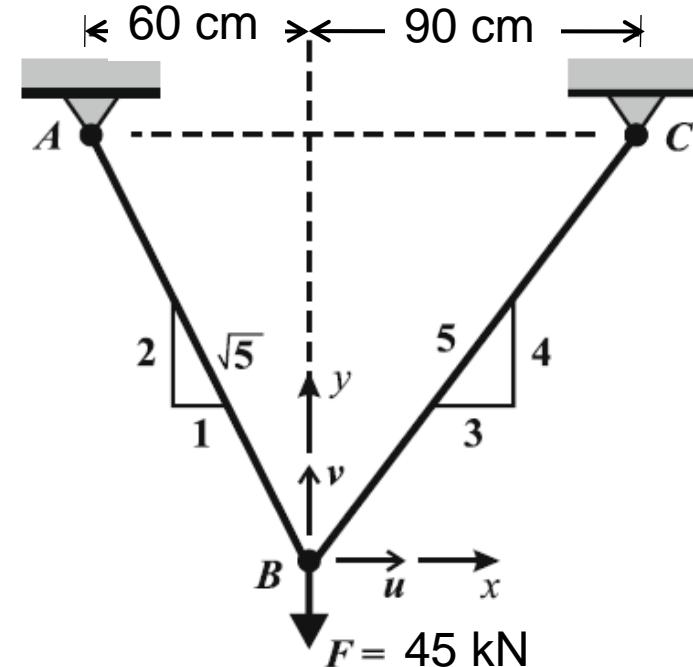


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Example 6 . Truss Deflection

Given: Aluminum truss ABC is loaded at joint B by a point load of $F = 45 \text{ kN}$.
The cross-sectional areas of the bars are: $A_{AB} = 325 \text{ mm}^2$ and $A_{BC} = 390 \text{ mm}^2$.
The modulus of aluminum is $E = 70 \text{ GPa}$.

Required: Determine the horizontal and vertical displacements of joint B , u , and v .



Statically Indeterminate (Redundant) Systems

In the introductory example, it was noted that some systems cannot be solved by *Statics* alone; this is the case when there are more unknowns than equilibrium equations. These systems are ***statically indeterminate or redundant***. When applying the force method, the idea of a redundant force must be introduced to complete such a problem. Exs 7–9, *are applications of the force method to redundant systems*.

Example 7. Two Parallel Bars

Given: Bars 1 and 2 are each attached to a rigid base and a rigid boss (*Fig.*). The boss is constrained to move vertically only. The bars have lengths, cross-sectional areas, and moduli as shown in the diagram. Downward load F is applied to the boss, which displaces (deflects) downward distance v . Assume the system remains elastic.

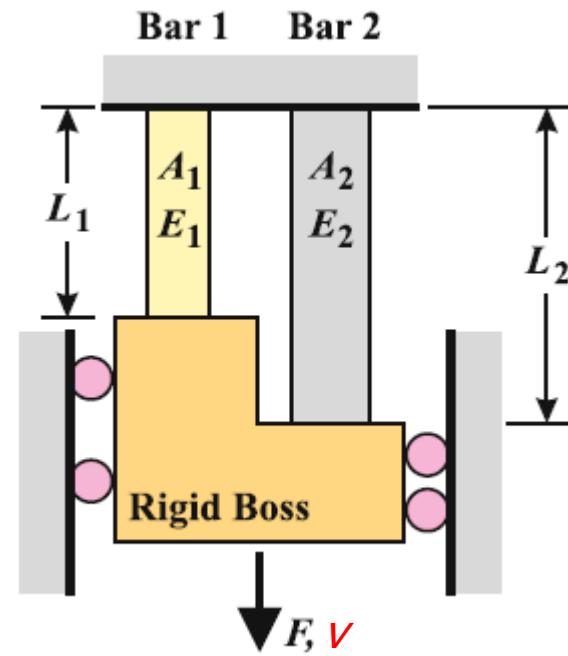
Required: For the particular case $L_2 = 2L_1$, $A_2 = 4A_1$, and $E_1 = E_2 = E$, determine expressions for (a) the stresses in each bar, σ_1 and σ_2 and (b) the downward deflection v of the rigid boss.

Solution:

Step 1. Equilibrium.

Step 2. Force–elongation.

Step 3. Compatibility



Example 8. Two In-Line Bars

Given: Another redundant system consists of two in-line bars, 1 and 2, which are joined together as shown, fixed at the top and bottom (Fig.). Load W is applied at the junction of the bars.

Required: For the particular case $L_2 = 2L_1$, $A_2 = 4A_1$, & $E_1 = E_2 = E$, determine

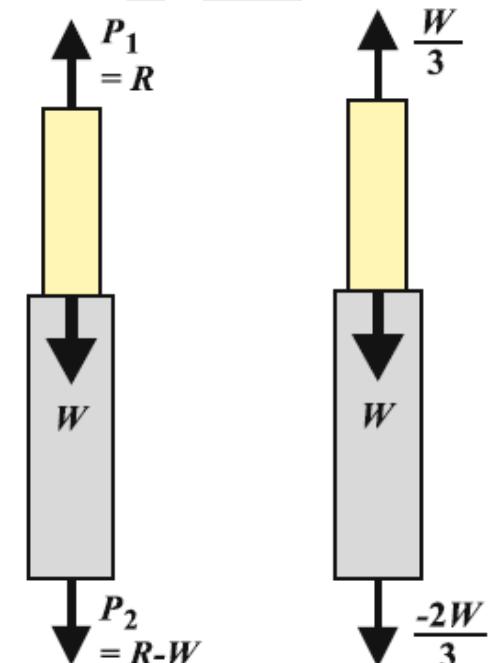
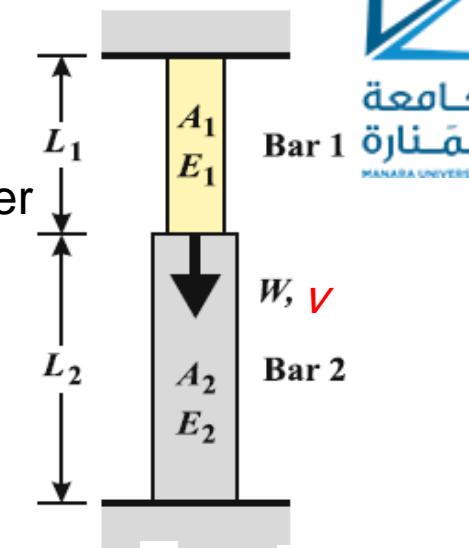
- the internal force and stress in each bar and
- the deflection v of the junction (where the load is applied).

Solution:

Step 1. Equilibrium,

Steps 2 .The force – elongation,

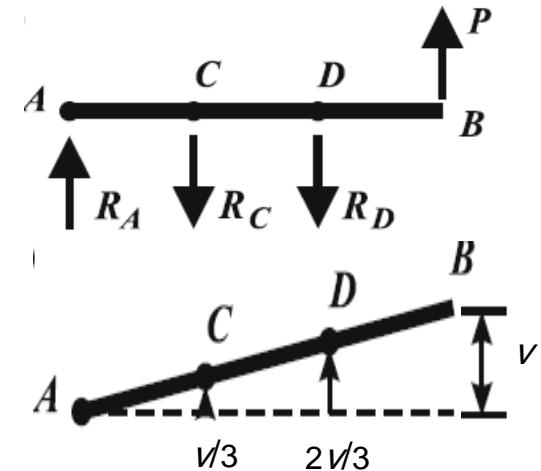
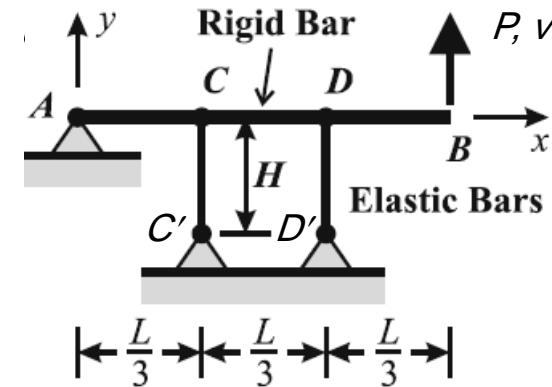
Step 3. The compatibility



Example 9. Rigid Bar

Given: Bar AB is rigid with length L , while bars C and D are elastic, both with modulus E , cross-sectional area A , and length H (Fig.). Force P is applied at point B which displaces upward by distance v .

Required: Determine the force in each bar, R_C and R_D , in terms of P .



2. Axial Members – Displacement Method

The *displacement method* is used here to solve four of the previous examples in *Section 4.1*, two statically determinate and two statically indeterminate (redundant). A new example, with many redundant members, is also solved.

From kinematic (displacement) relationships, the internal strains are first determined, which lead to the stresses via Hooke's Law, from which internal and external forces are determined using equilibrium. The steps are as follows:

1. Determine the elongation of each member in terms of the overall displacement of the assembly; the elongation of each member must be *compatible* with the elongations of the other members. If necessary use

$$\delta_{IJ} = (u_J - u_I) \cos \theta + (v_J - v_I) \sin \theta$$

2. Determine the strain in each member in terms of its elongation; e.g., $\varepsilon = \delta/L$.
3. Calculate the stresses from the elastic law; e.g., $\sigma = E\varepsilon$.
4. Apply the conditions of equilibrium to determine the internal and applied forces;

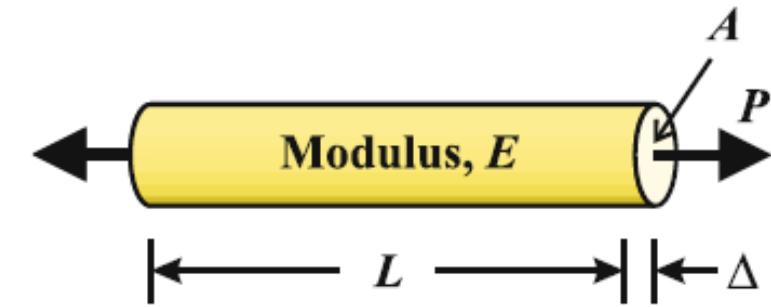
Axial Members and Pressure Vessels (Biaxial)

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Example 10. Uniform Bar in Tension

Given: The bar in *Example 4.1* elongates by Δ when axial force P is applied at its ends. The bar has constant cross-sectional area A and length L (*Fig.*). The material of the bar is elastic with modulus E .

Required: Determine the force P needed to elongate the bar by Δ .

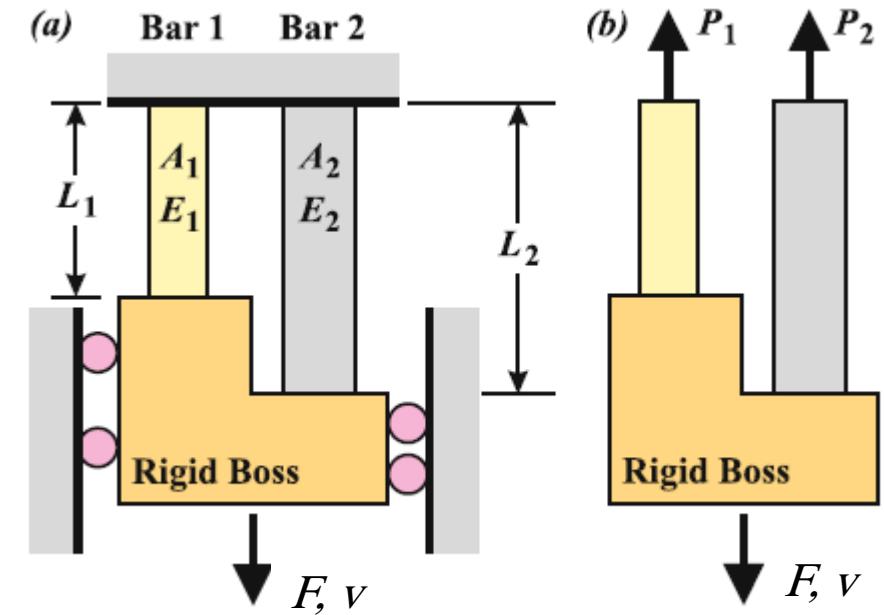


Axial Members and Pressure Vessels (Biaxial) العناصر المحورية وأوعية الضغط (ثنائية المحور)

Example 11. Parallel Bars With Applied Displacement

Given: The two-bar structure of *Example 4.7* is shown in Fig. The system is statically redundant. Due to load F , the rigid boss displaces downward by v .

Required: Using the displacement method, for the particular case $L_2 = 2L_1$, $A_2 = 4A_1$, and $E_1 = E_2 = E$, determine (a) the relationship between force F and displacement v and (b) the stress in each bar, σ_1 and σ_2 .

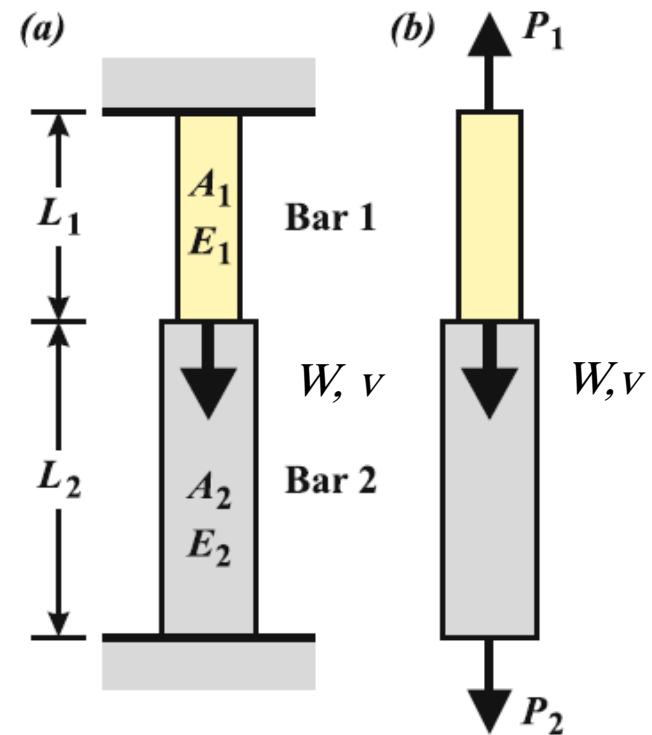


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Example 12. Two In-Line Bars

Given: The redundant system of Example 4.8 consists of two bars in series: Bars 1 and 2, fixed at their ends (Fig.). The applied load W at the junction displaces it by distance v .

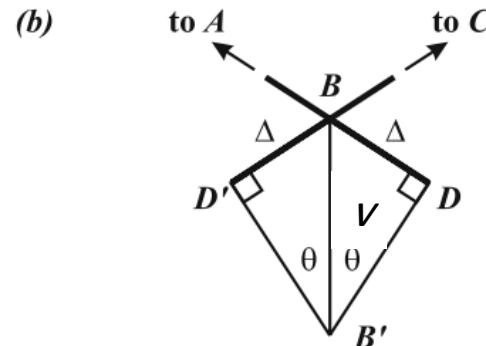
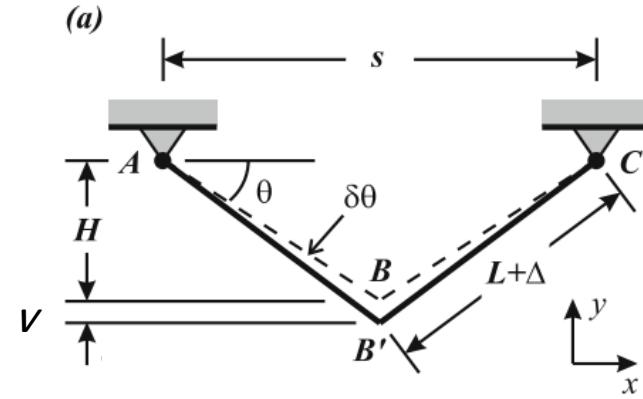
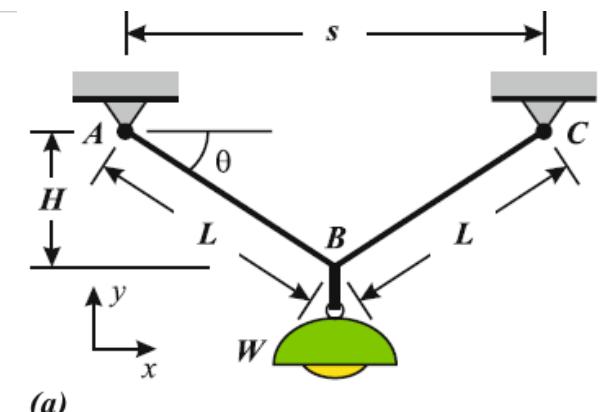
Required: For the particular case $L_2 = 2L_1$, $A_2 = 4A_1$, and $E_1 = E_2 = E$, determine (a) the load W to cause displacement v , and (b) how the load is distributed to the individual bars.



Example 13. Hanging Lamp

Given: A lamp weighing $W=60 \text{ N}$ is supported by two wires, both of length $L = 1.5 \text{ m}$ and diameter $D = 2.5 \text{ mm}$. The distance between the two cable mounts is $s = 2.4 \text{ m}$ so that point B is $H = 0.9 \text{ m}$ below horizontal line AC . The wires are made of steel with modulus $E = 207 \text{ GPa}$ and yield strength $S_y = 345 \text{ MPa}$. Assume the wire below point B is rigid and of sufficient strength.

Required: Using the displacement method, determine
 (a) the downward displacement v of the lamp and
 (b) the tension in each wire, $T_{AC} = T_{BC} = T$.



Example 14. Stiffness of a Wheel with Many Spok

Note: The displacement method is especially well suited for systems with many redundant members. For such systems, the force method is generally impractical.

Given: A bicycle wheel of radius R has N spokes, each of cross-sectional area A . The modulus of the spokes is E . The rim and hub are taken to be rigid. The weight and dynamic forces of the rider cause downward force F at the rigid wheel hub, displacing it downward by distance v (Fig.a–c).

Solution: Consider a triangular-shaped element $d\theta$ at angle θ to the horizontal (Fig.d), dashed triangle. The number of spokes represented by element $d\theta$, is:

$$dN = \frac{N}{2\pi} d\theta$$

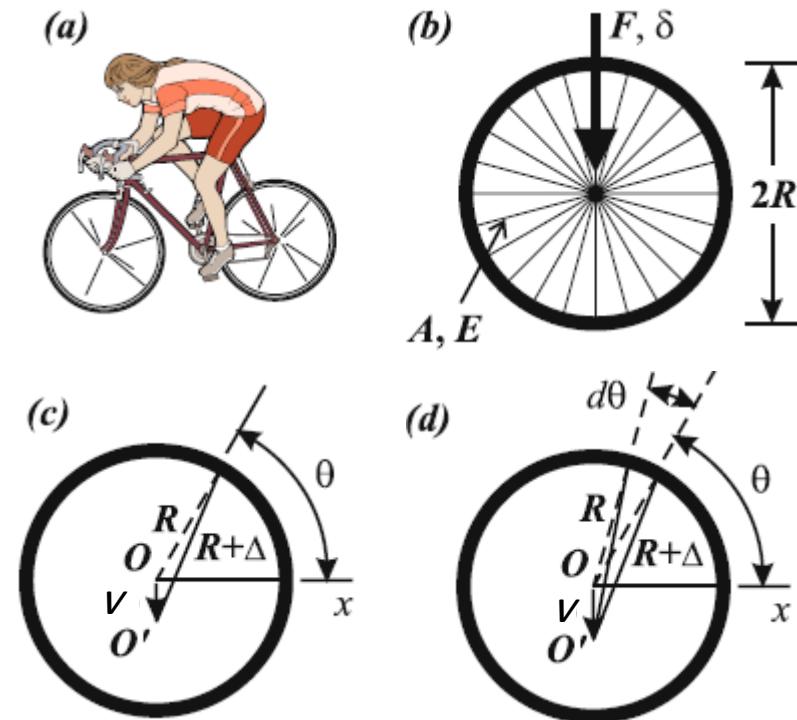
The cross-sectional area of the spokes in $d\theta$ is: $dA = A \frac{N}{2\pi} d\theta$

Step 1. Displacement–Elongation. $\Delta(\theta) = v \sin \theta \Rightarrow \varepsilon(\theta) = (v/R) \sin \theta$

Steps 2 and 3. Force–elongation. $dP(\theta) = \sigma(\theta)dA = E\varepsilon(\theta)dA = \frac{EAN}{2\pi R} v \sin \theta d\theta$

Step 4. Equilibrium. $F = \int_0^{2\pi} dP(\theta) \sin \theta = (EANv/2\pi R) \int_0^{2\pi} \sin^2 \theta d\theta = \frac{EAN}{2R} v$

Step 5. Results and Comments. $K = F/v = EAN/2R$ $\sigma(\theta) = (E v/R) \sin \theta$



spokes
spokes
 $(tension)$
 $(compression)$
} Pretension

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3. Thermal Loading

Changes in temperature cause materials to expand or contract. Consider a bar of length L that is free to expand. When the temperature is increased by an amount ΔT , the length of the bar increases by:

$$\Delta = L\alpha \Delta T$$

where α is the *coefficient of thermal expansion*.

The *thermal strain* ε_t of the bar is then:

$$\varepsilon_t = \frac{\Delta}{L} = \alpha \Delta T$$

Material	α (1/ $^{\circ}$ C)	E (GPa)
Steel	14×10^{-6}	200
Aluminum	23×10^{-6}	70
Concrete	7×10^{-6}	30

The coefficient of thermal expansion is a material property, its unit are the inverse of temperature (e.g., 1/ $^{\circ}$ C, 1/ $^{\circ}$ F).

The above *Table* provides representative material properties for steel, aluminum, and concrete. Although they actually vary with temperature, the coefficients α and modulus E are taken as constant here.

When two materials with different expansion coefficients must deform together, internal *thermal stresses* will develop within the system.

Example 15. Unconstrained Expansion of a Steel Bar

Given: An unconstrained (free to expand) steel bar of length $L = 1.0$ m is heated from room temperature (25°C) to 100°C .

Required: Determine (a) the thermal strain ε_t and (b) the elongation Δ of the bar.

Solution: Step 1. The thermal strain is: $\varepsilon_t = \alpha\Delta T = (14 \times 10^{-6}^\circ\text{C}^{-1})[(100 - 25)^\circ\text{C}] = 1.05 \times 10^{-3}$

Step 2. The elongation is: $\Delta = \varepsilon_t L = (1.05 \times 10^{-3})(1\text{m}) = 1.05 \text{ mm}$

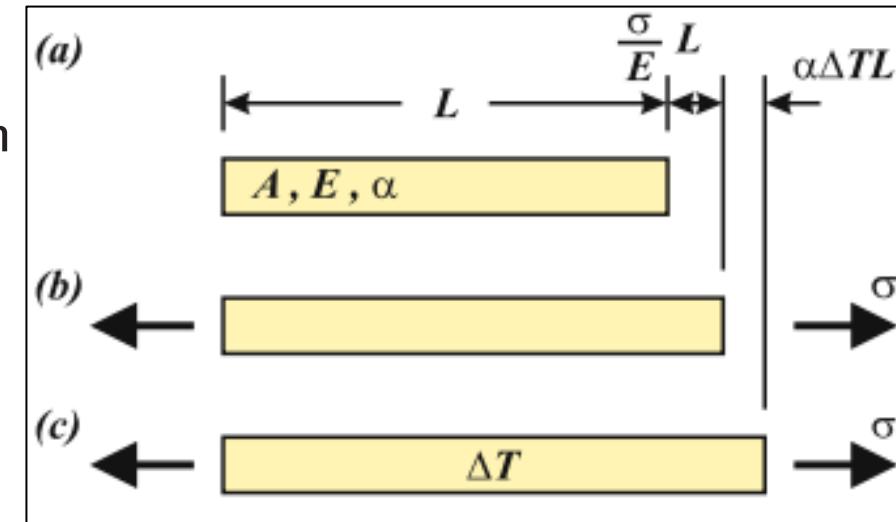
Thermal and Mechanical Loading (Temperature and Applied Stress)

The unconstrained next bar of length L is now subjected to a constant axial stress σ . The temperature is then increased by ΔT . The total strain in the bar is the sum of the *mechanical*/and *thermal strains*, ε_m and ε_t :

$$\varepsilon = \varepsilon_m + \varepsilon_t = \frac{\sigma}{E} + \alpha\Delta T$$

The change in length is:

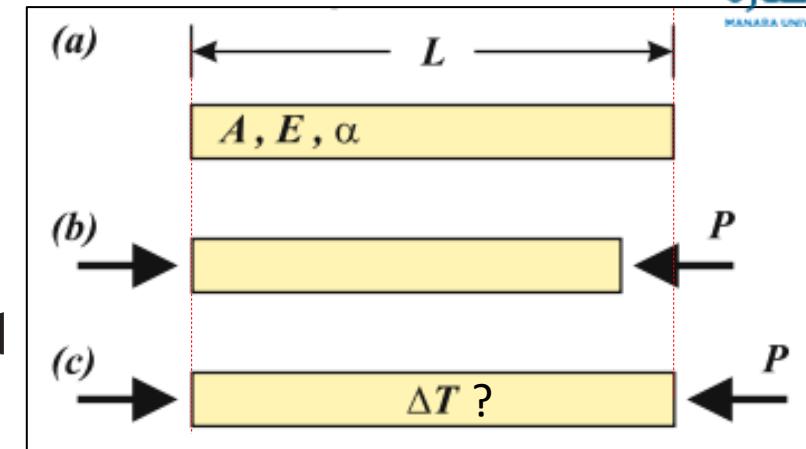
$$\Delta = \varepsilon L = \left(\frac{\sigma}{E} + \alpha\Delta T\right)L$$



Example 16. Steel Bar under Applied Stress and Temperature

Given: The Shown unconstrained steel bar of length $L = 1.0$ m and square cross-section of side $b = 20$ mm is subjected to a compressive axial load $P = 20$ kN. The modulus is $E = 200$ GPa.

Required: Determine the temperature increase ΔT that must be applied to the loaded bar to return it to its original length.



Solution: By applying temperature, the compressed bar is to expand to its original length, so the total elongation of the bar due to the mechanical and thermal loads is zero:

$$\Delta = \varepsilon L = \left(\frac{\sigma}{E} + \alpha \Delta T \right) L = 0 \Rightarrow \frac{\sigma}{E} + \alpha \Delta T = 0 \Rightarrow \frac{-P}{EA} + \alpha \Delta T = 0 \Rightarrow \Delta T = \frac{P}{\alpha EA} \Rightarrow$$

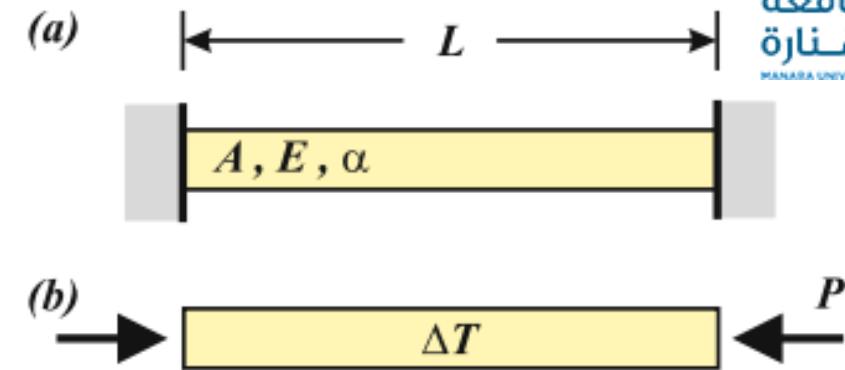
$$\Delta T = \frac{20 \times 10^3 \text{ [N]}}{14 \times 10^{-6} [\text{ }^{\circ}\text{C}^{-1}] \times (200 \times 10^9 [\text{N/m}^2]) \times (20 \times 20 \times 10^{-6} [\text{m}^2])} = 17.9 \text{ }^{\circ}\text{C}$$

Example 17. Aluminum Rod with Fixed Ends

Given: An aluminum rod fixed between two rigid supports

Required: Determine the stress in the rod when it is heated by 35°C .

Solution: The total change in length is zero:



$$\Delta = \varepsilon L = \left(\frac{\sigma}{E} + \alpha \Delta T \right) L = 0 \Rightarrow \frac{\sigma}{E} + \alpha \Delta T = 0 \Rightarrow \sigma = -E\alpha \Delta T \Rightarrow$$

$$\sigma = -(70 \times 10^9 [\text{Pa}]) \times (23 \times 10^{-6} [\text{ }^{\circ}\text{C}^{-1}]) (35 [\text{ }^{\circ}\text{C}]) = -56350 \times 10^3 [\text{Pa}] = -56.4 [\text{MPa}] \text{ (compression)}$$

Example 18. Loss of Prestress in Reinforced Concrete under Thermal Load

Given: A representative element of reinforced concrete (square cross-section 40×40 mm) surrounds a single high strength steel rebar (diameter $D = 10$ mm), as shown. The system is *prestressed* by tightening the rebar endcaps, which places the steel rebar in tension and the concrete in compression. No external load is applied to the system. The purpose of prestressing is to prevent tensile stresses in the concrete –and thus avoid fracture or cracking– by preloading the concrete in compression. Here, the rebar is under a tensile stress of $\sigma_{s,p} = 250$ MPa. Assume that the rebar and concrete remain the same length.

Required: Determine (a) the stress in the concrete after the prestressing process and (b) the loss of prestress in the concrete when the temperature increases from 20 to 40°C.

Solution: Step 1. Stress due to prestressing.

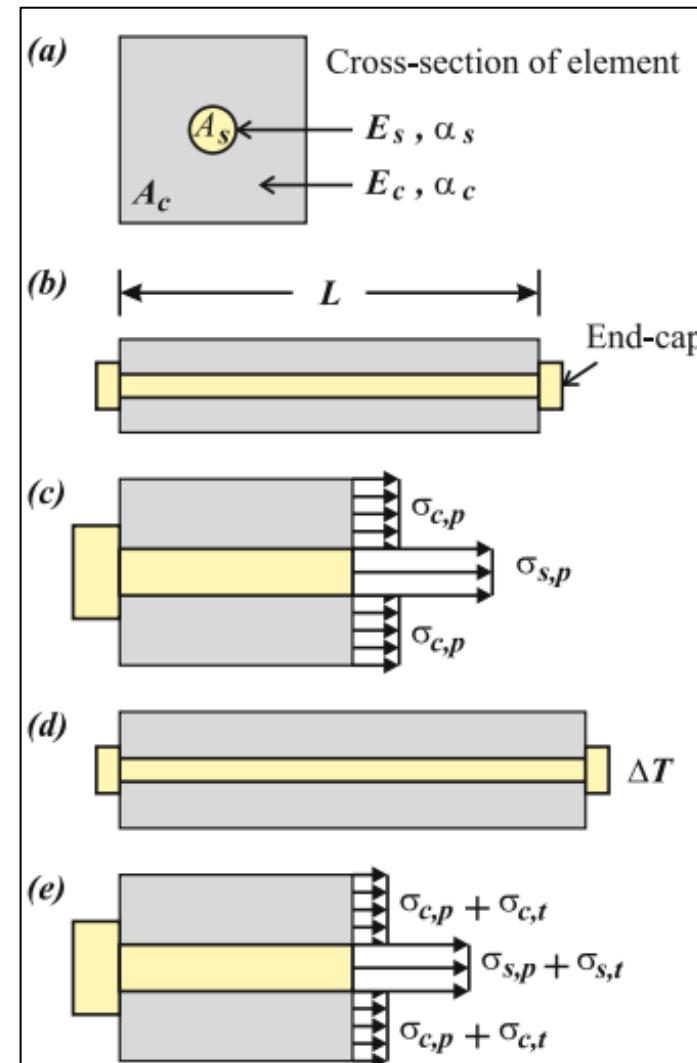
$$\text{The area of the steel is: } A_s = \frac{\pi}{4}(0.01 \text{ m})^2 = 78.54 \times 10^{-6} \text{ m}^2$$

The area of the concrete is thus:

$$A_c = (0.04 \times 0.04 \text{ m}^2) - A_s = 1600 \times 10^{-6} - 78.54 \times 10^{-6} = 1521 \times 10^{-6} \text{ m}^2$$

The stress in the concrete due to the mechanical prestress is found from equilibrium of (c):

$$\sigma_{s,p} A_s + \sigma_{c,p} A_c = 0 \Rightarrow \sigma_{c,p} = -\sigma_{s,p} A_s / A_c = -12.9 \text{ MPa} \text{ (compression)}$$



Step 2. Stress due to thermal loading. When temperature is applied, both materials expand figure (d). Unconstrained, the steel would expand more than the concrete. Here, since the materials are constrained to remain the same length, *thermal stresses* – stresses due to the thermal load – are *induced* in each material $\sigma_{S,t}$ and $\sigma_{C,t}$.

During thermal loading, no external mechanical load is applied to the system. Thus, the thermal stresses are in equilibrium: $\sigma_{S,t}A_S + \sigma_{C,t}A_C = 0 \Rightarrow \sigma_{C,t} = -\sigma_{S,t}A_S/A_C$

Consider the prestressed length as the reference for the thermally induced strain. Since the concrete and steel must expand or contract together, their strains due to thermal loading are the same. These strains are a combination of the free thermal expansion of each material, $\alpha\Delta T$, and the elastic strain due to the induced thermal stresses:

$$\varepsilon_{S,t} = \alpha_s \Delta T + \frac{\sigma_{S,t}}{E_s} = \varepsilon_{C,t} = \alpha_c \Delta T + \frac{\sigma_{C,t}}{E_c}$$

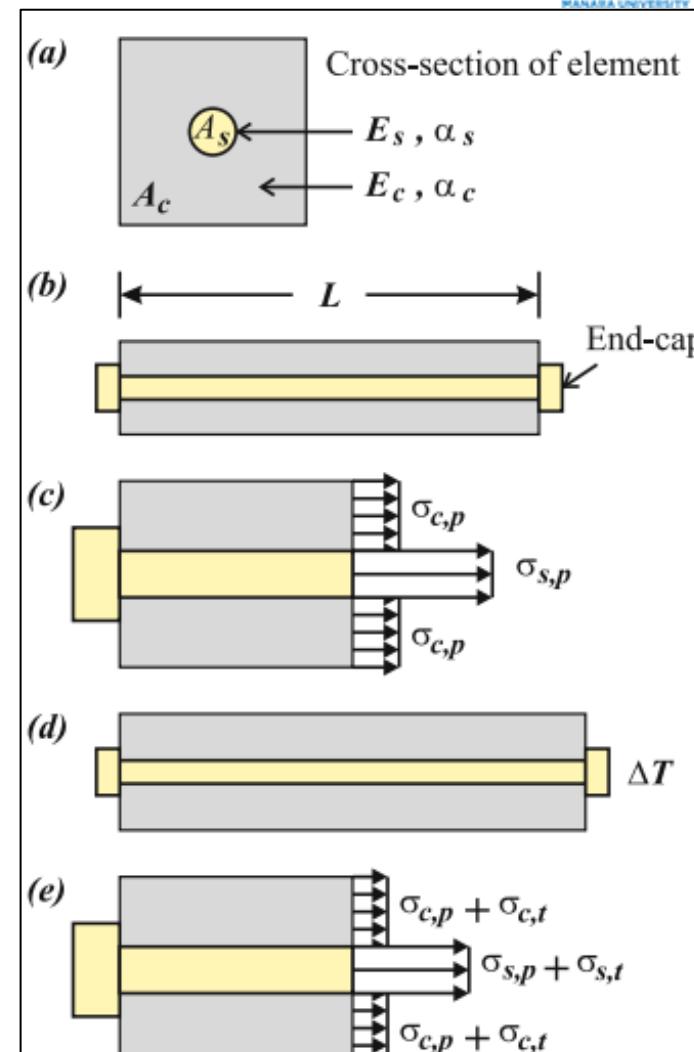
Rearranging and solving for the thermal stress in the steel gives:

$$\sigma_{S,t} = E_s(\alpha_c - \alpha_s)\Delta T + \frac{E_s}{E_c}\sigma_{C,t} = E_s(\alpha_c - \alpha_s)\Delta T - \sigma_{S,t} \frac{E_s A_s}{E_c A_c} \Rightarrow \sigma_{S,t} = \frac{E_s(\alpha_c - \alpha_s)}{1 + \frac{E_s A_s}{E_c A_c}} \Delta T$$

$$\sigma_{S,t} = \frac{200(7 - 14) \times 10^{-6}}{1 + [200(78.54)/30(1521)]} (40 - 20) = -20.83 \text{ MPa} \quad (\text{compression})$$

$$\sigma_{C,t} = -\sigma_{S,t}A_s/A_c = -(-20.83)(78.54/1521) = 1.08 \text{ MPa} \quad (\text{tension})$$

Step 3. Total stress: $\sigma_s = 250 - 20.83 = 229.17 \text{ MPa}$ $\sigma_c = -12.9 + 1.08 = -11.82 \text{ MPa}$



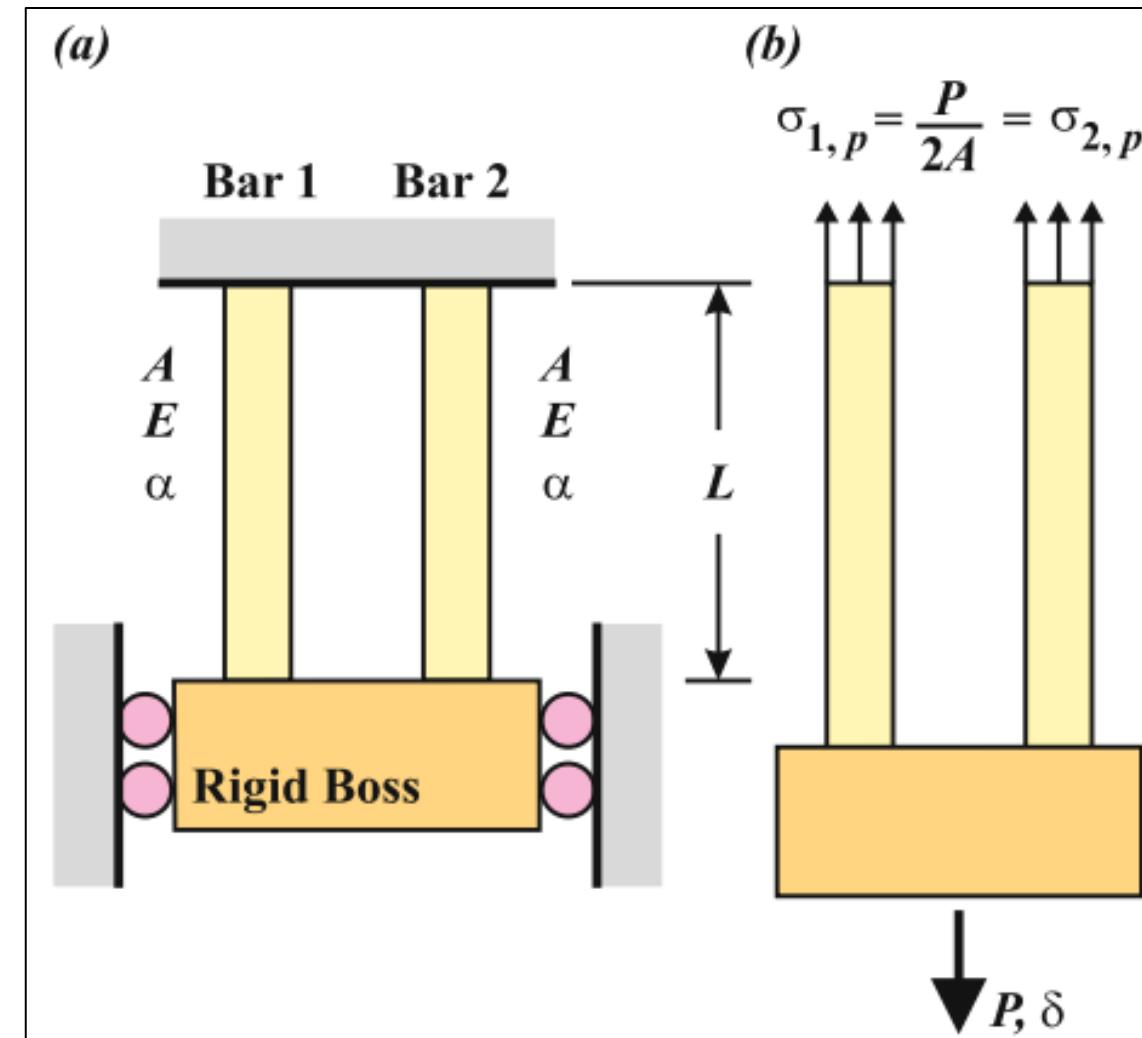
Example 19. Two-Bar Structure under Mechanical and Thermal Loads

Given: A two-bar structure, Bars 1 and 2. Each bar has length L , cross-sectional area A , modulus E , yield strength S_y , and thermal expansion coefficient α . The system is subjected to a tensile load P . Bar 2 is subjected to a thermal load ΔT greater than Bar 1. The bars are constrained to remain the same length. The material properties are assumed to be constant with temperature.

Required: (a) Determine the stress in each bar due to thermal and mechanical loading. (b) Determine the conditions to avoid yielding in terms of force P and temperature increase ΔT . Present the result on a plot.

Solution: Step 1. Mechanical loading. Since the bars are identical, due to applied load P , they support the same mechanical stress and have the same strain (Figure b):

$$\sigma_{1,P} = \sigma_{2,P} = \frac{P}{2A} \quad \text{and} \quad \varepsilon_{1,P} = \varepsilon_{2,P} = \frac{P}{2AE}$$



Step 2. Thermal loading. Apply temperature ΔT to Bar 2. Compatibility requires the same additional strain in each bar:

$$\varepsilon_{1,t} = \frac{\sigma_{1,t}}{E} = \varepsilon_{2,t} = \frac{\sigma_{2,t}}{E} + \alpha\Delta T$$

$$(c) \quad \sigma_1 = \frac{P}{2A} + \frac{E\alpha\Delta T}{2}$$

$$\sigma_2 = \frac{P}{2A} - \frac{E\alpha\Delta T}{2}$$

where $\sigma_{1,t}$ and $\sigma_{2,t}$ are the additional stresses induced in the bars by the increase in temperature of Bar 2. Equilibrium relates the *thermal stresses*:

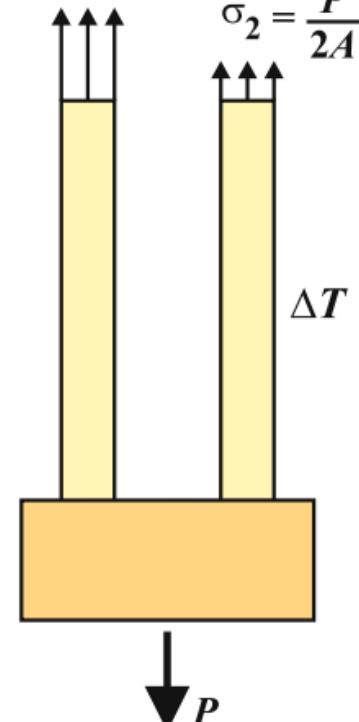
$$\sigma_{1,t}A + \sigma_{2,t}A = 0 \rightarrow \sigma_{1,t} = -\sigma_{2,t}$$

From compatibility of the thermal strains, and equilibrium, the induced thermal stresses are:

$$\sigma_{1,t} = \frac{E\alpha\Delta T}{2} \quad \text{and} \quad \sigma_{2,t} = -\frac{E\alpha\Delta T}{2}$$

Step 3. The total stress in each bar as a function of P and ΔT is found by superimposing the stresses from the mechanical and thermal cases (Figure c):

$$\sigma_1 = \sigma_{1,P} + \sigma_{1,t} = \frac{P}{2A} + \frac{E\alpha\Delta T}{2} \quad \text{and} \quad \sigma_2 = \sigma_{2,P} + \sigma_{2,t} = \frac{P}{2A} - \frac{E\alpha\Delta T}{2}$$



Step 4. Yielding. Since P and ΔT are positive in this case, then $\sigma_1 > \sigma_2$. Tensile yielding occurs when σ_1 reaches the yield strength S_y . To avoid yielding:

$$\sigma_1 = \frac{1}{2} \left[\frac{P}{A} + E\alpha\Delta T \right] < S_y$$

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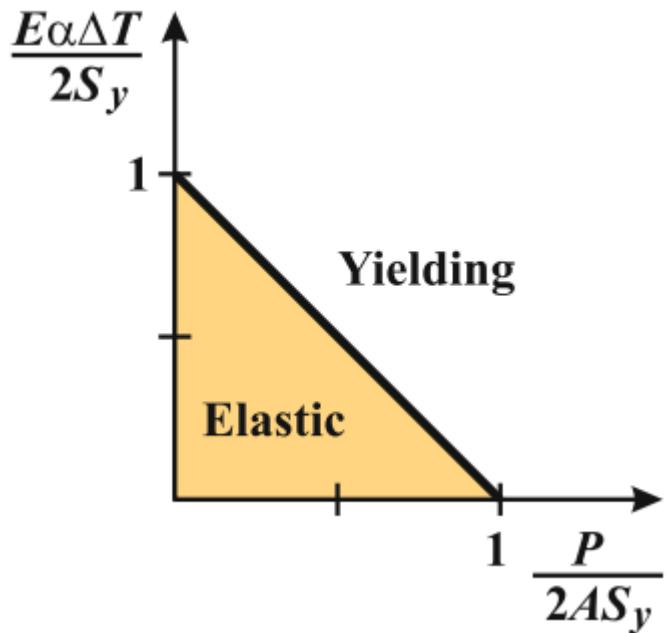
$$\sigma_1 = \frac{1}{2} \left[\frac{P}{A} + E\alpha\Delta T \right] < S_y \quad (d)$$

When $\Delta T = 0$, yielding occurs when: $P = P_y = 2AS_y$

When $P = 0$, yielding occurs when: $\Delta T = \Delta T_y = \frac{2S_y}{E\alpha}$

Normalizing σ_1 by the yield strength S_y , the equation to avoid yielding in Bar 1 reduces to

$$\frac{P}{2AS_y} + \frac{E\alpha\Delta T}{2S_y} < 1 \quad \text{or} \quad \frac{P}{P_y} + \frac{\Delta T}{\Delta T_y} < 1$$



A Temperature-Force Failure Map for the system can be plotted as shown in *Figure (d)*. The solid line is the boundary at which yielding occurs. Provided that the operating condition – temperature change ΔT and load P – lies within this boundary, yielding does not occur.