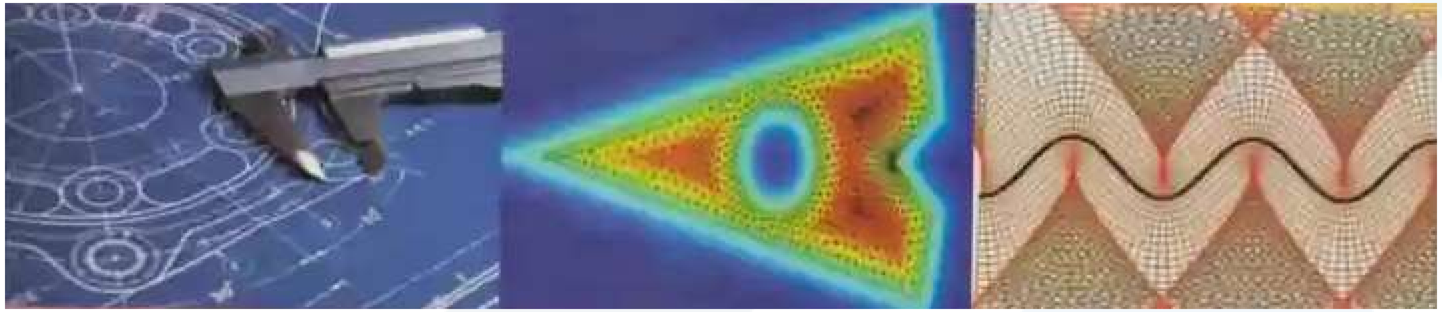
CEDC301: Engineering Mathematics

Lecture Notes 5: Series and Residues: Part B



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Chapter 3

Series and Residues

4. Zeros and Poles

Classification of Isolated Singular Points

- A classification is given depending on whether the principal part of its Laurent expansion contains zero, a finite number, or an infinite number of terms.

$z = z_0$	Laurent Series
Removable singularity	$a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$
Pole of order n	$\dfrac{a_{-n}}{(z-z_0)^n} + \dfrac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \cdots + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$
Simple pole	$\dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$
Essential singularity	$\cdots + \dfrac{a_{-2}}{(z-z_0)^2} + \dfrac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

- Example 14: Removable Singularity

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

$z = 0$ is a removable singularity of $f(z) = (\sin z)/z$.

- Example 15: Poles and Essential Singularity

$$\frac{\sin z}{z^2} = \overset{\substack{\text{principal part}\\ \downarrow}}{\frac{1}{z}} - \frac{z}{3!} + \frac{z^3}{5!} - \cdots$$

$|z| > 0$, we see that $a_{-1} \neq 0$, and so $z = 0$ is a simple pole of the function $f(z) = (\sin z)/z^2$.

The Laurent expansion of $f(z) = 1/(z-1)^2(z-3)$ valid for $0 < |z-1| < 2$

$$f(z) = \overbrace{-\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)}}^{\text{principal part}} - \frac{1}{8} - \frac{z-1}{16} - \cdots$$

since $a_{-2} \neq 0$, we conclude that $z = 1$ is a pole of order 2.

The principal part of Laurent series of the function $f(z) = e^{3/z}$ contains an infinite number of terms. Thus $z = 0$ is an essential singularity.

Zeros

- z_0 is a zero of a function f if $f(z_0) = 0$. An analytic function f has a zero of order n at $z = z_0$ if

$$f(z_0) = 0, \quad f'(z_0) = 0, \quad f''(z_0) = 0, \ \cdots, \quad f^{(n-1)}(z_0) = 0, \text{ but } f^{(n)}(z_0) \neq 0$$

- If an analytic function f has a zero of order n at $z = z_0$, it follows that the Taylor series expansion of f centered at z_0 must have the form:

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + a_{n+2}(z - z_0)^{n+2} + \cdots$$
$$= (z - z_0)^n[a_n + a_{n+1}(z - z_0) + a_{n+2}(z - z_0)^2 + \cdots]$$

- Theorem 11 (Zero of Order n): A function f that is analytic in some disk $|z - z_0| < R$ has a zero of order n at $z = z_0$ if and only if f can be written $f(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

- Example 16: Order of a Zero

  The analytic function $f(z) = z \sin z^2$ has a zero of order 3 at $z = 0$.

$$z \sin z^2 = z\left[z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \cdots\right] = z^3\left[1 - \frac{z^4}{3!} + \frac{z^8}{5!} - \cdots\right]$$

Poles

- Theorem 12 (Pole of Order n): A function f that is analytic in a deleted neighborhood of z_0, $0 < |z - z_0| < R$ has a pole of order n at $z = z_0$ if and only if f can be written $f(z) = \phi(z)/(z - z_0)^n$, where ϕ is analytic at $z = z_0$ and $\phi(z_0) \neq 0$.

- Theorem 13 (Pole of Order n): If the functions f and g are analytic at $z = z_0$ and f has a zero of order n at $z = z_0$ and $g(z_0) \neq 0$, then the function $F(z) = g(z)/f(z)$ has a pole of order n at $z = z_0$.

- Example 17: Order of Poles

$$f(z) = \frac{2z+5}{(z-1)(z+5)(z-2)^4}$$

The denominator has zeros of order 1 at $z = 1$ and $z = -5$, and a zero of order 4 at $z = 2$. Since the numerator is not zero at any of these points, it follows that f has simple poles at $z = 1$ and $z = -5$, and a pole of order 4 at $z = 2$.

$z = 0$ is a zero of order 3 of $f(z) = z \sin z^2 \Rightarrow F(z) = 1/(z \sin z^2)$ has a pole of order 3 at $z = 0$.

- If a function has a pole at $z = z_0$, then $|f(z)| \to \infty$ as $z \to z_0$ from any direction.

5. Residues and Residue Theorem

- If the complex function f has an isolated singularity at the point z_0, then f has a Laurent series representation:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

which converges for all z near z_0. More precisely, the representation is valid in some deleted neighborhood of z_0, $0 < |z - z_0| < R$.

Residue

The coefficient a_{-1} of $1/(z - z_0)$ in the Laurent series given above is called the residue of the function f at the isolated singularity z_0.

$$a_{-1} = Res\,(f(z), z_0)$$

- Example 18: Residues

  $z = 1$ is a pole of order 2 of the function $f(z) = 1/(z-1)^2(z-3)$. From the Laurent series we see that the coefficient of $1/(z-1)$ is $a_{-1} = Res\,(f(z), 1) = -¼$.

  $z = 0$ is an essential singularity of $f(z) = e^{3/z}$. From the Laurent series we see that the coefficient of $1/z$ is $a_{-1} = Res\,(f(z), 0) = 3$.

- Theorem 14 (Residue at a Simple Pole): If f has a simple pole at $z = z_0$, then:

$$Res\,(f(z), z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

- Theorem 15 (Residue at a Pole of Order n): If f has a pole of order n at $z = z_0$, then

$$Res\,(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

- Example 19: Residue at a Pole

  The function $f(z) = 1/(z-1)^2(z-3)$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$

$$Res\ (f(z),\,3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{1}{(z-1)^2} = \frac{1}{4}$$

$$Res\ (f(z),\,1) = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1}\frac{d}{dz}\frac{1}{z-3} = -\frac{1}{4}$$

- Suppose a function f can be written as a quotient $f(z) = g(z)/h(z)$, where g and h are analytic at $z = z_0$. If $g(z_0) \neq 0$ and if the function h has a zero of order 1 at z_0, then f has a simple pole at $z = z_0$ and

$$Res\ (f(z),\,z_0) = \frac{g(z_0)}{h'(z_0)}$$

- Example 20: Residue at a Pole

  The function $1/(z^4 + 1)$ has four simple poles

$$z_1 = e^{\pi i/4},\ z_2 = e^{3\pi i/4},\ z_3 = e^{5\pi i/4},\ z_4 = e^{7\pi i/4}$$

$$Res\ (f(z),\ z_1) = \frac{1}{4z_1^3} = \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$Res\ (f(z),\ z_2) = \frac{1}{4z_2^3} = \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

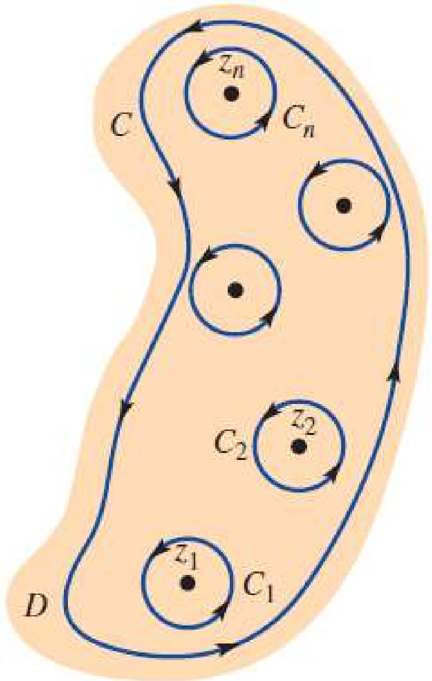
$$Res\ (f(z),\ z_3) = \frac{1}{4z_3^3} = \frac{1}{4}e^{-15\pi i/4} = \frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

$$Res\ (f(z),\ z_4) = \frac{1}{4z_4^3} = \frac{1}{4}e^{-21\pi i/4} = -\frac{1}{4\sqrt{2}} + \frac{1}{4\sqrt{2}}i$$

Residue Theorem

- **Theorem 16 (Cauchy's Residue Theorem):** Let D be a simply connected domain and C a simple closed contour lying entirely within D. If a function f is analytic on and within C, except at a finite number of singular points z_1, z_2, ..., z_n within C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res\ (f(z),\ z_k)$$



- **Example 21:** Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)} dz$, where

(a) C is the rectangle defined by $x = 0$, $x = 4$, $y = -1$, $y = 1$, and

(b) C is the circle $|z| = 2$.

(a) $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i[Res\,(f(z),1) + Res\,(f(z),3)] = 2\pi i\left[-\frac{1}{4}+\frac{1}{4}\right] = 0$

(b) $\oint_C \frac{1}{(z-1)^2(z-3)}\,dz = 2\pi i\,Res\,(f(z),1) = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2}i$

- Example 22: Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{2z+6}{z^2+4}\,dz$, where C is the circle $|z-i|=2$

$$\oint_C \frac{2z+6}{z^2+4}\,dz = 2\pi i\,Res\,(f(z),2i) = 2\pi i\,\frac{3+2i}{2i} = \pi(3+2i)$$

- Example 23: Evaluation by the Residue Theorem

Evaluate $\oint_C \tan z\,dz$, where C is the circle $|z|=2$

$\tan z$ has simple poles at the points where $\cos z = 0$. $z = (2n + 1)\pi/2$, $n = 0, 1, 2, \ldots$. Since only $-\pi/2$ and $\pi/2$ are within the circle $|z| = 2$,

$$\oint_C \tan z dz = 2\pi i[Res(f(z), -\pi/2) + Res(f(z),\pi/2)] = 2\pi i[-1-1] = -4\pi i$$

- Example 24: Evaluation by the Residue Theorem

  Evaluate $\oint_C e^{3/z} dz$, where C is the circle $|z| = 1$

  $$\oint_C e^{3/z} dz = 2\pi i\, Res(f(z), 0) = 6\pi i$$

- Note: L'Hôpital's rule is valid in complex analysis. If $f(z) = g(z)/h(z)$, where g and h are analytic at $z = z_0$, $g(z_0) = h(z_0) = 0$, and $h'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{g(z)}{h(z)} = \frac{g'(z_0)}{h'(z_0)}$$

6. Evaluation of Real Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$

- The basic idea here is to convert this integral into a complex integral where the contour C is the unit circle centered at the origin. $z = \cos\,\theta + i\,\sin\,\theta = e^{i\theta}$, $0 \leq \theta \leq 2\pi$

$$dz = ie^{i\theta}d\theta, \quad \cos\,\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\,\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$d\theta = \frac{dz}{iz}, \quad \cos\,\theta = \frac{1}{2}(z + z^{-1}), \quad \sin\,\theta = \frac{1}{2i}(z - z^{-1})$$

$$\oint_C F\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right)\frac{dz}{iz}$$

where C is $|z| = 1$.

▪ Example 25: A Real Trigonometric Integral

Evaluate $\int_0^{2\pi} \frac{d\theta}{(2+\cos\,\theta)^2}$

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}\,dz$$

$$f(z) = \frac{z}{(z^2+4z+1)^2} = \frac{z}{(z-z_0)^2(z-z_1)^2}$$

$z_0 = -2-\sqrt{3},\ z_1 = -2+\sqrt{3}$ only z_1 is inside the unit circle C,

$$\oint_C \frac{z}{(z^2+4z+1)^2}\,dz = 2\pi i\,Res\,(f(z), z_1)$$

$$Res\,(f(z),\,z_1) = \lim_{z\to z_1}\frac{d}{dz}(z-z_1)^2 f(z) = \lim_{z\to z_1}\frac{d}{dz}\frac{z}{(z-z_0)^2} = \frac{1}{6\sqrt{3}}$$

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}dz = \frac{4}{i}2\pi i\frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

$$\int_0^{2\pi}\frac{d\theta}{(2+\cos\,\theta)^2} = \frac{4\pi}{3\sqrt{3}}$$

Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

- When f is continuous on $(-\infty, \infty)$, $\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to\infty}\int_{-r}^{0} f(x)dx + \lim_{R\to\infty}\int_{0}^{R} f(x)dx$
- If both limits exist, the integral is said to be convergent; if one or both of the limits fail to exist, the integral is divergent.
- In the event that we know (a priori) that an integral $\int_{-\infty}^{\infty} f(x)dx$ converges:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

- This limit is called the Cauchy principal value of the integral and is written:

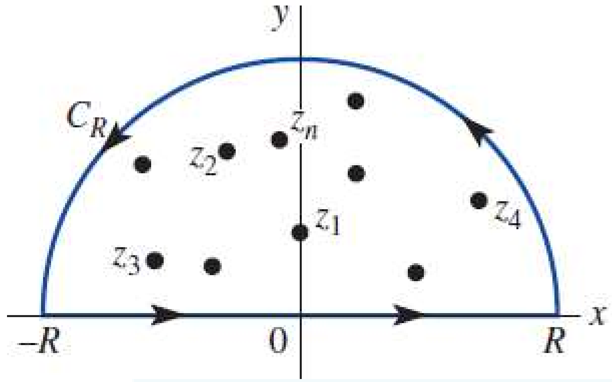
$$P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$$

- When an integral of the form $\int_{-\infty}^{\infty} f(x)dx$ converges, its Cauchy principal value is the same as the value of the integral. If the integral diverges, it may still possess a Cauchy principal value. For ex., the integral $\int_{-\infty}^{\infty} xdx$ diverge, but:

$$P.V.\int_{-\infty}^{\infty} xdx = \lim_{R\to\infty}\int_{-R}^{R} xdx = \lim_{R\to\infty}\left[\frac{R^2}{2}-\frac{(-R)^2}{2}\right] = 0$$

- To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where $f(x) = P(x)/Q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of:

the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = P(z)/Q(z)$ in the upper half-plane $Re(z) > 0$.



$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} Res\left(f(z), z_k\right)$$

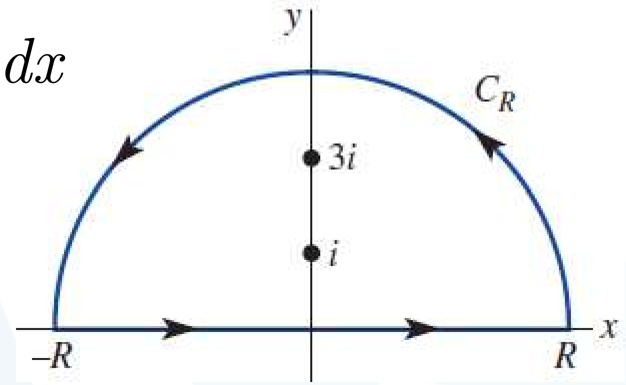
where z_k, $k = 1, 2, ..., n$, denotes poles in the upper half-plane.

If we can show that the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have:

$$P.V.\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} Res\left(f(z), z_k\right)$$

- Example 26: Cauchy $P.V.$ of an Improper Integral

Evaluate the Cauchy principal value of $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx$



$$f(z) = \frac{1}{(z^2+1)(z^2+9)} = \frac{1}{(z+i)(z-i)(z+3i)(z-3i)}$$

$$\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)}\,dz = I_1 + I_2$$

$$I_1 + I_2 = 2\pi i[Res(f(z), i) + Res(f(z), 3i)] = 2\pi i\left[\frac{1}{16i} + \left(-\frac{1}{48i}\right)\right] = \frac{\pi}{12}$$

On C_R, $\left|(z^2+1)(z^2+9)\right| = \left|z^2+1\right|\left|z^2+9\right| \geq \left||z|^2-1\right|\left||z|^2-9\right| = (R^2-1)(R^2-9)$

ML-inequality

$$|I_2| = \left|\int_{C_R} \frac{1}{(z^2+1)(z^2+9)}\,dz\right| \le \frac{\pi R}{(R^2-1)(R^2-9)} \underset{R\to\infty}{\longrightarrow} 0$$

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx = P.V.\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{12}$$

- **Theorem 17 (Behavior of Integral as $R \to \infty$):** Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \ge n + 2$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \le \theta \le \pi$, then $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$.

- **Example 27:** Cauchy $P.V.$ of an Improper Integral

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx$

$$Res\,(f(z),\,z_1) = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i \qquad Res\,(f(z),\,z_2) = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

$$P.V.\int_{-\infty}^{\infty} \frac{1}{x^4+1}\,dx = 2\pi i[Res\,(f(z),\,z_1) + Res\,(f(z),\,z_2)] = \frac{\pi}{\sqrt{2}}$$

Integrals of the Forms $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ or $\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$

$$\int_{-\infty}^{\infty} f(x)\,e^{i\alpha x}\,dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x\,dx$$

whenever both integrals on the right side converge. When $f(x) = P(x)/Q(x)$ is continuous on $(-\infty,\,\infty)$ we can evaluate both integrals at the same time by considering the integral $\oint_C f(z)e^{i\alpha z}dz$, where $\alpha > 0$ and C consists of:

the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z)$ in the upper half-plane $Re(z) > 0$.

- Theorem 18 (Behavior of Integral as $R \to \infty$): Suppose $f(z) = P(z)/Q(z)$, where the degree of $P(z)$ is n and the degree of $Q(z)$ is $m \geq n + 1$. If C_R is a semicircular contour $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $\alpha > 0$, then:

$$\int_{C_R} f(z)e^{i\alpha z}dz \to 0 \text{ as } R \to \infty$$

- Example 28: Using Symmetry

Evaluate the Cauchy principal value of $\int_0^\infty \frac{x \sin x}{x^2+9}dx$

$$\int_0^\infty \frac{x \sin x}{x^2+9}dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x \sin x}{x^2+9}dx$$

With $\alpha = 1$, we now form the contour integral $\oint_C \frac{z}{z^2+9} e^{iz} dz$ where C is the same contour as example 26

$$\int_{C_R} \frac{z}{z^2+9} e^{iz} dz + \int_{-R}^{R} \frac{x}{x^2+9} e^{ix} dx = 2\pi i \, Res\left(f(z)e^{iz}, 3i\right) = \frac{\pi}{e^3} i$$

$$\int_{C_R} f(z) e^{iz} dz \to 0 \text{ as } R \to \infty \Rightarrow P.V. \int_{-\infty}^{\infty} \frac{x}{x^4+9} e^{ix} dx = \frac{\pi}{e^3} i$$

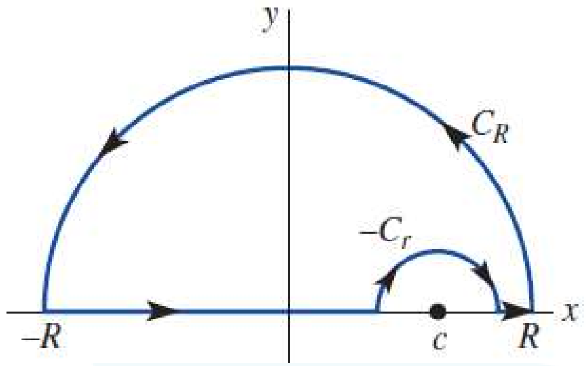
$$\int_{-\infty}^{\infty} \frac{x}{x^2+9} e^{ix} dx = \int_{-\infty}^{\infty} \frac{x \cos x}{x^2+9} dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} e^{ix} dx = \frac{\pi}{e^3} i$$

$$P.V. \int_{-\infty}^{\infty} \frac{x \cos x}{x^2+9} dx = 0, \quad P.V. \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{e^3}$$

$$\int_{0}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{1}{2} P.V. \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{2e^3}$$

Indented Contours

- When $f(x) = P(x)/Q(x)$ have poles on the real axis, we must modify the procedure used in previous Examples. For example, to evaluate $\int_{-\infty}^{\infty} f(x)dx$ by residues when $f(z)$ has a pole at $z = c$, where c is a real number, we use an indented contour.



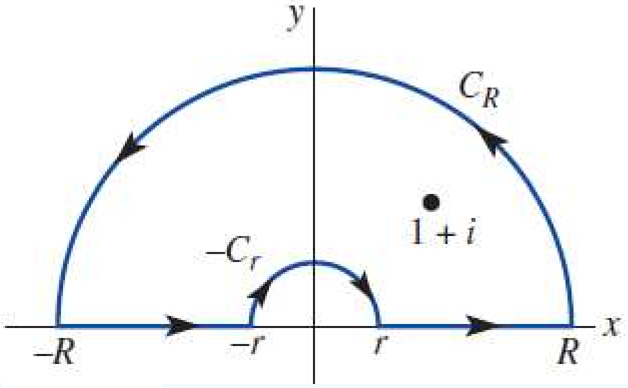
- Theorem 19 (Behavior of Integral as $r \to 0$): Suppose f has a simple pole at $z = c$ on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}$, $0 \le \theta \le \pi$, then:

$$\lim_{r \to 0} \int_{C_r} f(z)dz = \pi i\, Res\,(f(z), c)$$

- Example 29: Using an Indented Contour

Evaluate the Cauchy $P.V.$ of $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}dx$

The function $f(z) = 1/z(z^2 - 2z + 2)$ has simple poles at $z = 0$ and at $z = 1 + i$ in the upper half-plane.



$$\oint_C \frac{e^{iz}}{z(z^2-2z+2)}dz = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_{r}^{R} = 2\pi i\, Res\,(f(z)e^{iz}, 1+i)$$

where $\displaystyle\int_{-C_r} = -\int_{C_r}$

Taking the limits $R \to \infty$ and $r \to 0$, we find

$$P.V.\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2-2x+2)}dx - \pi i\, Res\,(f(z)e^{iz}, 0) = 2\pi i\, Res\,(f(z)e^{iz}, 1+i)$$

$$P.V.\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2-2x+2)}dx=\pi i\left(\frac{1}{2}\right)+2\pi i\left(\frac{e^{-1+i}}{4}(1+i)\right)$$

$$P.V.\int_{-\infty}^{\infty}\frac{\cos x}{x(x^2-2x+2)}dx=\frac{\pi}{2}e^{-1}(\sin 1+\cos 1)$$

$$P.V.\int_{-\infty}^{\infty}\frac{\sin x}{x(x^2-2x+2)}dx=\frac{\pi}{2}[1+e^{-1}(\sin 1-\cos 1)]$$

Integration along a Branch Cut

Branch Point at $z=0$ We will examine integrals of the form $\int_0^{\infty} f(x)dx$.
These integrals require a special type of contour because when $f(x)$ is converted to a complex function, the resulting integrand $f(z)$ has, in addition to poles, a nonisolated singularity at $z=0$.

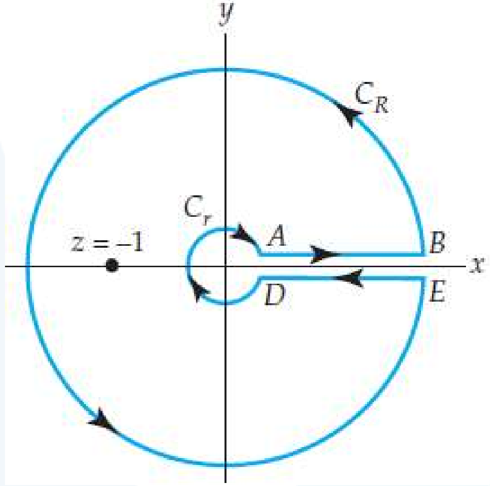
- Example 30: Integration along a Branch Cut

Evaluate $\displaystyle\int_0^{\infty} \frac{1}{\sqrt{x}(x+1)}\,dx$

$f(z) = \dfrac{1}{\sqrt{z}(z+1)}$ The origin is a branch point since $z^{1/2}$ has two values for any $z \neq 0$.

We can force $z^{1/2}$ to be single valued by choosing the positive x-axis as a branch cut ($0 < \theta < 2\pi$.

The integrand $f(z)$ is single valued and analytic on and within C, except for the simple pole at $z = -1 = e^{\pi i}$.



$$\oint_C \frac{1}{z^{1/2}(z+1)}\,dz = \int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} \quad = 2\pi i \, Res\left(f(z), -1\right)$$

On AB, $z = xe^{0i}$, and on ED, $z = xe^{(0+2\pi)i} = xe^{2\pi i}$

$$\int_{ED} f(z)dz = \int_R^r \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i}+1} e^{2\pi i}dx = -\int_R^r \frac{x^{-1/2}}{x+1}dx = \int_r^R \frac{x^{-1/2}}{x+1}dx$$

$$\int_{AB} f(z)dz = \int_r^R \frac{(xe^{0i})^{-1/2}}{xe^{0i}+1} e^{0i}dx = \int_r^R \frac{x^{-1/2}}{x+1}dx$$

$z = re^{i\theta}$ and $z = Re^{i\theta}$ on C_r and C_R, $\Rightarrow \left|\int_{C_r} f(z)dz\right| \le \frac{r^{-1/2}}{1-r}2\pi r = \frac{2\pi}{1-r}r^{1/2} \underset{r\to 0}{\to} 0$

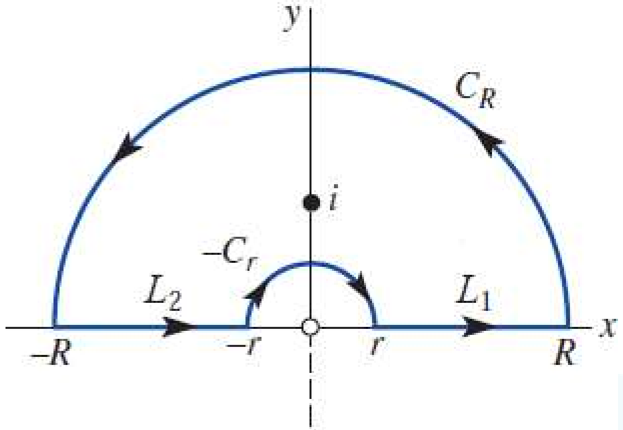
and $\left|\int_{C_R} f(z)dz\right| \le \frac{R^{-1/2}}{R-1}2\pi R = \frac{2\pi R}{R-1}\frac{1}{R^{1/2}} \underset{R\to\infty}{\to} 0$

$$2\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = 2\pi i\, Res\,(f(z),-1) = 2\pi i(-i) = 2\pi$$

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = \pi$$

- Example 31: Integration around a Point Cut

Evaluate $\displaystyle\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}\,dx$



$$f(z)=\frac{\log z}{(z^2+1)^2},\qquad |z|>0,\ -\frac{\pi}{2}<\arg z<\frac{\pi}{2}$$

The branch cut consists of the origin and the negative imaginary axis.

In order that the isolated singularity $z=i$ be inside the closed path, we require that $r<1<R$.

$$\oint_C \frac{\log z}{(z^2+1)^2}\,dz=\int_{C_R}+\int_{-R}^{-r}+\int_{-C_r}+\int_r^R \quad =2\pi i\,Res\,(f(z),i)$$

$$f(z)=\frac{\ln r+i\theta}{(r^2e^{i2\theta}+1)^2},\qquad (z=re^{i\theta})$$

On L_1, $z=xe^{0i}=x$, and on L_2, $z=xe^{\pi i}=-x$

$$\int_{L_2} f(z)dz = \int_{-R}^{-r} \frac{\ln(-x)+i\pi}{(x^2+1)^2}dx = -\int_{R}^{r} \frac{\ln x+i\pi}{(x^2+1)^2}dx = \int_{r}^{R} \frac{\ln x+i\pi}{(x^2+1)^2}dx$$

$$\int_{L_1} f(z)dz = \int_{r}^{R} \frac{\ln x}{(x^2+1)^2}dx$$

$z = re^{i\theta}$ & $z = Re^{i\theta}$ on C_r and C_R, $\Rightarrow \left|\int_{C_r} f(z)dz\right| \le \frac{-\ln r+\pi}{(1-r^2)^2}\pi r = \pi\frac{\pi r - r\ln r}{(1-r^2)^2} \underset{r\to 0}{\to} 0$

and $\left|\int_{C_R} f(z)dz\right| \le \frac{\ln R+\pi}{(R^2-1)^2}\pi R = \underset{r\to 0}{\to} 0$

$$2\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}dx + \int_r^R \frac{i\pi}{(x^2+1)^2}dx = 2\pi i\,Res\,(f(z),i) = 2\pi i\left(\frac{\pi}{8}+\frac{1}{4}i\right) = -\frac{\pi}{2}+\frac{\pi^2}{4}i$$

$$\int_0^{\infty} \frac{\ln x}{(x^2+1)^2}dx = -\frac{\pi}{4}, \qquad \int_0^{\infty} \frac{1}{(x^2+1)^2}dx = \frac{\pi}{4}$$