جَامعة المَنارة
كلية الهندسة

قسم الروبوت و الأنظمة الذكية

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نظم التحكم اللاخطي
Nonlinear Control systems

مدرس المقرر

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Introduction

- The subject of nonlinear control deals with the analysis and the design of nonlinear control systems, i.e., of control systems containing at least one nonlinear component.
- In the analysis, a nonlinear closed-loop system is assumed to have been designed, and we wish to determine the characteristics of the system's behavior.
- In the design, we are given a nonlinear plant to be controlled and some specifications of closed-loop system behavior, and our task is to construct a controller so that the closed loop system meets the desired characteristics.
- In practice, of course, the issues of design and analysis are intertwined, because the design of a nonlinear control system usually involves an iterative process of analysis and design.

Why Nonlinear Control ?

- Linear control is a mature subject with a variety of powerful methods and a long history of successful industrial applications. Thus, it is natural for one to wonder why so many researchers and designers, from such broad areas as aircraft and spacecraft control, robotics, process control, and biomedical engineering, have recently showed an active interest in the development and applications of nonlinear control methodologies. Many reasons can be cited for this interest:

Why Nonlinear Control ?

- **Improvement of existing control systems**
- Linear control methods rely on the key assumption of small range operation for the linear model to be valid. When the required operation range is large, a linear controller is likely to perform very poorly or to be unstable, because the nonlinearities in the system cannot be properly compensated for.
- Nonlinear controllers, on the other hand, may handle the nonlinearities in large range operation directly. This point is easily demonstrated in robot motion control problems.
- When a linear controller is used to control robot motion, it neglects the nonlinear forces associated with the motion of the robot links. The controller's accuracy thus quickly degrades as the speed of motion increases, because many of the dynamic forces involved, such as centripetal forces, vary as the square of the speed.

Why Nonlinear Control ?

- **Improvement of existing control systems**
- Therefore, in order to achieve a pre-specified accuracy in robot tasks such as pick-and-place, arc welding and laser cutting, the speed of robot motion, and thus productivity, has to be kept low. On the other hand, a conceptually simple nonlinear controller, commonly called computed torque controller, can fully compensate the nonlinear forces in the robot motion and lead to high accuracy control for a very large range of robot speeds and a large workspace.

Why Nonlinear Control ?

- **Analysis of hard nonlinearities**

Another assumption of linear control is that the system model is indeed linearizable.

However, in control systems there are many nonlinearities whose discontinuous nature does not allow linear approximation. These so-called "hard nonlinearities" include friction, saturation, dead-zones, backlash, and hysteresis, and are often found in control engineering.

Their effects cannot be derived from linear methods, and nonlinear analysis techniques must be developed to predict a system's performance in the presence of these inherent nonlinearities. Because such nonlinearities frequently cause undesirable behavior of the control systems, such as instabilities or limit cycles, their effects must be predicted and properly compensated for.

Why Nonlinear Control ?

- **Dealing with model uncertainties**

In designing linear controllers, it is usually necessary to assume that the parameters of the system model are reasonably well known. However, many control problems involve uncertainties in the model parameters. This may be due to a slow time variation of the parameters (e.g., of ambient air pressure during an aircraft flight), or to an abrupt change in parameters (e.g,, in the inertial parameters of a robot when a new object is grasped). A linear controller based on inaccurate or obsolete values of the model parameters may exhibit significant performance degradation or even instability. Nonlinearities can be intentionally introduced into the controller part of a control system so that model uncertainties can be tolerated. Two classes of nonlinear controllers for this purpose are robust controllers and adaptive controllers.

Why Nonlinear Control ?

- **Design Simplicity**
- Good nonlinear control designs may be simpler and more intuitive than their linear counterparts. This a priori paradoxical result comes from the fact that nonlinear controller designs are often deeply rooted in the physics of the plants. To take a very simple example, consider a swinging pendulum attached to a hinge, in the vertical plane. Starting from some arbitrary initial angle, the pendulum will oscillate and progressively stop along the vertical. Although the pendulum's behavior could be analyzed close to equilibrium by linearizing the system, physically its stability has very little to do with the eigenvalues of some linearized system matrix:

it comes from the fact that the total mechanical energy of the system is progressively dissipated by various friction forces (e.g., at the hinge), so that the pendulum comes to rest at a position of minimal energy.

Why Nonlinear Control ?

- **Design Simplicity**
- There may be other related or unrelated reasons to use nonlinear control techniques, such as cost and performance optimality. In industrial settings, ad-hoc extensions of linear techniques to control advanced machines with significant nonlinearities may result in unduly costly and lengthy development periods, where the control code comes with little stability or performance guarantees and is extremely hard to transport to similar but different applications.
- Linear control may require high quality actuators and sensors to produce linear behavior in the specified operation range, while nonlinear control may permit the use of less expensive components with nonlinear characteristics. As for performance optimality, we can cite bang-bang type controllers, which can produce fast response, but are inherently nonlinear.

Why Nonlinear Control ?

- **Design Simplicity**
- modern technology, such as high-speed high-accuracy robots or high-performance aircrafts, is demanding control systems with much more stringent design specifications.
- Nonlinear control occupies an increasingly conspicuous position in control engineering, as reflected by the ever-increasing number of papers and reports on nonlinear control research and applications.

Nonlinear System Behavior

- Physical systems are inherently nonlinear. Thus, all control systems are nonlinear to a certain extent. Nonlinear control systems can be described by nonlinear differential equations. However, if the operating range of a control system is small, and if the involved nonlinearities are smooth, then the control system may be reasonably approximated by a linearized system, whose dynamics is described by a set of linear differential equations.

Nonlinear System Behavior

- **NONLINEARITIES**
- Nonlinearities can be classified as inherent (natural) and intentional (artificial).
- Inherent nonlinearities are those which naturally come with the system's hardware and motion. Examples of inherent nonlinearities include centripetal forces in rotational motion, and Coulomb friction between contacting surfaces. Usually, such nonlinearities have undesirable effects, and control systems have to properly compensate for them.

Nonlinear System Behavior

- **NONLINEARITIES**

Intentional nonlinearities, on the other hand, are artificially introduced by the designer. Nonlinear control laws, such as adaptive control laws and bang-bang optimal control laws, are typical examples of intentional nonlinearities.

Nonlinearities can also be classified in terms of their mathematical properties, as continuous and discontinuous. Because discontinuous nonlinearities cannot be locally approximated by linear functions, they are also called "hard" nonlinearities.

- Hard nonlinearities (such as, e.g., backlash, hysteresis, or stiction) are commonly found in control systems, both in small range operation and large range operation.
- Whether a system in small range operation should be regarded as nonlinear or linear depends on the magnitude of the hard nonlinearities and on the extent of their effects on the system performance.

Nonlinear System Behavior

- **LINEAR SYSTEMS**
- Linear control theory has been predominantly concerned with the study of linear time- invariant (LTI) control systems, of the form

$$\mathbf{x} = \mathbf{A}\mathbf{x}$$

with **x** being a vector of states and **A** being the system matrix. LTI systems have quite simple properties, such as a linear system has a unique equilibrium point if **A** is nonsingular;

- the equilibrium point is stable if all eigenvalues of **A** have negative real parts, regardless of initial conditions;
- the transient response of a linear system is composed of the natural modes of the system, and the general solution can be solved analytically;
- in the presence of an external input **u(t)**, i.e., with

Nonlinear System Behavior

- LINEAR SYSTEMS

$$x=Ax+Bu$$

the system response has a number of interesting properties. First, it satisfies the principle of superposition. Second, the asymptotic stability of the system implies bounded-input bounded-output stability in the presence of **u**. Third, a sinusoidal input leads to a sinusoidal output of the same frequency.

- NONLINEAR SYSTEMS

Nonlinear System Behavior

- **AN EXAMPLE OF NONLINEAR SYSTEM BEHAVIOR**

however, is much more complex. Due to the lack of linearity and of the associated superposition property, nonlinear systems respond to external inputs quite differently from linear systems, as the following example illustrates.

Nonlinear System Behavior

- **AN EXAMPLE OF NONLINEAR SYSTEM BEHAVIOR**
- **Example** : A simplified model of the motion of an underwater vehicle can be written

$$\mathbf{v' + |v|v = u}$$

where **v** is the vehicle velocity and **u** is the control input (the thrust provided by a propeller). The nonlinearity **|v|v** corresponds to a typical "square-law" drag. Assume that we apply a unit step input in thrust **u**, followed **5** seconds later by a negative unit step input. The system response is plotted in Figure. We see that the system settles much faster in response to the positive unit step than it does in response to the subsequent negative unit step.

Nonlinear System Behavior

- **Example** :
- Intuitively, this can be interpreted as reflecting the fact that the "apparent damping" coefficient **|v|** is larger at high speeds than at low speeds. Assume now that we repeat the same experiment but with larger steps, of amplitude **10**. Predictably, the difference between the settling times in response to the positive and negative steps is even more marked. Furthermore, the settling speed **v_s** in response to the first step is not **10** times that obtained in response to the first unit step in the first experiment, as it would be in a linear system. This can again be understood intuitively, by writing that

Nonlinear System Behavior

- **Example** :

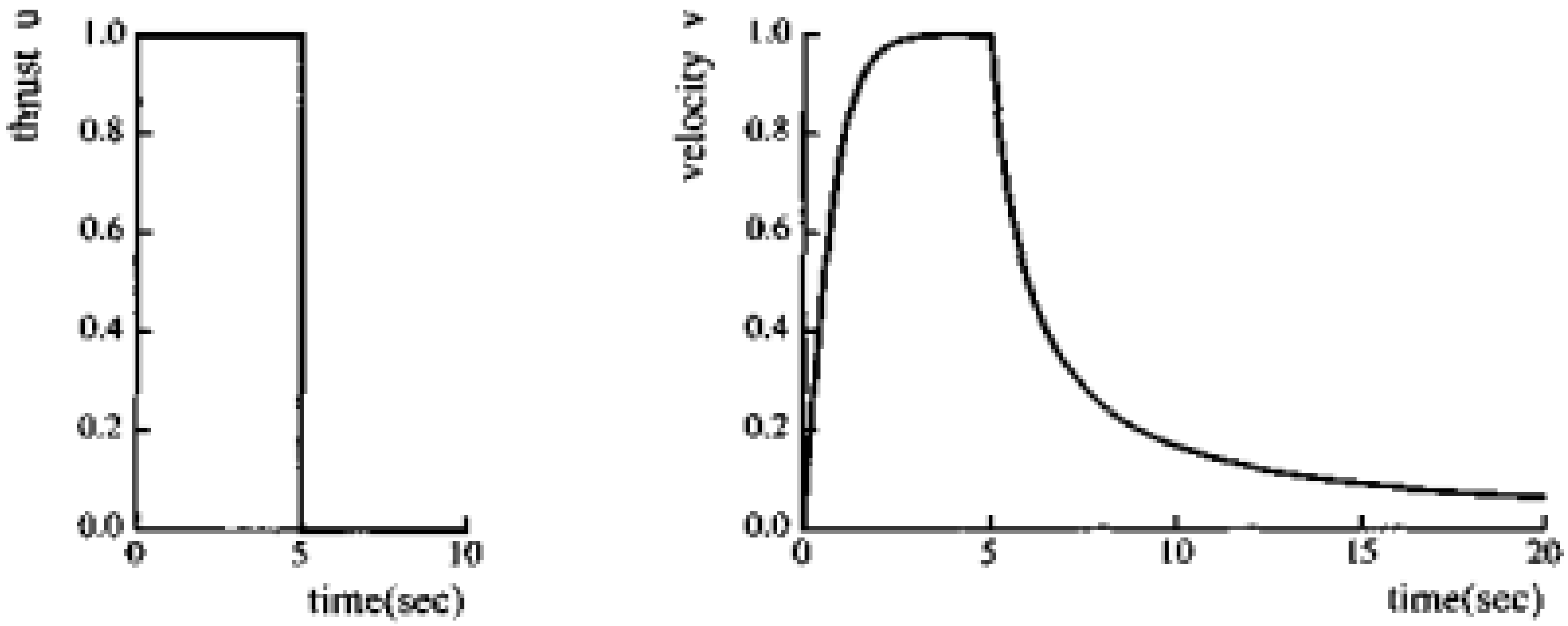


Figure 1.1 : Response of system (*1.3*) to unit steps

Nonlinear System Behavior

- **Example** :

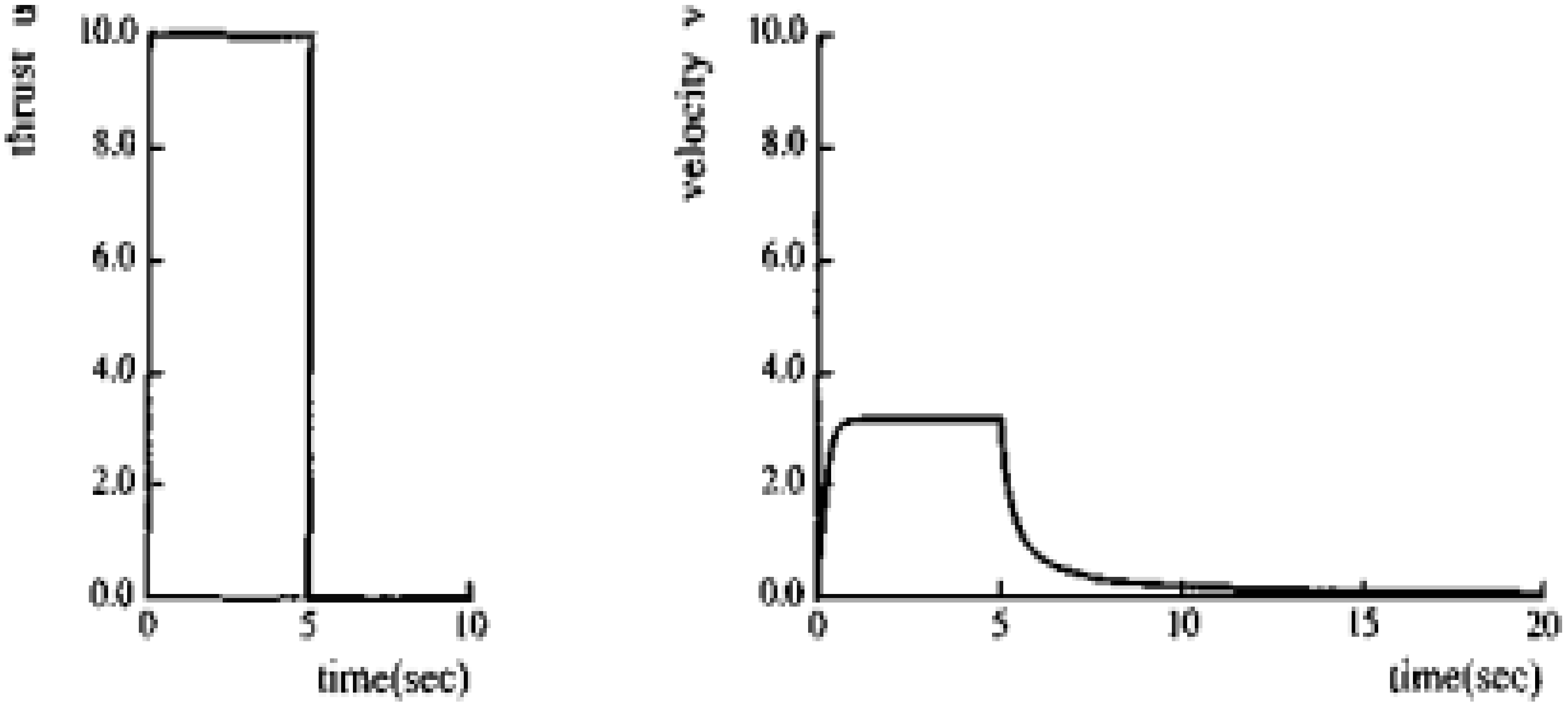


Figure 1.2 : Response of system (1.3) to steps of amplitude 10

Nonlinear System Behavior

- **Example** :

$u = 1 \rightarrow 0 + |v_s|\, v_s = 1 \rightarrow v_s = 1$

$u = 10 \rightarrow 0 + |v_s|\, v_s = 10 \rightarrow v_s = \sqrt{10} \approx 3.2$

Carefully understanding and effectively controlling this nonlinear behavior is particularly important if the vehicle is to move in a large dynamic range and change speeds continually, as is typical of industrial remotely-operated underwater vehicles (R.O.V.'s).

Nonlinear System Behavior

SOME COMMON NONLINEAR SYSTEM BEHAVIORS

Let us now discuss some common nonlinear system properties, so as to familiarize ourselves with the complex behavior of nonlinear systems and provide a useful background for our study.

Nonlinear System Behavior

SOME COMMON NONLINEAR SYSTEM BEHAVIORS

Multiple Equilibrium Points

Nonlinear System Behavior Nonlinear systems frequently have more than one equilibrium point (an equilibrium point is a point where the system can stay forever without moving, as we shall formalize later). This can be seen by the following simple example.

Nonlinear System Behavior

SOME COMMON NONLINEAR SYSTEM BEHAVIORS

Multiple Equilibrium Points   Example

A first-order system Consider the first order system

$$\dot{x} = -x + x^2$$

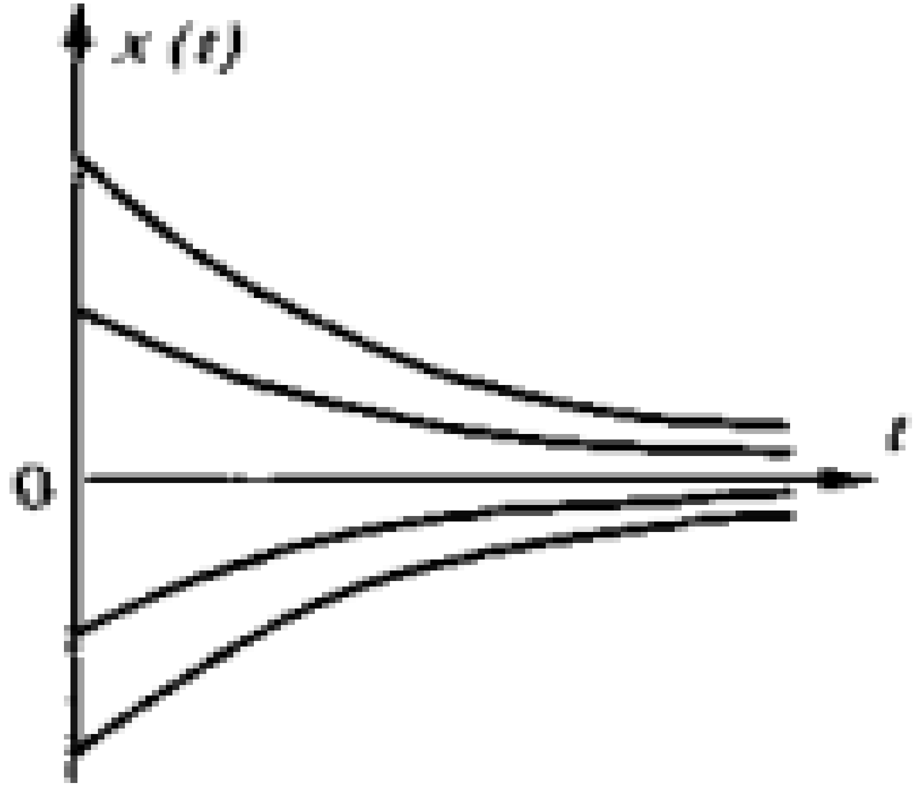
with initial condition $x(0) = x_0$. Its linearization is

$$\dot{x} = -x$$

The solution of this linear equation is $\mathbf{x(t) = x_0 e^{-t}}$. It is plotted in Figure for various initial conditions. The linearized system clearly has a unique equilibrium point at **x =0.**

Nonlinear System Behavior

Multiple Equilibrium Points   Example



(a)

Equilibrium point

- Equilibrium point doesn't have to be unique.

Ex: $\ddot{y} + \dot{y} + \sin y = 0$ (pendulum)

$$\left.\begin{aligned} y &= x_1 \\ \dot{y} &= x_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \sin x_1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{2s} &= 0 \\ \sin x_{1s} &= 0 \end{aligned}$$

$$\text{Equilibrium point } x_s = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \quad n = 0, 1, \cdots$$

Ex: $\ddot{y} + \dot{y} - y + y^3 = 0$ (Rayleigh eq.)

$$\left.\begin{aligned} \dot{x}_1 &= x_2 = 0 \\ \dot{x}_2 &= -x_2 - x_1 - x_1^3 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{2s} &= 0 \\ x_{1s}(1 - x_{1s}^2) &= 0 \end{aligned}$$

$$x_s^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x_s^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_s^3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Nonlinear System Behavior

- **Limit Cycles**

Nonlinear systems can display oscillations of fixed amplitude and fixed period without external excitation. These oscillations are called limit cycles, or self-excited oscillations. This important phenomenon can be simply illustrated by a famous oscillator dynamics, first studied in the 1920's by the Dutch electrical engineer Balthasar Van der Pol.

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**

The second-order nonlinear differential equation

$$m\ddot{x} + 2c(x^2 - l)\,\dot{x} + kx = 0$$

where **m, c** and **k** are positive constants, is the famous Van der Pol equation. It can be regarded as describing a mass-spring-damper system with a position-dependent damping coefficient **2c(x2 - 1)** (or, equivalently, an RLC electrical circuit with a nonlinear resistor).

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**
- For large values of **x**, the damping coefficient is positive and the damper removes energy from the system. This implies that the system motion has a convergent tendency. However, for small values of **x**, the damping coefficient is negative and the damper adds energy into the system. This suggests that the system motion has a divergent tendency.

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**
- Therefore, because the nonlinear damping varies with **x**, the system motion can neither grow unboundedly nor decay to zero. Instead, it displays a sustained oscillation independent of initial conditions, as illustrated in Figure. This so-called limit cycle is sustained by periodically releasing energy into and absorbing energy from the environment, through the damping term. This is in contrast with the case of a conservative mass spring system, which does not exchange energy with its environment during its vibration.

Nonlinear System Behavior

- **Limit Cycles   Example**
- **Van der Pol Equation**

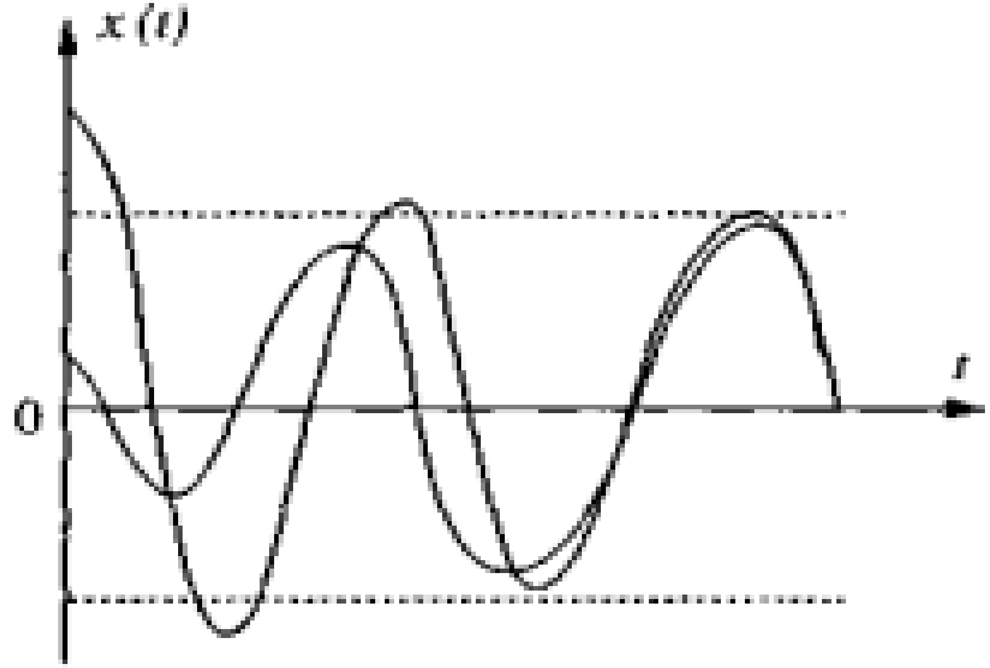


Figure 1.4 : Responses of the Van der Pol oscillator

Nonlinear System Behavior

- **Limit Cycles**
- Of course, sustained oscillations can also be found in linear systems, in the case of marginally stable linear systems (such as a mass-spring system without damping) or in the response to sinusoidal inputs. However, limit cycles in nonlinear systems are different from linear oscillations in a number of fundamental aspects. First, the amplitude of the self-sustained excitation is independent of the initial condition, as seen in Figure, while the oscillation of a marginally stable linear system has its amplitude determined by its initial conditions. Second, marginally stable linear systems are very sensitive to changes in system parameters (with a slight change capable of leading either to stable convergence or to instability), while limit cycles are not easily affected by parameter changes.

Nonlinear System Behavior

- **Limit Cycles**

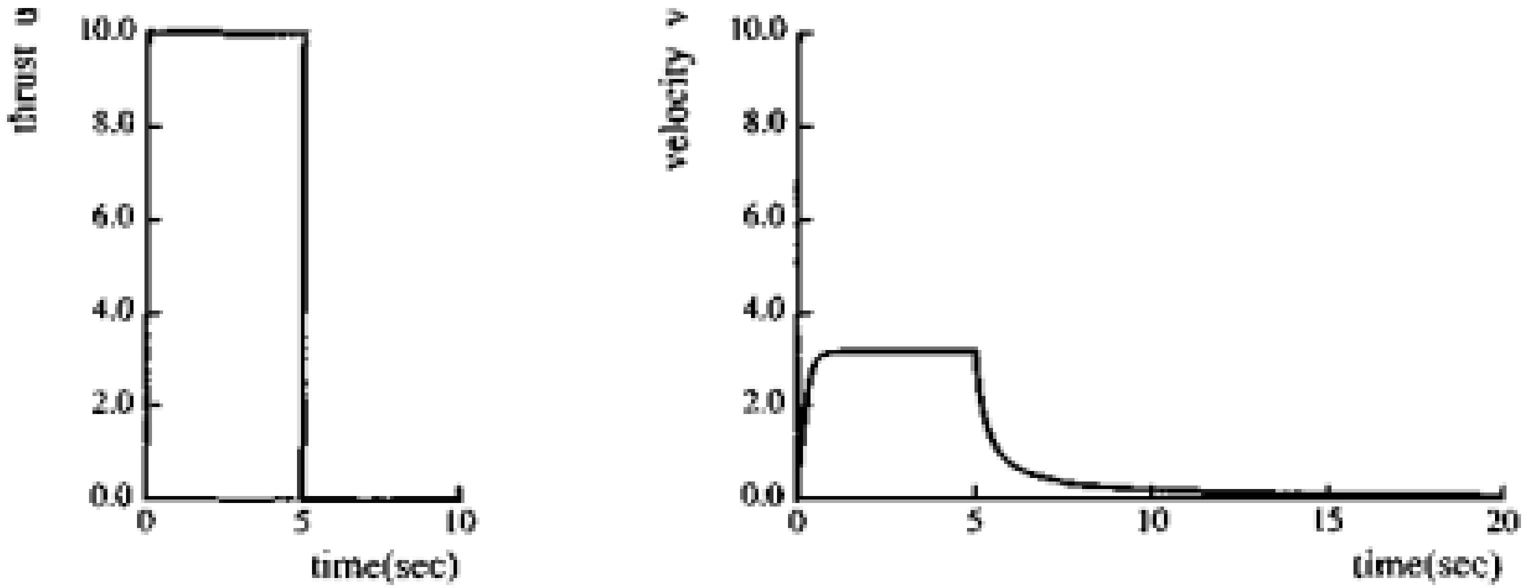


Figure 1.2 : Response of system (1.3) to steps of amplitude 10

Nonlinear System Behavior

- **Limit Cycles**
- Limit cycles represent an important phenomenon in nonlinear systems. They can be found in many areas of engineering and nature. Aircraft wing fluttering, a limit cycle caused by the interaction of aerodynamic forces and structural vibrations, is frequently encountered and is sometimes dangerous. The hopping motion of a legged robot is another instance of a limit cycle. Limit cycles also occur in electrical circuits, e.g., in laboratory electronic oscillators. As one can see from these examples, limit cycles can be undesirable in some cases, but desirable in other cases. An engineer has to know how to eliminate them when they are undesirable, and conversely how to generate or amplify them when they are desirable. To do this, however, requires an understanding of the properties of limit cycles and a familiarity with the tools for manipulating them.

Nonlinear System Behavior

- **Bifurcations**
- As the parameters of nonlinear dynamic systems are changed, the stability of the equilibrium point can change (as it does in linear systems) and so can the number of equilibrium points. Values of these parameters at which the qualitative nature of the system's motion changes are known as critical or bifurcation values. The phenomenon of bifurcation, i.e., quantitative change of parameters leading to qualitative change of system properties, is the topic of bifurcation theory.

Nonlinear System Behavior

- **Bifurcations**
- For instance, the smoke rising from an incense stick (smokestacks and cigarettes are old-fashioned) first accelerates upwards (because it is lighter than the ambient air), but beyond some critical velocity breaks into swirls. More prosaically, let us consider the system described by the so-called undamped Duffing equation

$$\ddot{x} + \alpha x + x^3 = 0$$

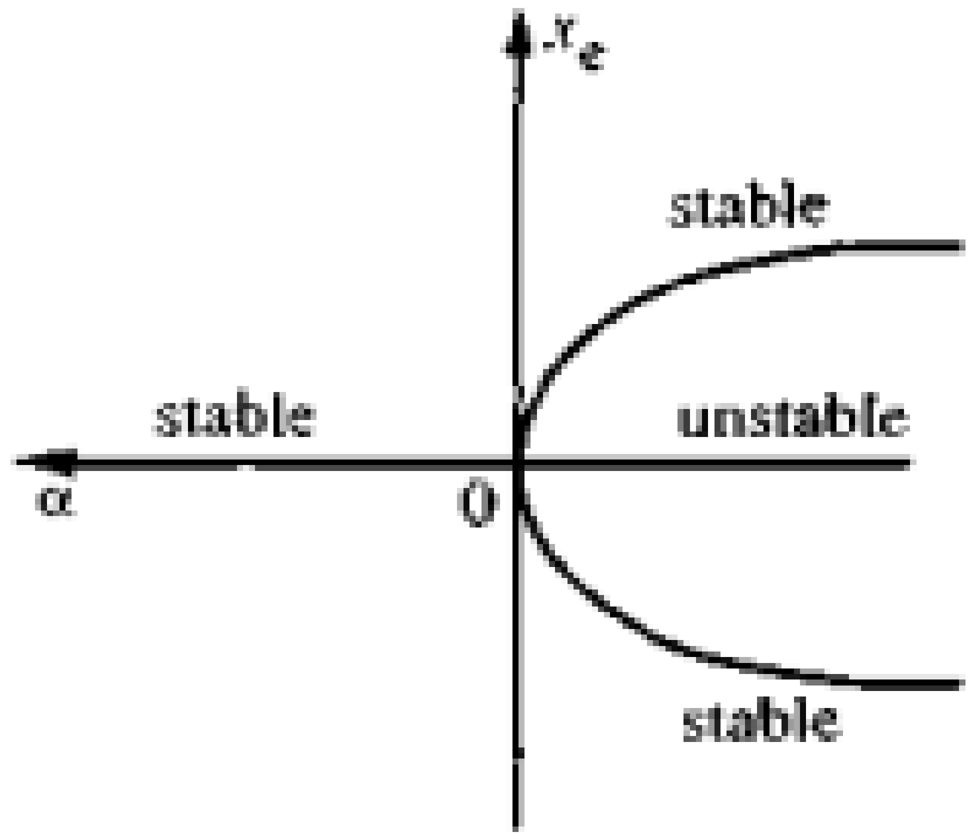
(the damped Duffing equation is $\ddot{x} + c\dot{x} + \alpha x + \beta x^3 = 0$, which may represent a mass-damper-spring system with a hardening spring).

Nonlinear System Behavior

- **Bifurcations**
- We can plot the equilibrium points as a function of the parameter **α**. As **α** varies from positive to negative, one equilibrium point splits into three points ($\mathbf{x_e = 0, \sqrt{\alpha}, -\sqrt{\alpha}}$), as shown in Figure (a). This represents a qualitative change in the dynamics and thus **α = 0** is a critical bifurcation value. This kind for bifurcation is known as a pitchfork, due to the shape of the equilibrium point plot in Figure (a).

Nonlinear System Behavior

- **Bifurcations**



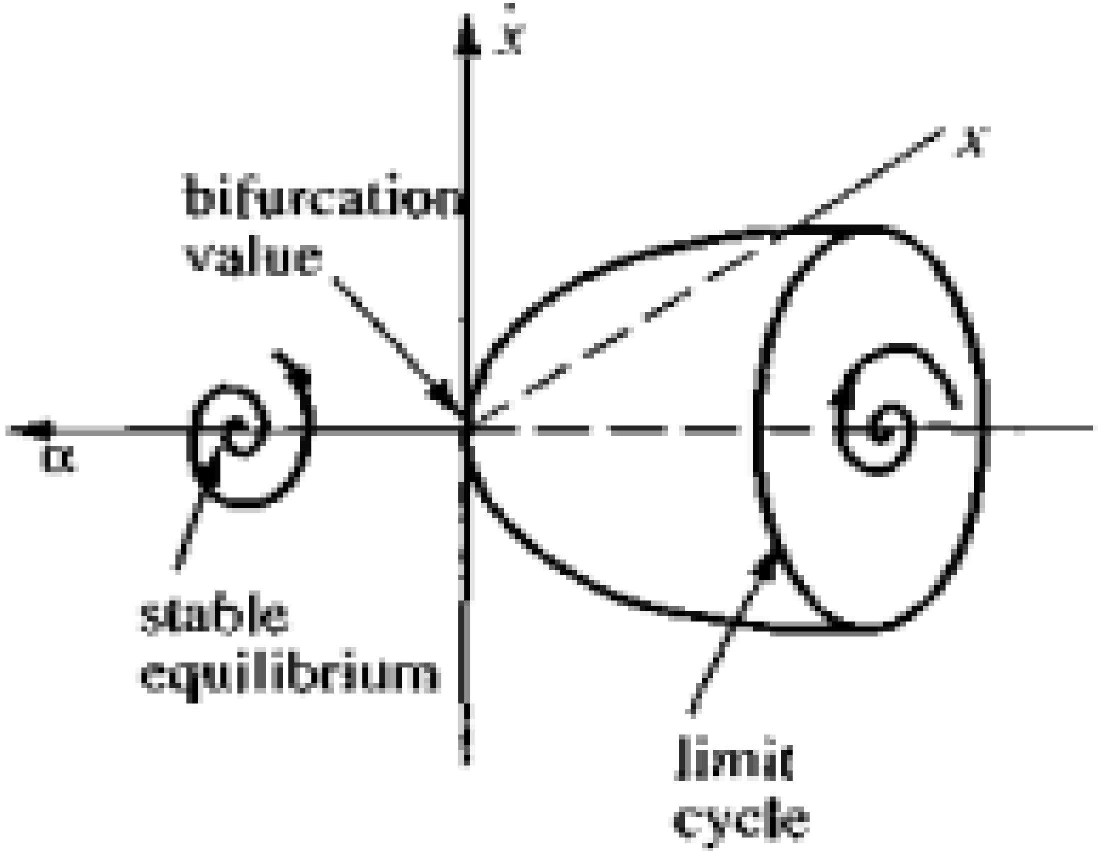
(a)

Nonlinear System Behavior

- **Bifurcations**
- Another kind of bifurcation involves the emergence of limit cycles as parameters are changed. In this case, a pair of complex conjugate eigenvalues $\mathbf{P_1 = \Upsilon + j\omega}$, $\mathbf{P_2 = \Upsilon - j\omega}$ cross from the left-half plane into the right-half plane, and the response of the unstable system diverges to a limit cycle. Figure (b) depicts the change of typical system state trajectories (states are $\mathbf{x}$ and $\dot{\mathbf{x}}$) as the parameter a is varied. This type of bifurcation is called a Hopf bifurcation**.**

Nonlinear System Behavior

- **Bifurcations**



(b)

Nonlinear System Behavior

- **Chaos**
- For stable linear systems, small differences in initial conditions can only cause small differences in output. Nonlinear systems, however, can display a phenomenon called chaos, by which we mean that the system output is extremely sensitive to initial conditions. The essential feature of chaos is the unpredictability of the system output. Even if we have an exact model of a nonlinear system and an extremely accurate computer, the system's response in the long-run still cannot be well predicted.

Nonlinear System Behavior

- **Chaos**
- Chaos must be distinguished from random motion. In random motion, the system model or input contain uncertainty and, as a result, the time variation of the output cannot be predicted exactly (only statistical measures are available). In chaotic motion, on the other hand, the involved problem is deterministic, and there is little uncertainty in system model, input, or initial conditions.

Nonlinear System Behavior

- **Chaos**
- As an example of chaotic behavior, let us consider the simple nonlinear system

$$\ddot{x} + 0.1\dot{x} + x^5 = 6 \sin t$$

- which may represent a lightly-damped, sinusoidally forced mechanical structure undergoing large elastic deflections. Figure shows the responses of the system corresponding to two almost identical initial conditions, namely $\mathbf{x(0) = 2, \dot{x}(0) = 3}$ (thick line) and $\mathbf{x(0) = 2.01, \dot{x}(0) = 3.01}$ (thin line). Due to the presence of the strong nonlinearity in $\mathbf{x^5}$, the two responses are radically different after some time.

Nonlinear System Behavior

- **Chaos**

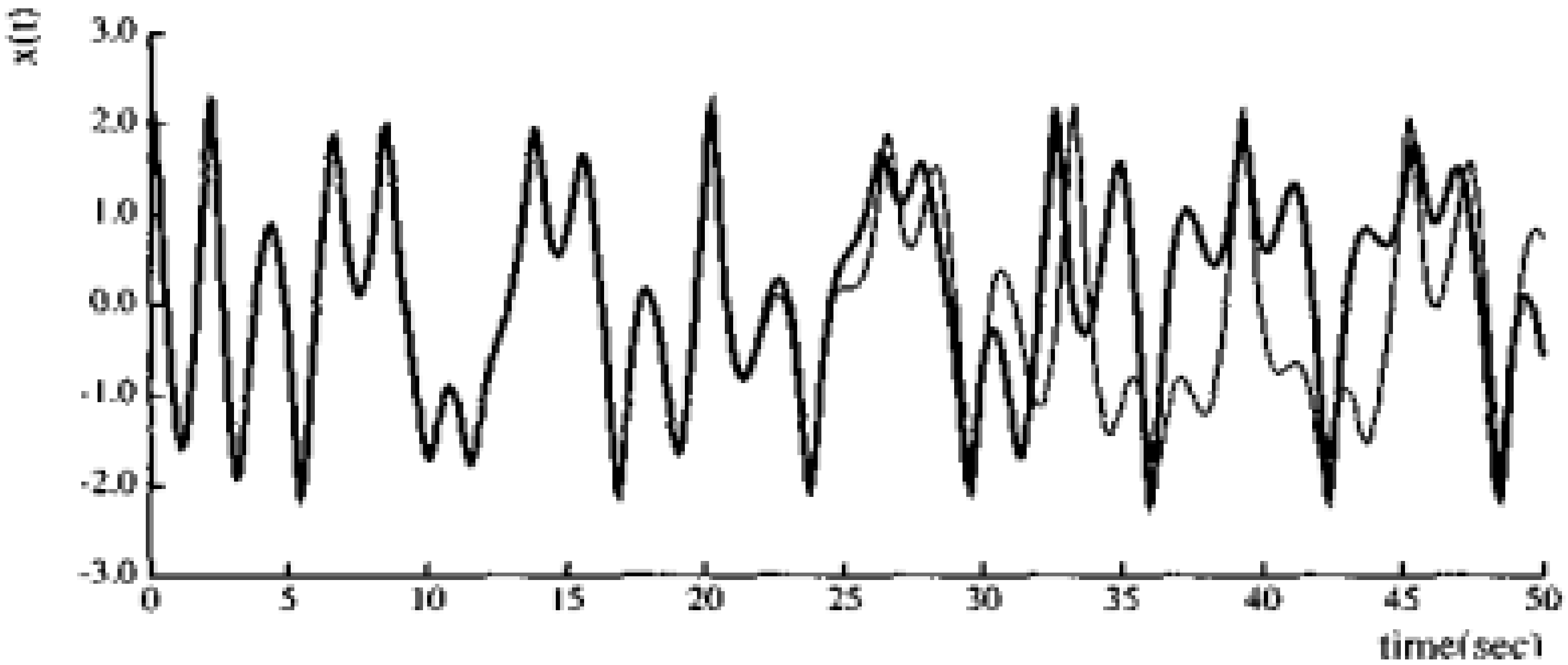


Figure 1.6 : Chaotic behavior of a nonlinear system

Nonlinear System Behavior

- **Chaos**
- Chaotic phenomena can be observed in many physical systems. The most commonly seen physical problem is turbulence in fluid mechanics (such as the swirls of our incense stick). Atmospheric dynamics also display clear chaotic behavior, thus making long-term weather prediction impossible. Some mechanical and electrical systems known to exhibit chaotic vibrations include buckled elastic structures, mechanical systems with play or backlash, systems with aeroelastic dynamics, wheelrail dynamics in railway systems, and, of course, feedback control devices.

Nonlinear System Behavior

- **Chaos**
- Chaos occurs mostly in strongly nonlinear systems. This implies that, for a given system, if the initial condition or the external input cause the system to operate in a highly nonlinear region, it increases the possibility of generating chaos. Chaos cannot occur in linear systems. Corresponding to a sinusoidal input of arbitrary magnitude, the linear system response is always a sinusoid of the same frequency. By contrast, the output of a given nonlinear system may display sinusoidal, periodic, or chaotic behaviors, depending on the initial condition and the input magnitude.

Nonlinear System Behavior

- **Chaos**
- In the context of feedback control, it is of course of interest to know when a nonlinear system will get into a chaotic mode (so as to avoid it) and, in case it does, how to recover from it. Such problems are the object of active research.

Nonlinear System Behavior

- **Other behaviors**
- Other interesting types of behavior, such as jump resonance, subharmonic generation, asynchronous quenching, and frequency-amplitude dependence of free vibrations, can also occur and become important in some system studies. However, the above description should provide ample evidence that nonlinear systems can have considerably richer and more complex behavior than linear systems.

Analysis methods

- 1. Linearisation Technique
- 2. Phase Plane Method
- 3. Describing Function Method
- 4. Lyapunov Stability Analysis

In Design NL.sys

- 1. Linear
- 2. Optimal
- 3. Adaptive
- 4. Robust

- Linear

$$\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2}$$

2nd order
not
2nd degree

$$\sin t\left(\frac{dy}{dt}\right)$$

- Nonlinear

$$\left(\frac{dy}{dt}\right)^2$$

$$\sin x\left(\frac{dy}{dt}\right)$$

$$xy$$

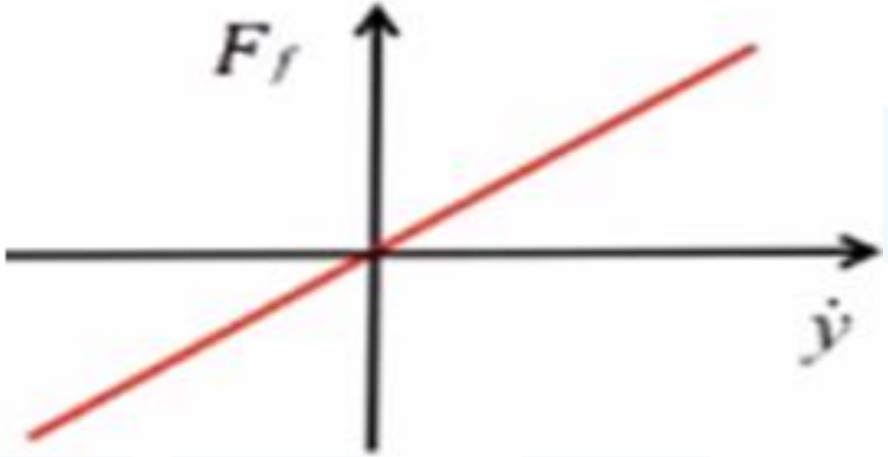
$$x^3$$

Friction force

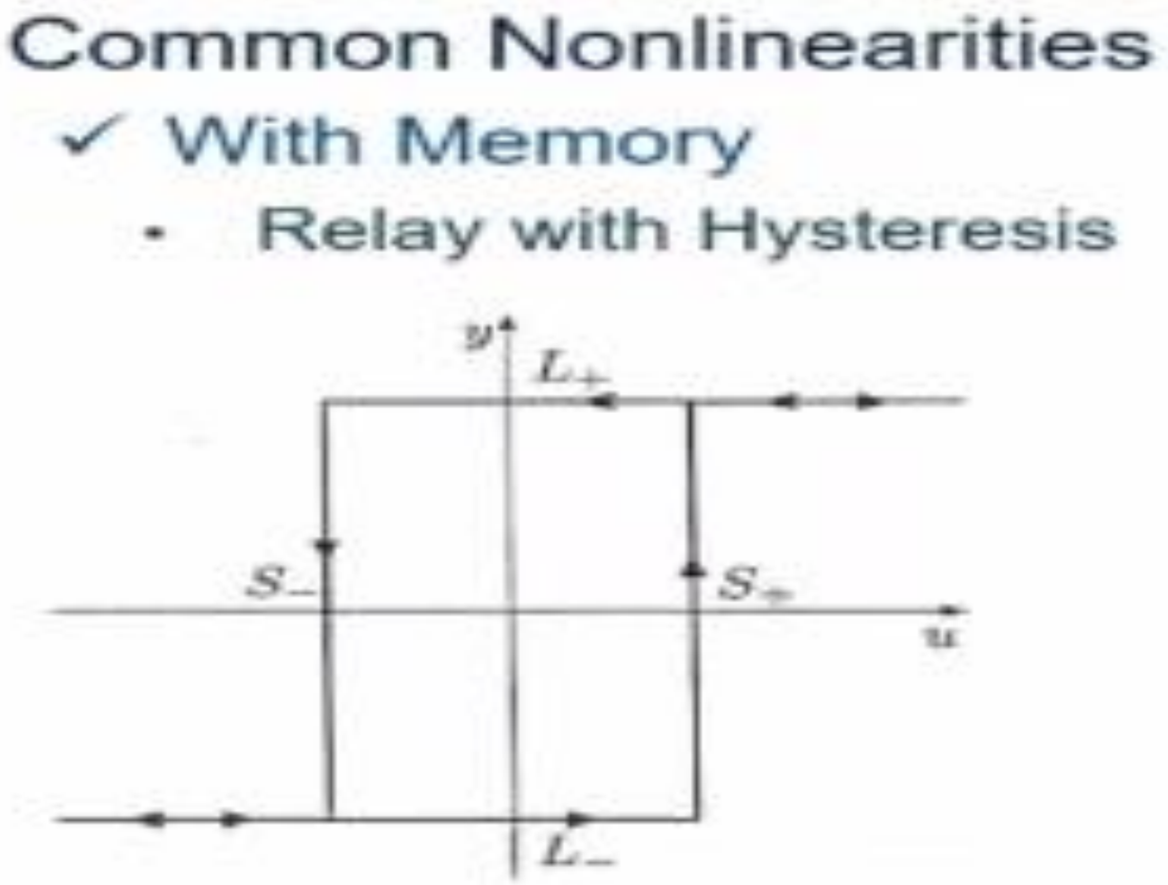
Nonlinear System Examples

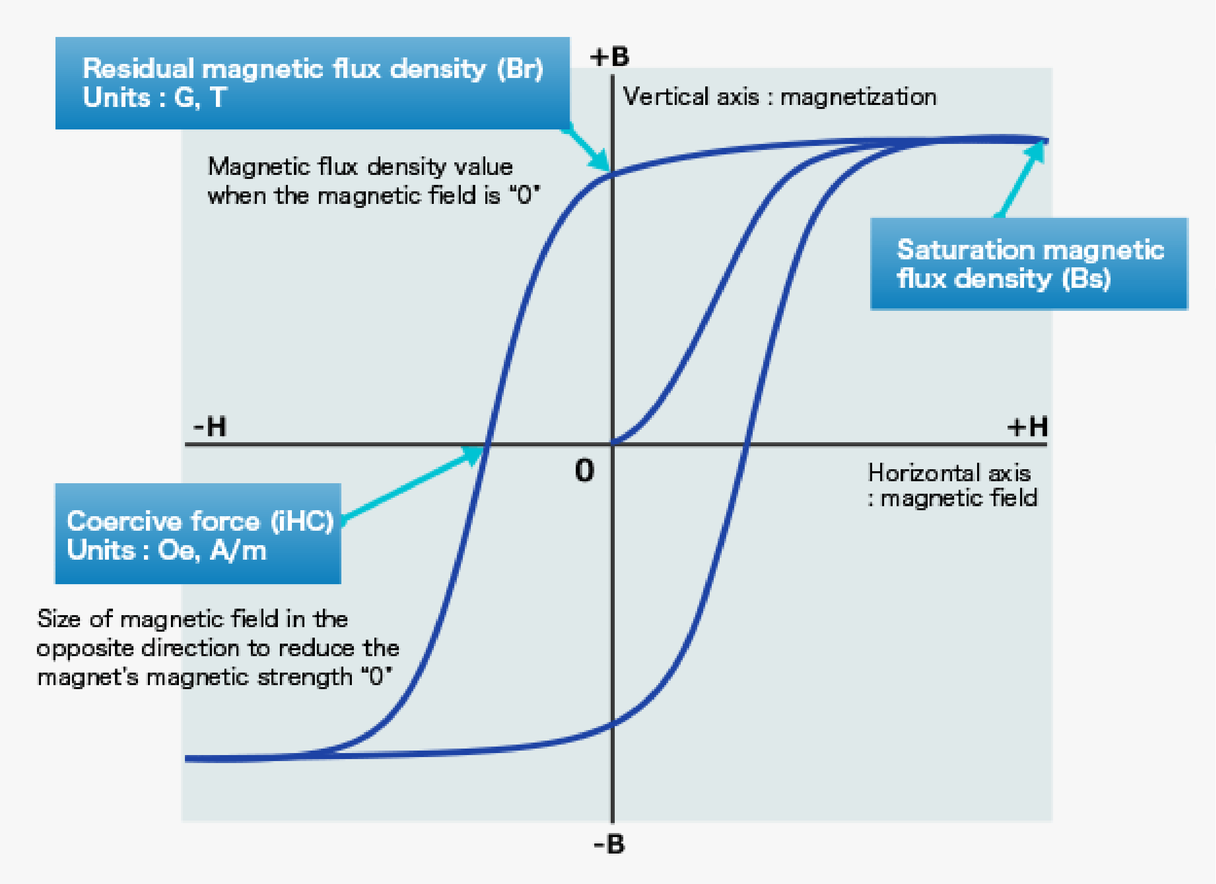
✓ Mass-Spring System

- Friction Force F_f
  - Viscous Friction $F_f = c\,\dot{y}$

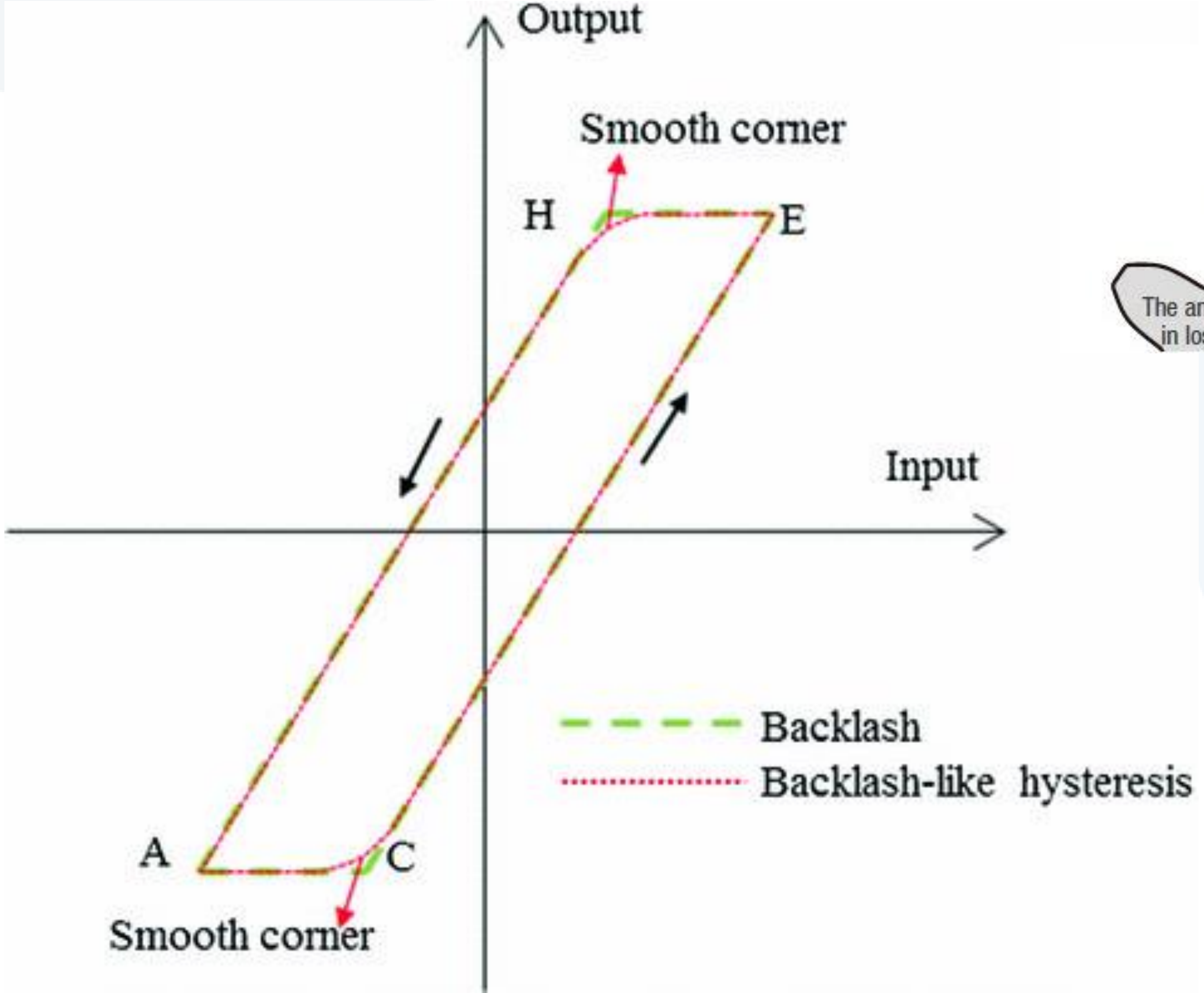


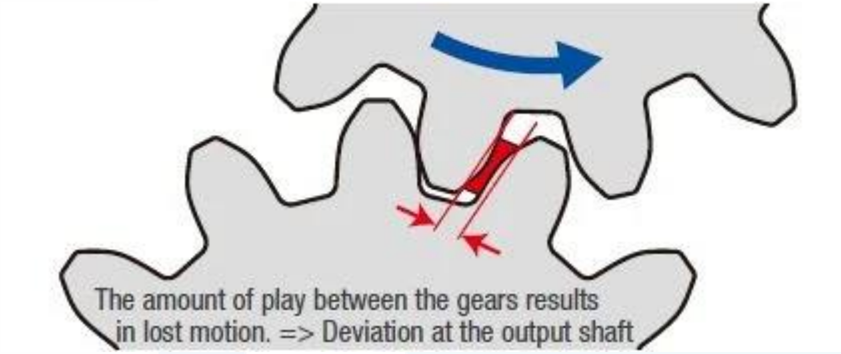
hysteresis


Common Nonlinearities
✓ With Memory
• Relay with Hysteresis
y
L+
S−
S+
u
L−

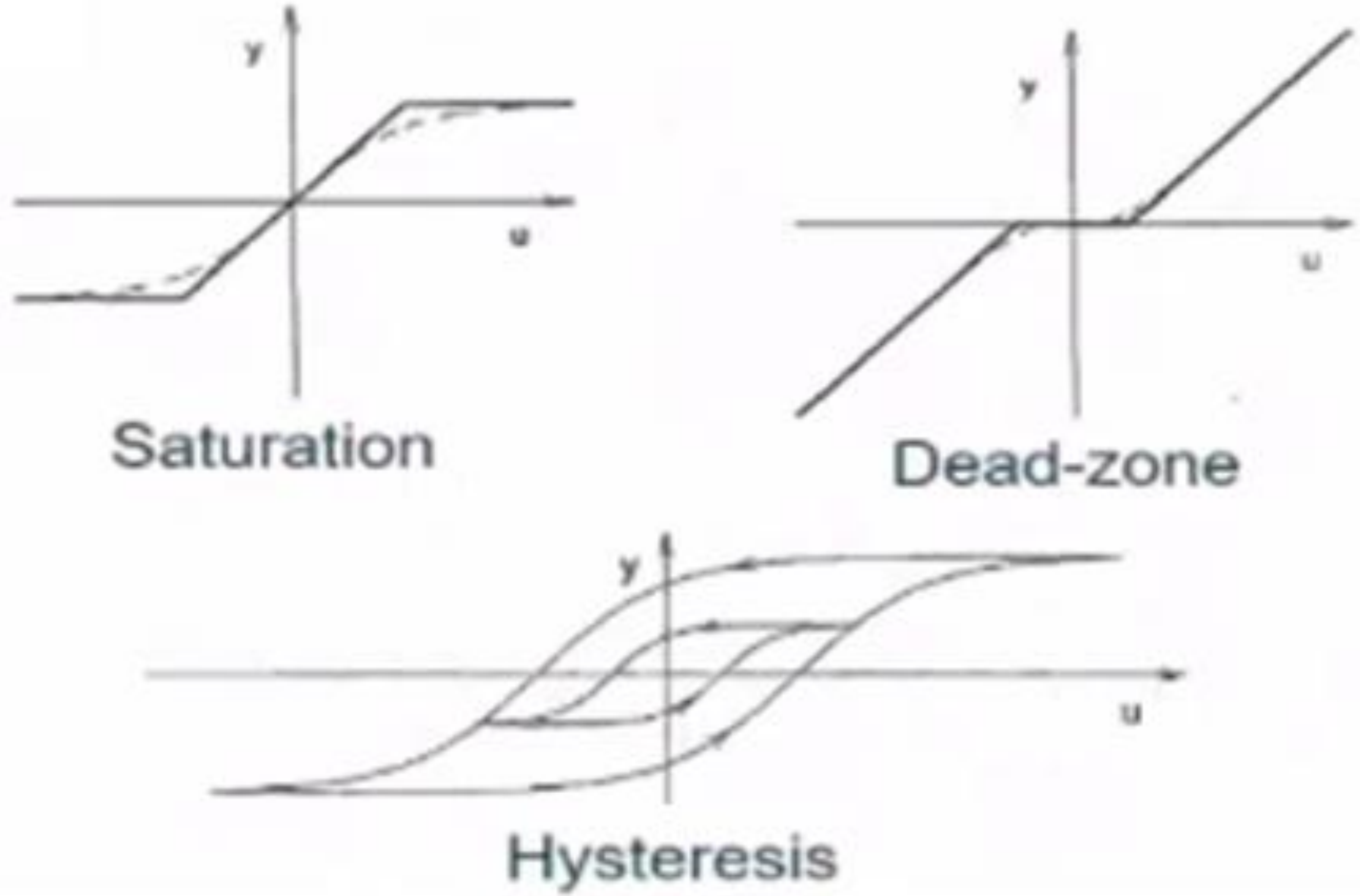

Residual magnetic flux density (Br)
Units : G, T
Magnetic flux density value
when the magnetic field is "0"
+B
Vertical axis : magnetization
Saturation magnetic
flux density (Bs)
-H
+H
0
Horizontal axis
: magnetic field
Coercive force (iHC)
Units : Oe, A/m
Size of magnetic field in the
opposite direction to reduce the
magnet's magnetic strength "0"
-B

Backlash

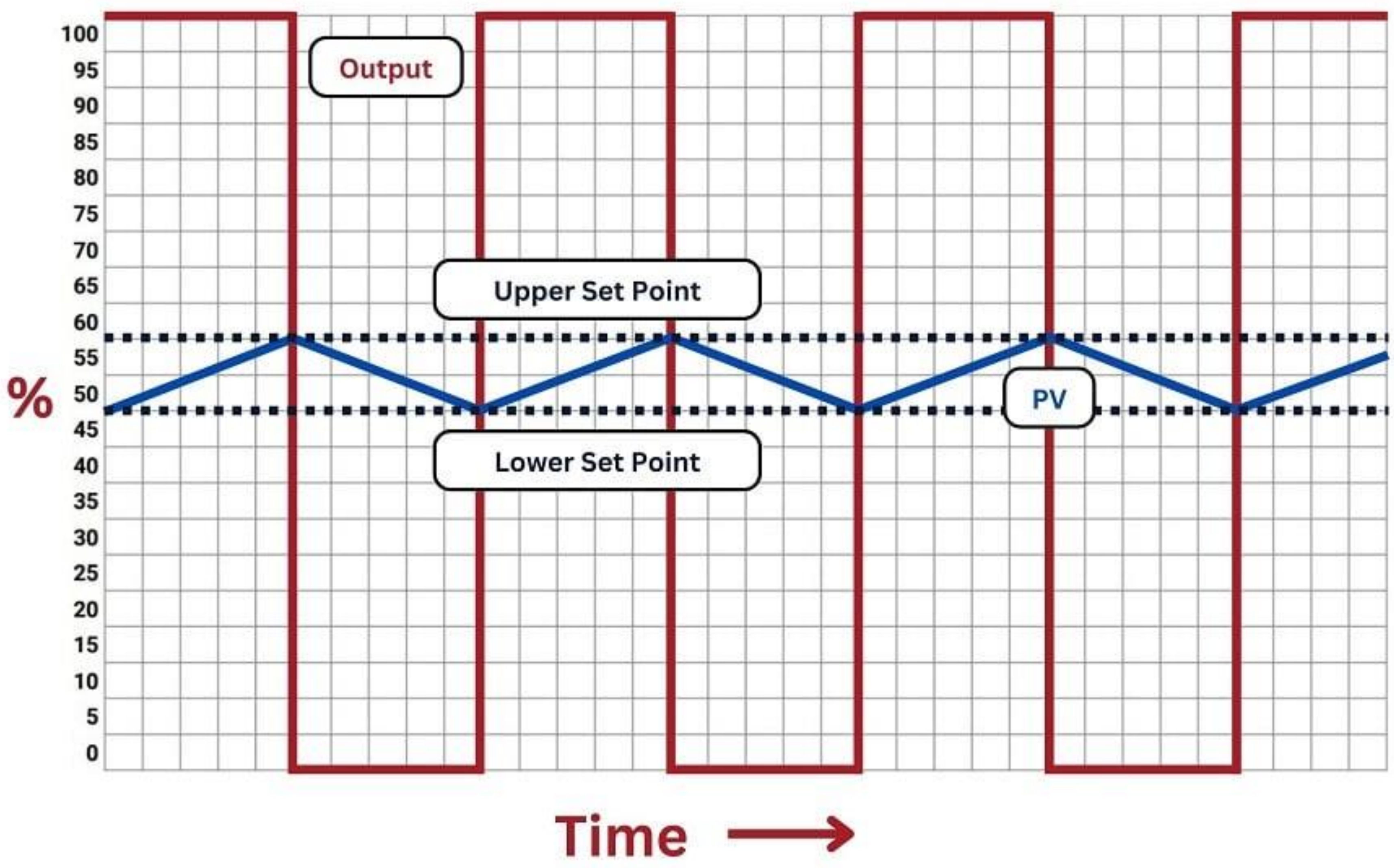

Output
Input
Smooth corner
H
E
A
C
Smooth corner
Backlash
Backlash-like hysteresis


The amount of play between the gears results in lost motion. => Deviation at the output shaft

Saturation –dead zone

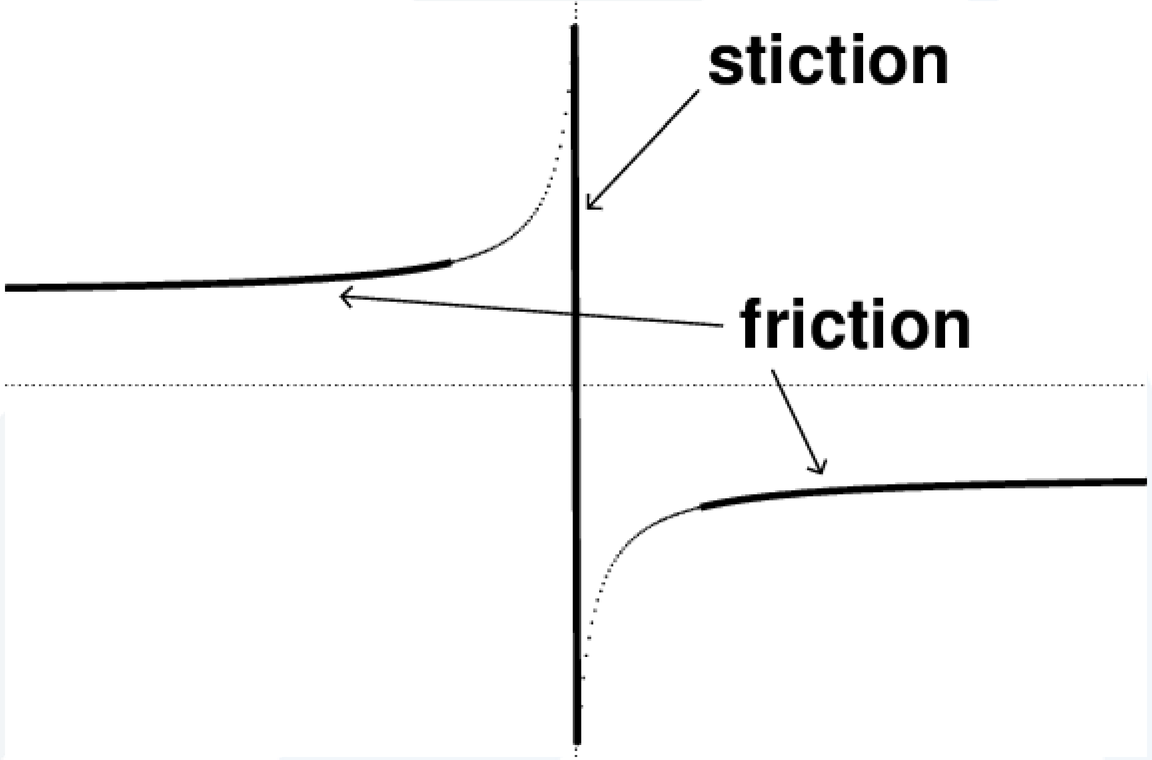


On-Off or Bang Bang



Stiction

- Stiction or high static friction is a common problem in spring-diaphragm type control valves, which are widely used in the process industry



Equilibrium point

- Equilibrium point doesn't have to be unique.

Ex: $\ddot{y} + \dot{y} + \sin y = 0 \quad \textbf{(pendulum)}$

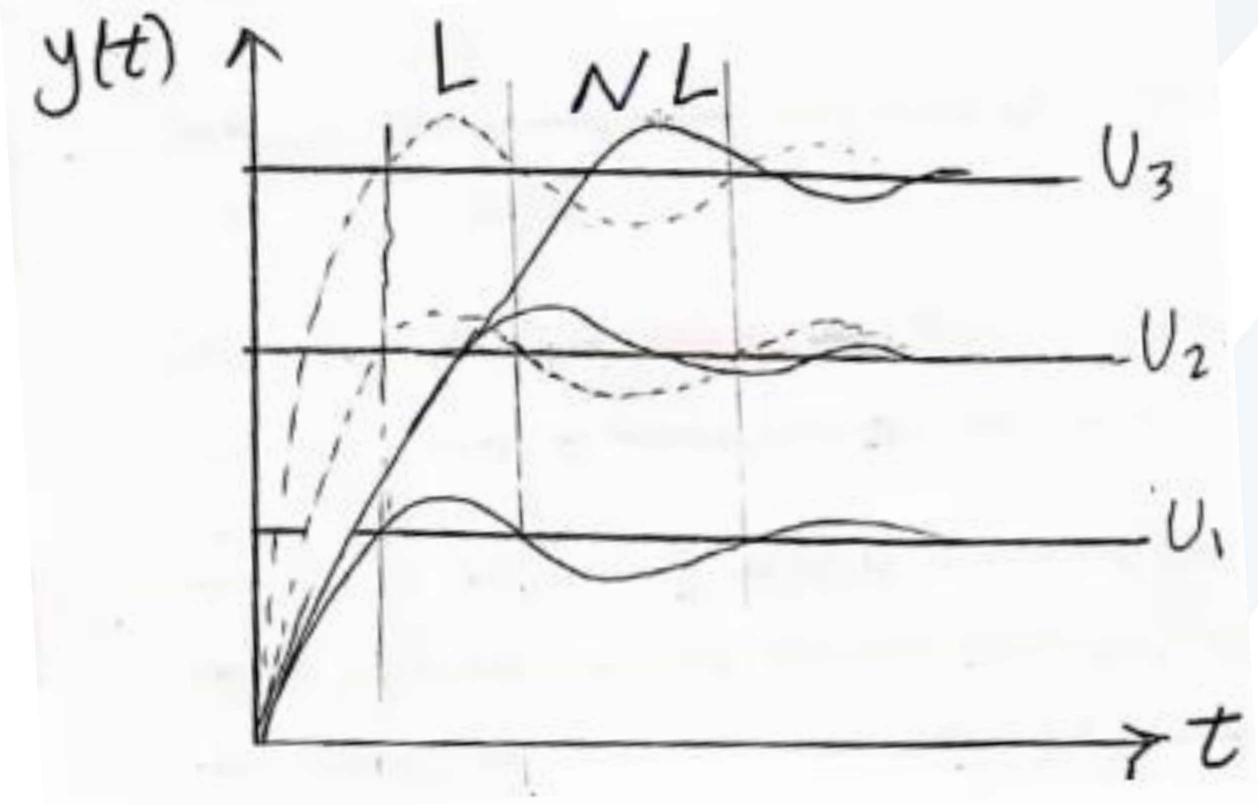
$$\left.\begin{aligned} y &= x_1 \\ \dot{y} &= x_2 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \sin x_1 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{2s} &= 0 \\ \sin x_{1s} &= 0 \end{aligned}$$

$$\text{Equilibrium point } x_s = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}, \quad n = 0, 1, \cdots$$

Ex: $\ddot{y} + \dot{y} - y + y^3 = 0 \quad \textbf{(Rayleigh eq.)}$

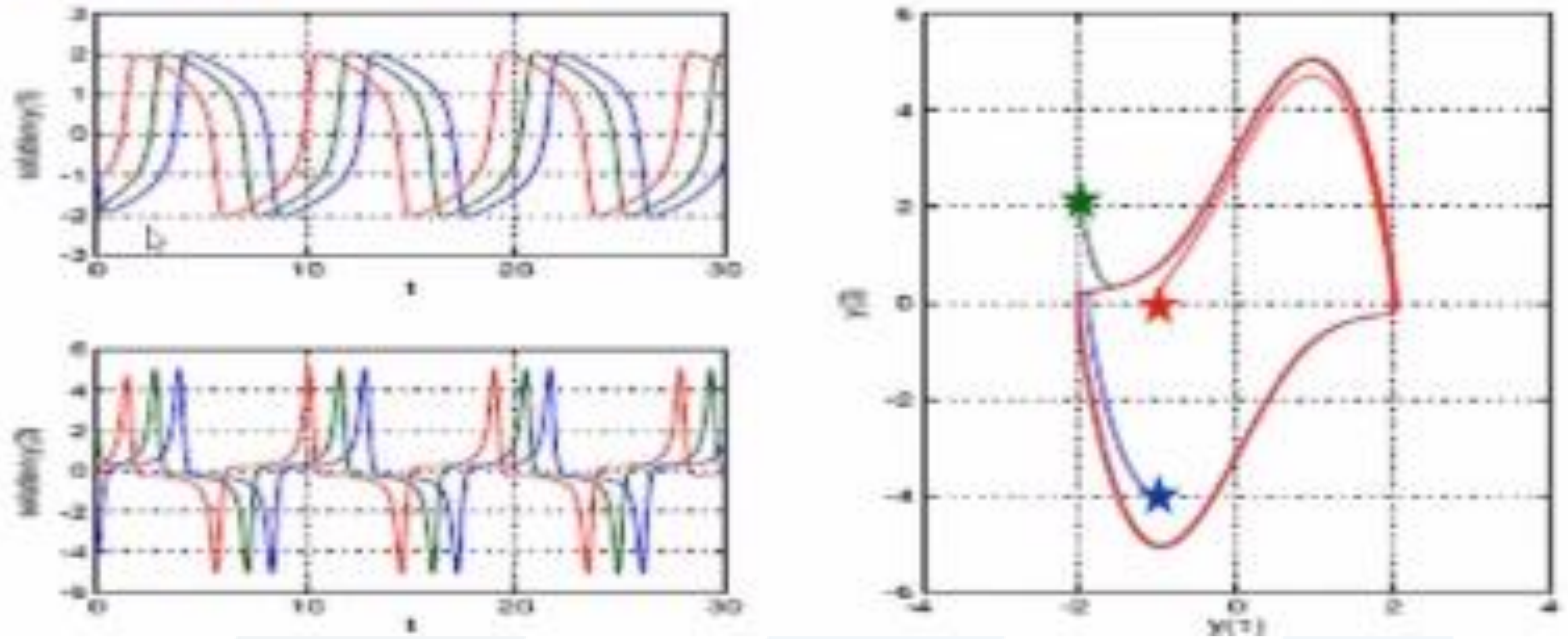
$$\left.\begin{aligned} \dot{x}_1 &= x_2 = 0 \\ \dot{x}_2 &= -x_2 - x_1 - x_1^3 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_{2s} &= 0 \\ x_{1s}(1 - x_{1s}^2) &= 0 \end{aligned}$$

$$x_s^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad x_s^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_s^3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$



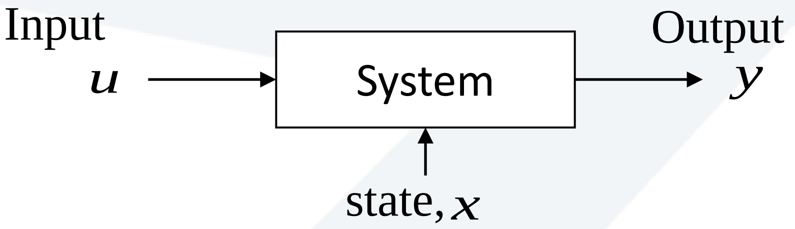
y(t)
L
NL
U3
U2
U1
t

- Limit Cycle
  - ✓ Stable oscillations whose amplitude does not depend on the initial conditions.



1. Phenomena of Nonlinear Dynamics

- Linear vs. Nonlinear



$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}\right\} \cdots\cdots (1)$$

$$\left.\begin{aligned} \dot{x} &= f(x,u) \\ y &= \varphi(x) \end{aligned}\right\} \cdots\cdots (2)$$

$$f : R^n \times R^m \rightarrow R^n$$
$$\varphi : R^n \rightarrow R^p$$

Definitions : <u>Linear</u> : when the superposition holds
<u>Nonlinear</u> : otherwise

Stability & Output of systems

- Stability depends on the system's parameter (linear)
- Stability depends on the initial conditions, input signals as well as the system parameters (nonlinear).
- Output of a linear system has the same frequency as the input although its amplitude and phase may differ.
- Output of a nonlinear system usually contains additional frequency components and may, in fact, not contain the input frequency.

Time invariant vs. Time varying

- Time invariant vs. Time varying
  - System (1) is time invariant ⇔ parameters are constant

    - Linear time varying system

$$\left.\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned}\right\} \quad \cdots \quad (3)$$

- System (2) is time invariant ⇔ no function has t as its argument.

  - Nonlinear time varying system

$$\left.\begin{aligned} \dot{x} &= f(x,u,t) \\ y &= \varphi(x,t) \end{aligned}\right\} \quad \cdots \quad (4)$$