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Lecture 3: Orthogonality - Projection

CEDC102: Linear Algebra

Manara University

2023-2024

- Length and Dot Product in $\mathbb{R}^n$
- Projection matrices
- Orthonormal Bases: Gram-Schmidt Process

Length and Dot Product in $\mathbb{R}^n$

- Length:

The length of a vector $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is given by

$$\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- Note: The length of a vector is also called its norm.

- Notes: Properties of length

$(1)\ \|\boldsymbol{v}\| \geq 0$

$(2)\ \|\boldsymbol{v}\| = 1 \Rightarrow \boldsymbol{v}$ is called a unit vector

$(3)\ \|\boldsymbol{v}\| = 0$ iff $\boldsymbol{v} = 0$

- Ex :

(*a*) In $\mathbb{R}^5$, the length of $\boldsymbol{v} = (0, -2, 1, 4, -2)$ is given by

$$\|\boldsymbol{v}\| = \sqrt{0^2 + (-2)^2 + 1^2 + 4^2 + (-2)^2} = \sqrt{25} = 5$$

(*b*) In $\mathbb{R}^3$ the length of $\boldsymbol{v} = \left(\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ is given by

$$\|\boldsymbol{v}\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2} = \sqrt{\frac{17}{17}} = 1 \qquad (\boldsymbol{v} \text{ is a unit vector})$$

- A standard unit vector in $\mathbb{R}^n$:

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\} = \{(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,0,\ldots,1)\}$$

- Ex :

  the standard unit vector in $\mathbb{R}^2$: $\{\boldsymbol{i}, \boldsymbol{j}\} = \{(1, 0), (0, 1)\}$

  the standard unit vector in $\mathbb{R}^3$: $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

- Notes: (Two nonzero vectors are parallel)

  $\boldsymbol{u} = c\boldsymbol{v}$

  (1) $c > 0 \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ have the same direction

  (2) $c < 0 \Rightarrow \boldsymbol{u}$ and $\boldsymbol{v}$ have the opposite direction

- Theorem : (Length of a scalar multiple)

  Let $\boldsymbol{v}$ be a vector in $\mathbb{R}^n$ and c be a scalar, then $\|c\boldsymbol{v}\| = |c|\,\|\boldsymbol{v}\|$

- Theorem : (Unit vector in the direction of $\boldsymbol{v}$)

  If $\boldsymbol{v}$ is a nonzero vector in $\mathbb{R}^n$, then the vector $\boldsymbol{u} = \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ has length 1 and has the same direction as $\boldsymbol{v}$.

  This vector $\boldsymbol{u}$ is called the unit vector in the direction of $\boldsymbol{v}$.

- Note: The process of finding the unit vector in the direction of $\boldsymbol{v}$ is called normalizing the vector $\boldsymbol{v}$.

- Ex : (Finding a unit vector)

  Find the unit vector in the direction of $\boldsymbol{v} = (3, -1, 2)$, and verify that this vector has length 1.

  Sol:

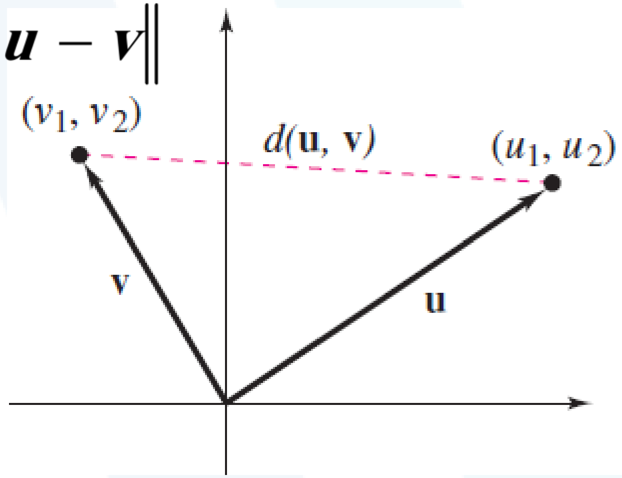
$$\|\boldsymbol{v}\| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$$

$$\Rightarrow \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} = \frac{(3\,,-1\,,2)}{\sqrt{3^2+(-1)^2+2^2}} = \frac{1}{\sqrt{14}}(3\,,-1\,,2) = \left(\frac{3}{\sqrt{14}}\,,\frac{-1}{\sqrt{14}}\,,\frac{2}{\sqrt{14}}\right)$$

$$\sqrt{\left(\frac{3}{\sqrt{14}}\right)^2+\left(\frac{-1}{\sqrt{14}}\right)^2+\left(\frac{2}{\sqrt{14}}\right)^2} = \sqrt{\frac{14}{14}} = 1 \Rightarrow \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \text{ is a unit vector}$$

- **Distance between two vectors:**

The distance between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ is: $d(\boldsymbol{u},\ \boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\|$



- **Notes: (Properties of distance)**

(1) $d(\boldsymbol{u},\ \boldsymbol{v}) \geq 0$

(2) $d(\boldsymbol{u},\ \boldsymbol{v}) = 0$ if and only if $\boldsymbol{u} = \boldsymbol{v}$

(3) $d(\boldsymbol{u},\ \boldsymbol{v}) = d(\boldsymbol{v},\ \boldsymbol{u})$

- Ex : (Distance between 2 vectors)

The distance between $\boldsymbol{u} = (0, 2, 2)$ and $\boldsymbol{v} = (2, 0, 1)$ is

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \|(0-2), 2-0, 2-1)\| = \sqrt{(-2)^2 + 2^2 + 1^2} = 3$$

- Dot product in $\mathbb{R}^n$:

The dot product of $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is the scalar quantity

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

- Ex : (Finding the dot product of two vectors)

The dot product of $\boldsymbol{u} = (1, 2, 0, -3)$ and $\boldsymbol{v} = (3, -2, 4, 2)$ is

$$\boldsymbol{u} \cdot \boldsymbol{v} = (1)(3) + (2)(-2) + (0)(4) + (-3)(2) = -7$$

- Theorem : (Properties of the dot product)

If $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ are vectors in $\mathbb{R}^n$ and c is a scalar, then the following properties are true.

(1) $\boldsymbol{u}.\boldsymbol{v} = \boldsymbol{v}.\boldsymbol{u}$

(2) $\boldsymbol{u}.(\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u}.\boldsymbol{v} + \boldsymbol{u}.\boldsymbol{w}$

(3) $c(\boldsymbol{u}.\boldsymbol{v}) = (c\boldsymbol{u}).\boldsymbol{v} = \boldsymbol{u}.(c\boldsymbol{v})$

(4) $\boldsymbol{v}.\boldsymbol{v} \geq 0$, and $\boldsymbol{v}.\boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$

(5) $\boldsymbol{v}.\boldsymbol{v} = \|\boldsymbol{v}\|^2$

- Euclidean n-space:

$\mathbb{R}^n$ was defined to be the set of all order n-tuples of real numbers. When $\mathbb{R}^n$ is combined with the standard operations of vector addition, scalar multiplication, vector length, and the dot product, the resulting vector space is called Euclidean n-space.

- Ex : (Finding dot products)

$$\mathbf{u} = (2, -2),\ \mathbf{v} = (5, 8),\ \mathbf{w} = (-4, 3)$$

$(a)\ \mathbf{u}.\mathbf{v}$ $\quad (b)\ (\mathbf{u}.\mathbf{v})\mathbf{w}$ $\quad (c)\ \mathbf{u}.(2\mathbf{v})$ $\quad (d)\ \|\mathbf{w}\|^2$ $\quad (e)\ \mathbf{u}.(\mathbf{v} - 2\mathbf{w})$

Sol:

$$(a)\ \mathbf{u}.\mathbf{v} = (2)(5) + (-2)(8) = -6$$

$$(b)\ (\mathbf{u}.\mathbf{v})\mathbf{w} = -\mathbf{w} = -6(-4, 3) = (24, -18)$$

$$(c)\ \mathbf{u}.(2\mathbf{v}) = 2(\mathbf{u}.\mathbf{v}) = 2(-6) = -12$$

$$(d)\ \|\mathbf{w}\|^2 = \mathbf{w}.\mathbf{w} = (-4)(-4) + (3)(3) = 25$$

$$(e)\ (\mathbf{v} - 2\mathbf{w}) = (5 - (-8), 8 - 6) = (13, 2)$$

$$\mathbf{u}.(\mathbf{v} - 2\mathbf{w}) = (2)(13) + (-2)(2) = 22$$

- Ex : (Using the properties of the dot product)

Given $\boldsymbol{u}.\boldsymbol{u} = 39,\ \boldsymbol{u}.\boldsymbol{v} = -3,\ \boldsymbol{v}.\boldsymbol{v} = 79$

Find $(\boldsymbol{u} + 2\boldsymbol{v}).(3\boldsymbol{u} + \boldsymbol{v})$

Sol:

$$\begin{aligned}(\boldsymbol{u} + 2\boldsymbol{v}).(3\boldsymbol{u} + \boldsymbol{v}) &= \boldsymbol{u}.(3\boldsymbol{u} + \boldsymbol{v}) + 2\boldsymbol{v}.(3\boldsymbol{u} + \boldsymbol{v}) \\ &= \boldsymbol{u}.(3\boldsymbol{u}) + \boldsymbol{u}.\boldsymbol{v} + (2\boldsymbol{v}).(3\boldsymbol{u}) + (2\boldsymbol{v}).\boldsymbol{v} \\ &= 3(\boldsymbol{u}.\boldsymbol{u}) + \boldsymbol{u}.\boldsymbol{v} + 6(\boldsymbol{v}.\boldsymbol{u}) + 2(\boldsymbol{v}.\boldsymbol{v}) \\ &= 3(\boldsymbol{u}.\boldsymbol{u}) + 7(\boldsymbol{u}.\boldsymbol{v}) + 2(\boldsymbol{v}.\boldsymbol{v}) \\ &= 3(39) + 7(-3) + 2(79) = 254\end{aligned}$$

- **Theorem : (The Cauchy - Schwarz inequality)**

  If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$, then $|\boldsymbol{u}\cdot\boldsymbol{v}| \le \|\boldsymbol{u}\|\|\boldsymbol{v}\|$

- **Ex 8: (An example of the Cauchy - Schwarz inequality)**

  Verify the Cauchy - Schwarz inequality for $\boldsymbol{u} = (1, -1, 3)$ and $\boldsymbol{v} = (2, 0, -1)$

Sol:

$$\boldsymbol{u}\cdot\boldsymbol{u} = 11,\ \boldsymbol{u}\cdot\boldsymbol{v} = -1,\ \boldsymbol{v}\cdot\boldsymbol{v} = 5$$

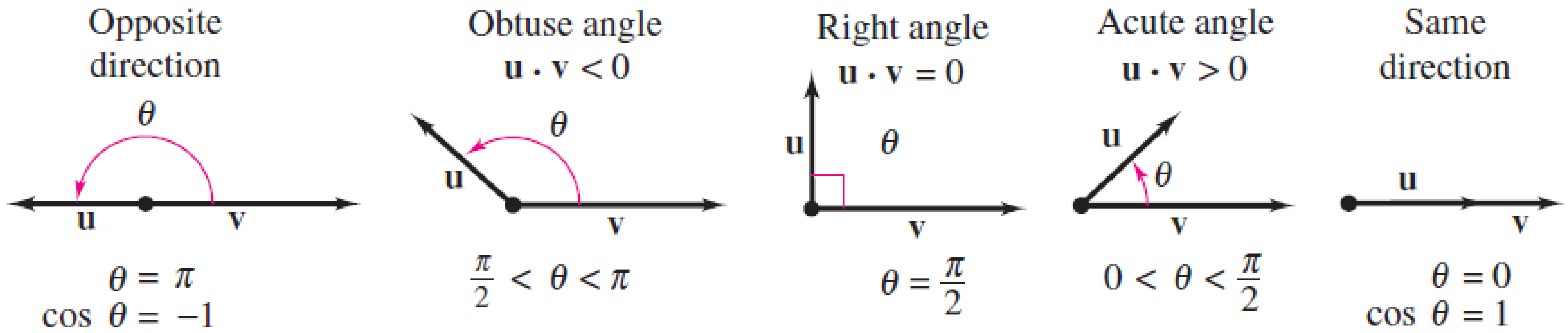
$$|\boldsymbol{u}\cdot\boldsymbol{v}| = |-1| = 1$$

$$\|\boldsymbol{u}\|\|\boldsymbol{v}\| = \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} = \sqrt{11}\sqrt{5} = \sqrt{55}$$

$$\Rightarrow |\boldsymbol{u}\cdot\boldsymbol{v}| \le \|\boldsymbol{u}\|\|\boldsymbol{v}\|$$

- The angle between two vectors in $\mathbb{R}^n$:

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|},\ 0 \le \theta \le \pi$$



$\cos\theta < 0$ $\quad$ $\cos\theta = 0$ $\quad$ $\cos\theta > 0$

- Note:

The angle between the zero vector and another vector is not defined.

- Ex : (Finding the angle between two vectors)

$$\boldsymbol{u} = (-4, 0, 2, -2),\ \boldsymbol{v} = (2, 0, -1, 1)$$

Sol:

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}.\boldsymbol{u}} = \sqrt{(-4)^2 + 0^2 + 2^2 + (-2)^2} = \sqrt{24}$$

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v}.\boldsymbol{v}} = \sqrt{(2)^2 + 0^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\boldsymbol{u}.\boldsymbol{v} = (-4)(2) + (0)(0) + (2)(-1) + (-2)(1) = -12$$

$$\Rightarrow \cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{-12}{\sqrt{24}\sqrt{6}} = \frac{-12}{\sqrt{144}} = -1$$

$\Rightarrow \theta = \pi$ $\boldsymbol{u}$ and $\boldsymbol{v}$ have opposite directions ($\boldsymbol{u} = -2\boldsymbol{v}$)

Orthogonal vectors

Two vectors are orthogonal when their dot product is zero: $v \cdot w = v^T \cdot w = 0$

Think of Pythagoras: right triangle with sides v and w.

Orthogonal vectors $v^T \cdot w = 0$ and $||v||^2 + ||w||^2 = ||v + w||^2$

The right side is $(v + w)^T \cdot (v + w)$ This equals $v^T \cdot v + w^T \cdot w$
when $w^T \cdot v = v^T \cdot w = 0$

Orthogonal subspaces

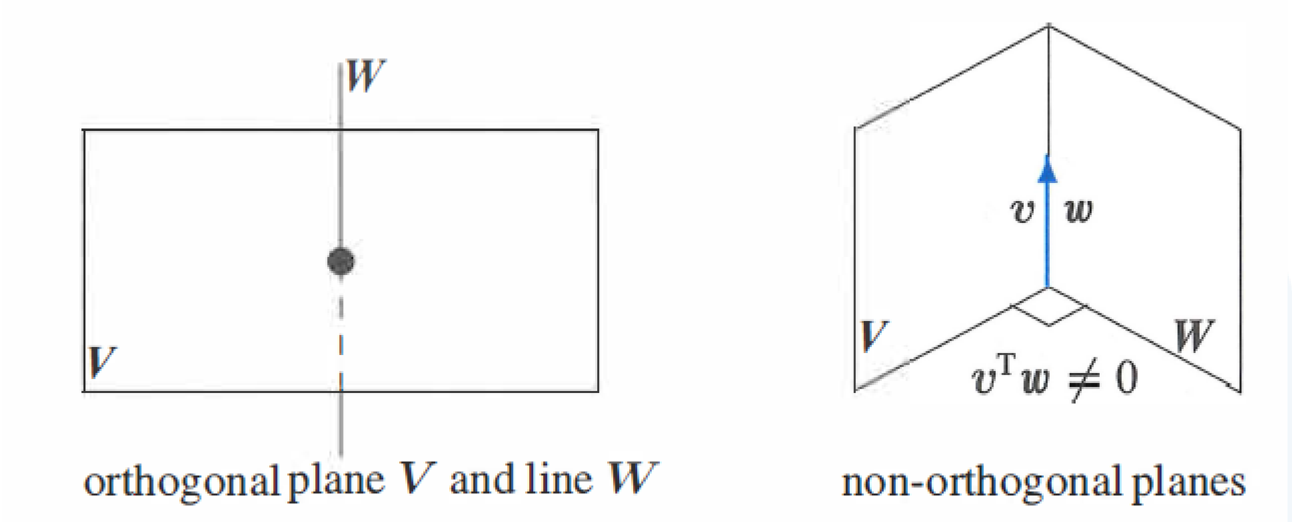
Two subspaces $\boldsymbol{V}$ and $\boldsymbol{W}$ of a vector space are orthogonal if every vector v in V is perpendicular to every vector $\boldsymbol{w}$ in W:

Orthogonal subspaces $\quad v^T \cdot w = 0$ for all v in V and $\boldsymbol{w}$ in W

Example 1 The floor of your room (extended to infinity) is a subspace V. The line where two walls meet is a subspace W (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line of the walls is perpendicular to every vector in the floor.

Example 2 Two walls look perpendicular but those two subspaces are not orthogonal! The meeting line is in both V and W -and this line is not perpendicular to itself. Two planes (dimensions 2 and 2 in $\mathbb{R}^3$) cannot be orthogonal subspaces.

When a vector is in two orthogonal subspaces, it must be zero. It is perpendicular to itself.



The crucial examples for linear algebra come from the four fundamental subspaces.

Row space is orthogonal to the Nullspace, Because $Ax = 0$: Every vector x in the nullspace is perpendicular to every row of A,

$$Ax = \begin{bmatrix} \text{row } 1 \\ \vdots \\ \text{row } m \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$(\textbf{row 1}) \cdot x$ is zero

$(\textbf{row } m) \cdot x$ is zero

Every row has a zero dot product with x. Then x is also perpendicular to every combination of the rows.

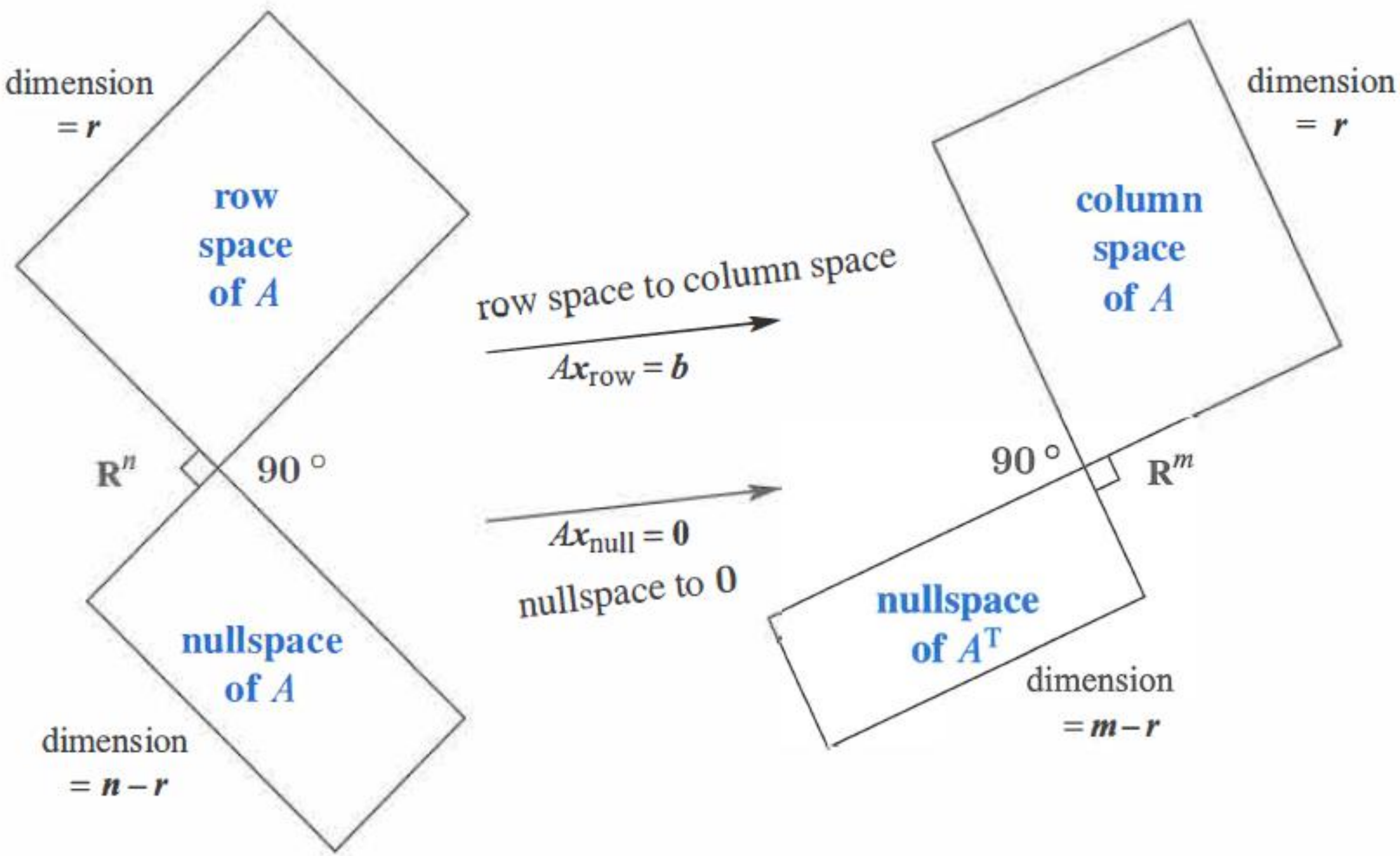
The whole row space $C(A^T)$ is orthogonal to $N(A)$.

Every vector y in the nullspace of A^T is perpendicular to every column of A.

The left nullspace $\boldsymbol{N(A^T)}$ and the column space $\boldsymbol{C(A)}$ are orthogonal in $\mathbb{R}^m$

$$C(A) \perp N(A^{\mathrm{T}})$$

$$A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} (\textbf{column 1})^{\mathrm{T}} \\ \cdots \\ (\textbf{column } n)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ 0 \end{bmatrix}.$$


dimension
= r
row
space
of A
R^n
90 °
nullspace
of A
dimension
= n − r
row space to column space
Ax_row = b
Ax_null = 0
nullspace to 0
dimension
= r
column
space
of A
90 °
R^m
nullspace
of A^T
dimension
= m − r

- **Orthogonal Complements:**

  If W is a subspace of $\mathbb{R}^n$, then the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W is called the orthogonal complement of W and is denoted by the symbol $W^{\perp}$

- **Theorem:**

  If W is a subspace of $\mathbb{R}^n$, then:

  (a) $W^{\perp}$ is a subspace of V

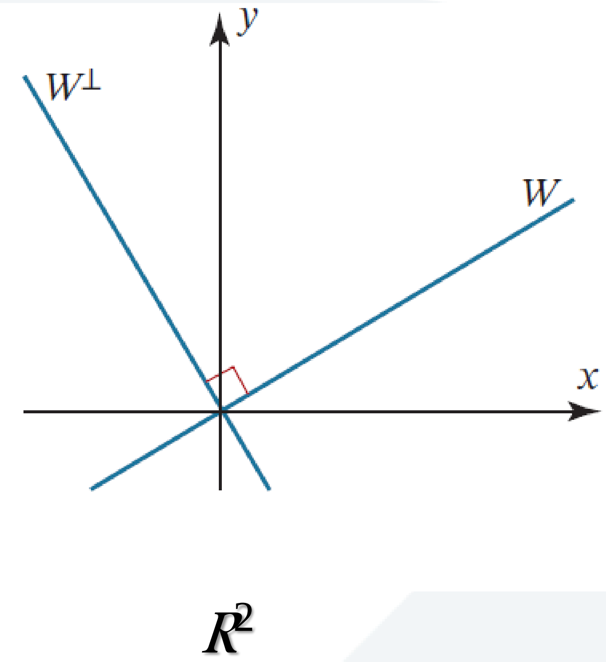
  (b) $W^{\perp} \cap W = \{\mathbf{0}\}$

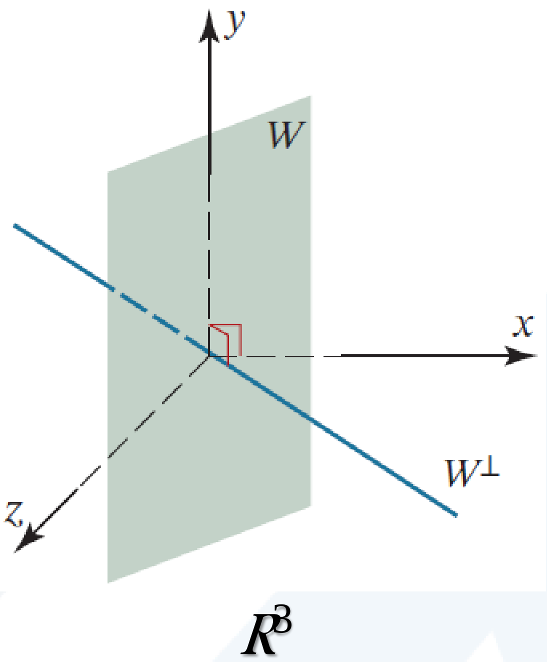
  (c) $R^n = W \oplus W^{\perp}$

- **Note:**

  If W is a subspace of $\mathbb{R}^n$, then $(W^{\perp})^{\perp} = W$

- Ex :





- Theorem : If A is an $m \times n$ matrix, then:

  (a) $N(A)$ and the $C(A^T)$ are orthogonal complements in $\mathbb{R}^n$

  (b) $N(A^T)$ and the $C(A)$ are orthogonal complements in $\mathbb{R}^m$

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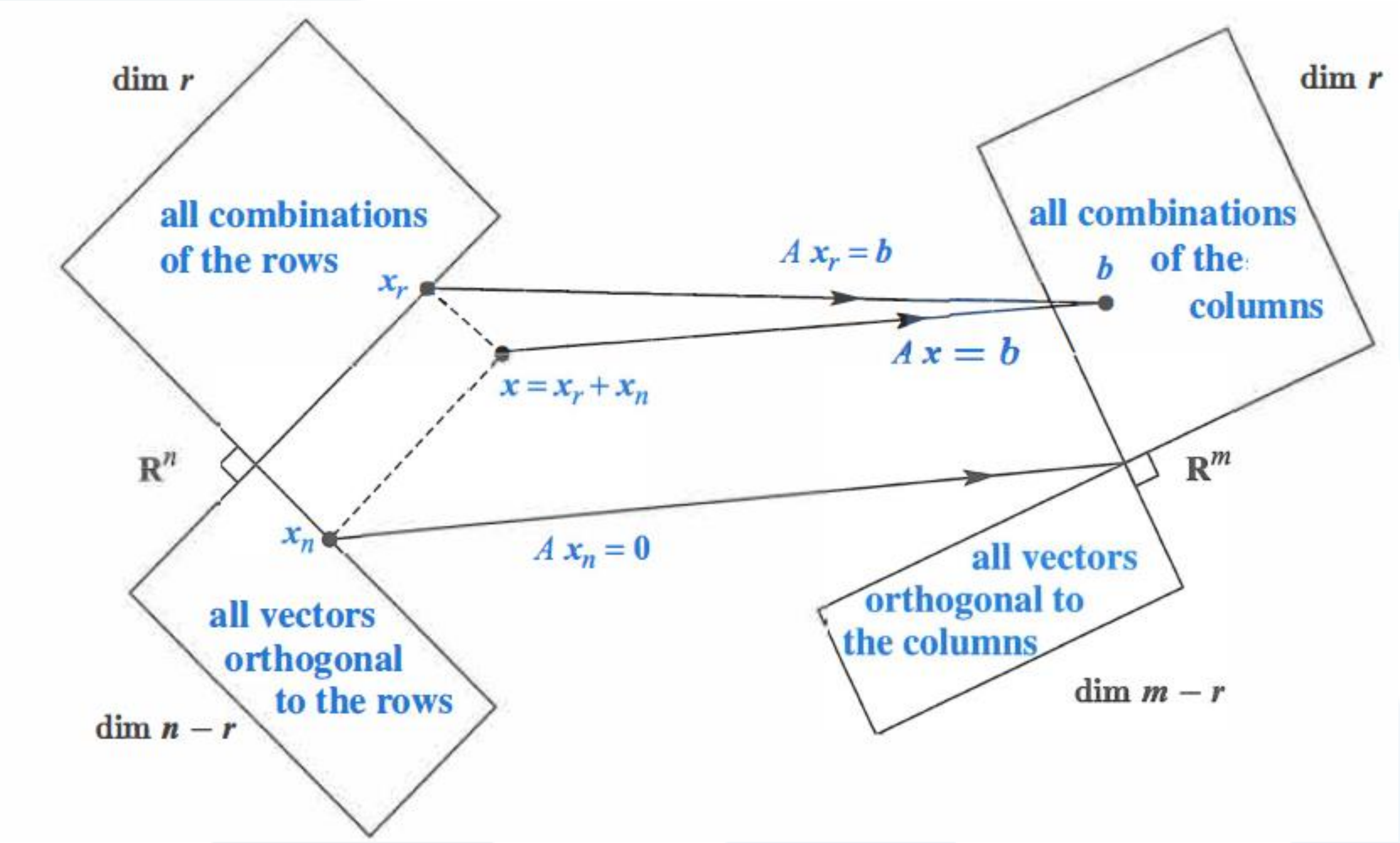


- Ex :

Find the orthogonal complement of the subspace W of R^4 spanned by the two column vectors v_1 and v_2 of the matrix A

$$A = \begin{array}{cc} \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ v_1, \; v_2 \end{array}$$

Sol:

$NS(A^T)$ and the $CS(A)$ are orthogonal complements $\Rightarrow$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{u_1,} + t \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{u_2}$$

$W = \text{span}\{v_1, v_2\}$ and $W^{\perp} = \text{span}\{u_1, u_2\}$

- Summary of equivalent conditions for square matrices:

If A is an $n \times n$ matrix, then the following conditions are equivalent:

(1) A is invertible

(2) $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $n \times 1$ matrix $\mathbf{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(4) A is row-equivalent to I_n

(5) $\text{rank}(A) = n$

(6) The n row vectors of A are linearly independent.

(7) The n column vectors of A are linearly independent.

(8) The column vectors of A span R^n

(9) The row vectors of A span R^n

(10) The column vectors of A form a basis for R^n

(11) The row vectors of A form a basis for R^n

(12) $\text{rank}(A) = n$

(13) $\text{nullity}(A) = 0$

(14) The orthogonal complement of the null space of A is R^n

(15) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$

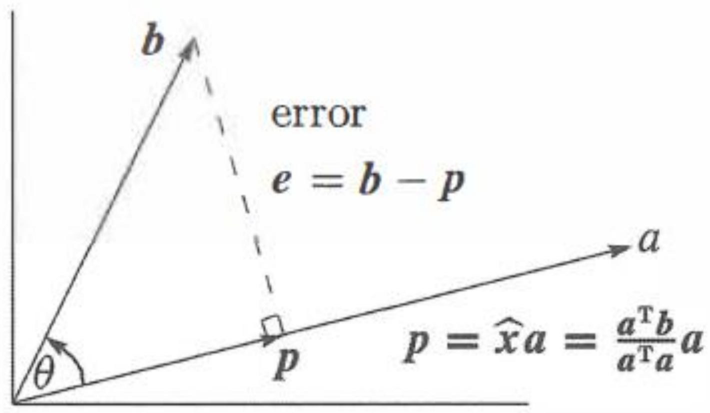
- ## Orthogonal projections :

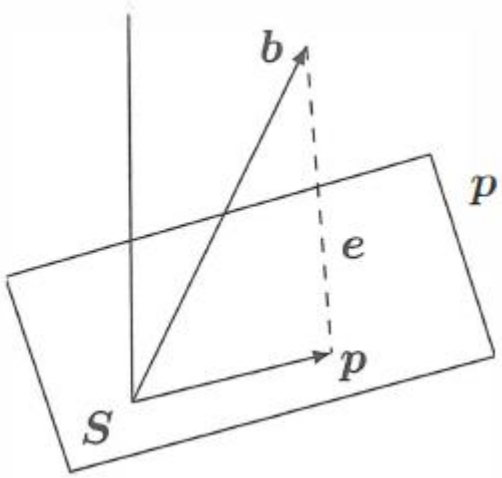
Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two vectors in an inner product space V, such that $\boldsymbol{a} \neq 0$. Then the orthogonal projection of b onto $\boldsymbol{a}$ is given by $\text{proj}_a b = a\dfrac{a^T b}{a^T a}$

Projecting $\boldsymbol{b}$ onto $\boldsymbol{a}$ with error $\boldsymbol{e} = \boldsymbol{b} - \widehat{x}\boldsymbol{a}$

$\boldsymbol{a} \cdot (\boldsymbol{b} - \widehat{x}\boldsymbol{a}) = 0 \quad \text{or} \quad \boldsymbol{a} \cdot \boldsymbol{b} - \widehat{x}\boldsymbol{a} \cdot \boldsymbol{a} = 0$

$$\widehat{x} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{a} \cdot \boldsymbol{a}} = \frac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}.$$





$$\boldsymbol{p} = A\widehat{\boldsymbol{x}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = P\boldsymbol{b}$$

The projection $\boldsymbol{p}$ of $\boldsymbol{b}$ onto a line and onto $\boldsymbol{S}$ = column space of A.

The projection of b onto the line through a is the vector $\boldsymbol{p} = \widehat{x}\boldsymbol{a} = \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a}$

Special case 1: If $\boldsymbol{b} = \boldsymbol{a}$ then $\widehat{x} = 1$. The projection of $\boldsymbol{a}$ onto $\boldsymbol{a}$ is itself. $P\boldsymbol{a} = \boldsymbol{a}$.

Special case 2: If $\boldsymbol{b}$ is perpendicular to $\boldsymbol{a}$ then $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = 0$. The projection is $\boldsymbol{p} = \mathbf{0}$.

Projection matrix:

Now comes the projection matrix. In the formula for p, what matrix is multiplying b? You can see the matrix better if the number x is on the right side of a:

Projection matrix: Proj $p = Pb = a\dfrac{a^T b}{a^T a}$ the matrix is $P = \dfrac{aa^T}{a^T a}$

Properties of projection matrix

- P is a column times a row!
- The column is a, the row is a^T, divide by the number $a^T a$.
- The projection matrix P is m by m,
- its rank is one.
- the column space of P is the line through a
- P is symmetric $P^T = P$
- $P^2 = P$. Projecting a second time doesn't change anything,

Why project?

Because $Ax = b$ may have no solution

Instead : Solve $A\hat{x} = p$ (projection of b onto the column space of A)

***Problem:* Find the combination $p = \widehat{x}_1 a_1 + \cdots + \widehat{x}_n a_n$ closest to a given vector b.**

Find the vector $\widehat{x}$, find the projection $p = A\widehat{x}$, find the projection matrix P.

The key: ***This error vector $b - A\widehat{x}$ is perpendicular to the subspace.***

$$\begin{bmatrix} - & a_1^{\mathrm{T}} & - \\ & \vdots & \\ - & a_n^{\mathrm{T}} & - \end{bmatrix} \begin{bmatrix} \\ b - A\hat{x} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ 0 \\ \\ \end{bmatrix}$$

The matrix with those rows ${a^T}_i$ is A^T . The n equations are exactly $A^T(b - A\hat{x}) = 0$

Note: $e = (b - A\hat{x}) \in N(A^T)$ witch is perpendicular to $C(A)$

Rewrite the last equation $A^T A\hat{x} = A^T b$

The combination $p = \widehat{x_1}a_1 + \cdots + \widehat{x_n}a_n$ that is the closest to b comes from $\hat{x}$

Find $\hat{x}$ $(n \times 1)$ $\qquad A^T(b - A\hat{x}) = 0$ or $\qquad A^TA\hat{x} = A^Tb$

This symmetric matrix A^TA is n by n. It is invertible if the a's are independent.
The solution is $\hat{x} = (A^TA)^{-1}A^Tb$. The projection of b onto the subspace is p:

Find p $(m\times1)$ $\qquad p = A\hat{x} = A(A^TA)^{-1}A^Tb$

The next formula picks out the projection matrix that is multiplying b

Find P $(m\times m)$ $\qquad \mathrm{P} = A(A^TA)^{-1}A^T$

The matrix A is rectangular. It has no inverse matrix.

Orthonormal Bases: Gram-Schmidt Process

- Orthogonal:

A set S of vectors in an inner product space V is called an orthogonal set if every pair of vectors in the set is orthogonal.

$$S=\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\cdots,\ \boldsymbol{v}_n\}\subseteq V \qquad <\boldsymbol{v}_i,\boldsymbol{v}_j>=0,\quad i\neq j$$

- Orthonormal:

An orthogonal set in which each vector is a unit vector is called orthonormal

$$S=\{\boldsymbol{v}_1,\ \boldsymbol{v}_2,\cdots,\ \boldsymbol{v}_n\}\subseteq V \qquad <\boldsymbol{v}_i,\boldsymbol{v}_j>=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$$

- Note:

If S is a basis, then it is called an orthogonal basis or an orthonormal basis.

- Ex : (A nonstandard orthonormal basis for R^3)

Show that the following set is an orthonormal basis.

$$S = \left\{ \overset{v_1}{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)}, \overset{v_2}{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3} \right)}, \overset{v_3}{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right)} \right\}$$

Sol:

$$v_1 \cdot v_2 = -\frac{1}{6} + \frac{1}{6} + 0 = 0$$

$$v_1 \cdot v_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$v_2 \cdot v_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

$$\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

$$\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{9}} = 1$$

$$\|v_3\| = \sqrt{v_3 \cdot v_3} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$$

- Ex : (An orthonormal basis for P_3)

with the inner product $\langle p, q\rangle = a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$

the standard basis $B = \{1, x, x^2, x^3\}$ is orthonormal

- Ex : (An Orthogonal Set in $C[0, 2\pi]$)

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx$$

Show that the set $S = \{1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx\}$ is orthogonal

Sol: $\langle 1, \sin nx\rangle = \int_0^{2\pi} \sin nx\, dx = 0, \quad \langle 1, \cos nx\rangle = \int_0^{2\pi} \cos nx\, dx = 0$

$$\langle \sin mx, \cos nx\rangle = \int_0^{2\pi} \sin mx \cos nx\, dx = 0$$

$$\langle \sin mx, \sin nx\rangle = \int_0^{2\pi} \sin mx \sin nx\, dx = 0, \quad m \neq n$$

$$\langle \cos mx, \cos nx \rangle = \int_0^{2\pi} \cos mx \cos nx \, dx = 0, \quad m \neq n$$

The set S is orthogonal but not orthonormal

An orthonormal set can be formed by normalizing each vector in S

$$\|1\| = \sqrt{\langle 1, 1 \rangle} = \sqrt{\int_0^{2\pi} dx} = \sqrt{2\pi}$$

$$\|\sin nx\| = \sqrt{\langle \sin nx, \sin nx \rangle} = \sqrt{\int_0^{2\pi} \sin^2 nx \, dx} = \sqrt{\pi}$$

$$\|\cos nx\| = \sqrt{\langle \cos nx, \cos nx \rangle} = \sqrt{\int_0^{2\pi} \cos^2 nx \, dx} = \sqrt{\pi}$$

So the set $\left\{ \dfrac{1}{\sqrt{2\pi}}, \dfrac{1}{\sqrt{\pi}} \sin x, \dfrac{1}{\sqrt{\pi}} \cos x, \cdots, \dfrac{1}{\sqrt{\pi}} \sin nx, \dfrac{1}{\sqrt{\pi}} \cos nx \right\}$ is orthonormal

- **Theorem : (Orthogonal sets are linearly independent)**

If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

- **Corollary to Theorem :**

If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

- **Ex : (Using orthogonality to test for a basis)**

Show that the following set is a basis for R^4

$$S = \{\underset{v_1}{(2, 3, 2, -2)},\ \underset{v_2}{(1, 0, 0, 1)},\ \underset{v_3}{(-1, 0, 2, 1)},\ \underset{v_4}{(-1, 2, -1, 1)}\}$$

Sol:

$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$: nonzero vectors

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 + 0 + 0 - 2 = 0 \qquad \mathbf{v}_2 \cdot \mathbf{v}_3 = -1 + 0 + 0 + 1 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -2 + 0 + 4 - 2 = 0 \qquad \mathbf{v}_2 \cdot \mathbf{v}_4 = -1 + 0 + 0 + 1 = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_4 = -2 + 6 - 2 - 2 = 0 \qquad \mathbf{v}_3 \cdot \mathbf{v}_4 = 1 + 0 - 2 + 1 = 0$$

$\Rightarrow S$ is orthogonal $\Rightarrow S$ is a basis for R^4

- Theorem : (Coordinates relative to an orthonormal basis)

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogoal/orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n \qquad \text{orthogonal}$$

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n \qquad \text{orthonormal}$$

- Note:

Coordinate vector of a vector w in V relative to an orthogonal/ orthonormal basis S is

$$[u]_S = \left(\frac{\langle u, v_1\rangle}{\|v_1\|^2}, \frac{\langle u, v_2\rangle}{\|v_2\|^2}, \cdots, \frac{\langle u, v_n\rangle}{\|v_n\|^2}\right)^T \qquad [u]_S = (\langle u, v_1\rangle, \langle u, v_2\rangle, \cdots, \langle u, v_n\rangle)^T$$

- Ex : (Representing vectors relative to an orthonormal basis)

Find the coordinates of vector $u = (5, -5, 2)$ relative to the following orthonormal basis

$$S = \{(\tfrac{3}{5}, \tfrac{4}{5}, 0), (-\tfrac{4}{5}, \tfrac{3}{5}, 0), (0, 0, 1)\}$$

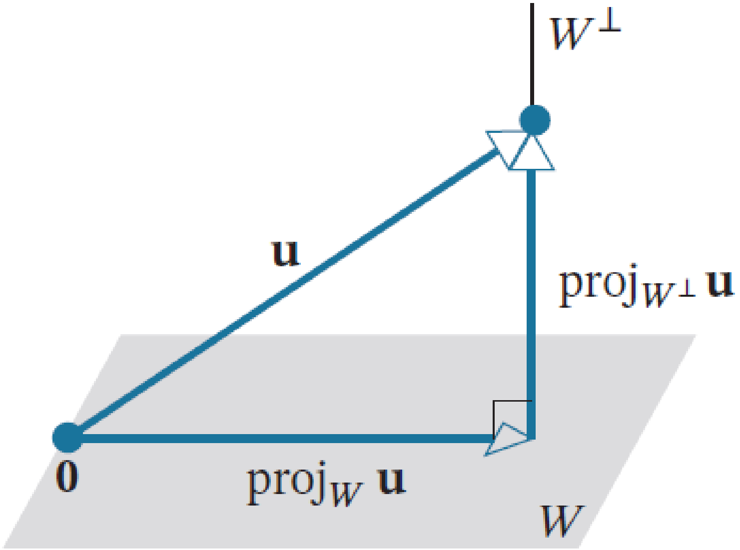
Sol:

$$< u, v_1 > = u . v_1 = (5, -5, 2) . (\tfrac{3}{5}, \tfrac{4}{5}, 0) = -1$$
$$< u, v_2 > = u . v_2 = (5, -5, 2) . (-\tfrac{4}{5}, \tfrac{3}{5}, 0) = -7 \quad \Rightarrow [u]_S = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$
$$< u, v_3 > = u . v_3 = (5, -5, 2) . (0, 0, 1) = 2$$

- Theorem : (Projection Theorem)

If W is a finite-dimensional subspace of an inner product space V, then every vector $\boldsymbol{u}$ in V can be expressed in exactly one way as $\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$, where $\boldsymbol{w}_1$ is in W and $\boldsymbol{w}_2$ is in $W^{\perp}$

$$\boldsymbol{u} = \text{proj}_W \boldsymbol{u} + \text{proj}_{W^{\perp}} \boldsymbol{u} = \text{proj}_W \boldsymbol{u} + (\boldsymbol{u} - \text{proj}_W \boldsymbol{u})$$



- Theorem : (Projection Theorem)

Let W be a finite-dimensional subspace of an inner product space V. If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$ is an orthogonal/orthonormal basis for W, then

$$\operatorname{proj}_W \boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 + \cdots + \frac{\langle \boldsymbol{u}, \boldsymbol{v}_r \rangle}{\|\boldsymbol{v}_r\|^2}\boldsymbol{v}_r$$ orthogonal

$$\operatorname{proj}_W \boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 + \langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 + \cdots + \langle \boldsymbol{u}, \boldsymbol{v}_r \rangle \boldsymbol{v}_r$$ orthonormal

- Ex : (Calculating Projections)

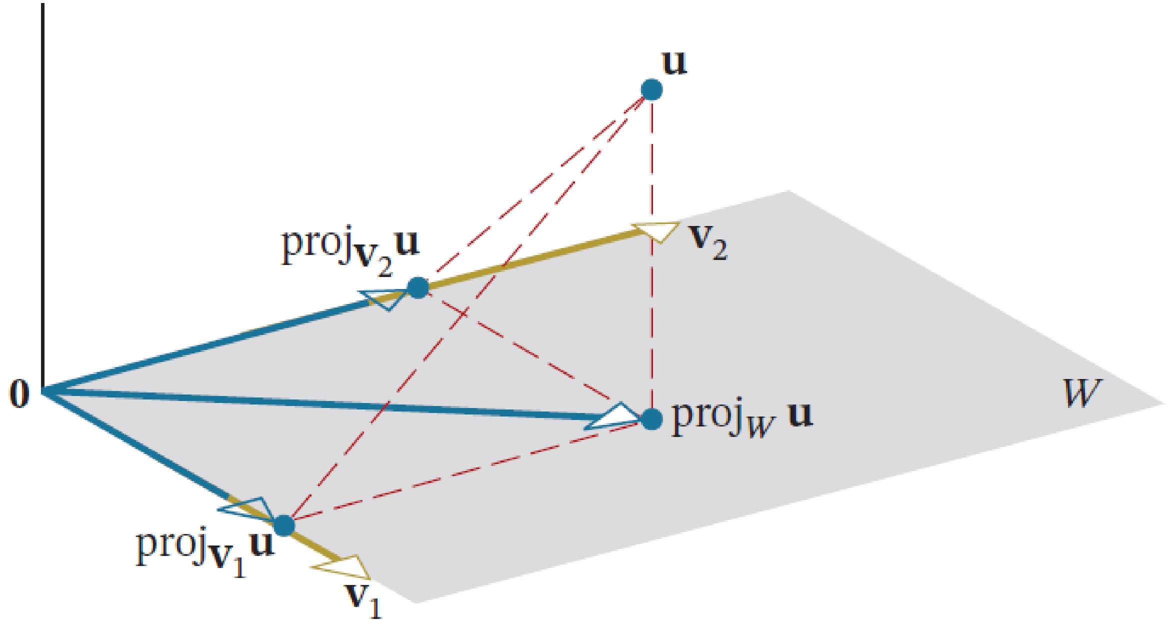
Let R^3 have the Euclidean inner product, and let W be the subspace spanned by the orthonormal vectors $\boldsymbol{v}_1 = (0, 1, 0)$ and $\boldsymbol{v}_2 = (-4/5, 0, 3/5)$. The orthogonal projection of $\boldsymbol{u} = (1, 1, 1)$ on W is

$$\text{proj}_W \boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v}_1 \rangle \boldsymbol{v}_1 + \langle \boldsymbol{u}, \boldsymbol{v}_2 \rangle \boldsymbol{v}_2 = (1)(0, 1, 0) + (-1/5)(-4/5, 0, 3/5) = (4/25, 1, -3/25)$$

The component of $\boldsymbol{u}$ orthogonal to W is

$$\text{proj}_{W^\perp} \boldsymbol{u} = \boldsymbol{u} - \text{proj}_W \boldsymbol{u} = (1, 1, 1) - (4/25, 1, -3/25) = (21/25, 0, 28/25)$$

- A geometric interpretation of orthogonal projections in R^3



- Theorem : (Projection Theorem)

Every nonzero finite-dimensional inner product space has an orthonormal basis

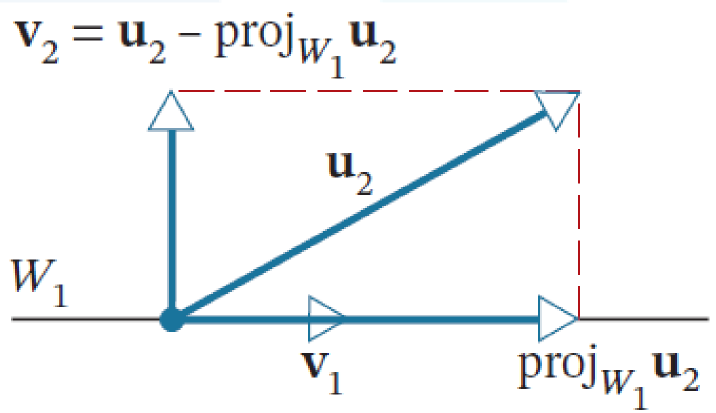
Proof (Gram-Schmidt orthonormalization construction)

Let W be any nonzero finite-dimensional subspace of an inner product space, and suppose that $\{u_1, u_2, \ldots, u_r\}$ is any basis for W

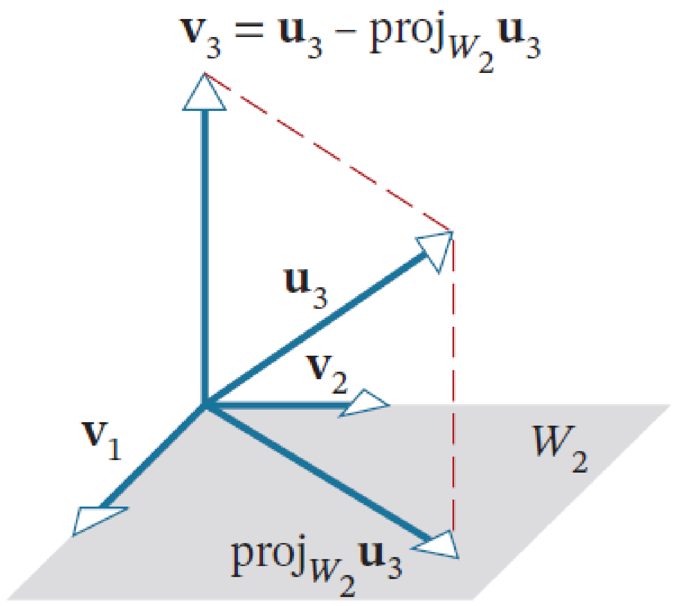
Step 1: Let $v_1 = u_1$

Step 2: $v_2 = u_2 - \text{proj}_{W_1} u_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$

$W_1 = \text{span}(v_1)$ and $v_2 \perp v_1,\ v_2 \neq 0$



Step 3: $\boldsymbol{v}_3 = \boldsymbol{u}_3 - \text{proj}_{W_2}\boldsymbol{u}_3 = \boldsymbol{u}_3 - \dfrac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1 \rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle}\boldsymbol{v}_1 - \dfrac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2 \rangle}{\langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle}\boldsymbol{v}_2$



$W_2 = \text{span}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ and $\boldsymbol{v}_3 \perp W_2,\ \boldsymbol{v}_3 \neq \mathbf{0}$

Continuing in this way we will produce after r steps an orthogonal set of nonzero vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r\}$

By normalizing these basis vectors we can obtain an orthonormal basis

- Theorem : (Gram-Schmidt orthonormalization process)

(1) Let $B = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is a basis for an inner product space V

(2) Let $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, where

$$v_1 = u_1$$

$$v_2 = u_2 - \text{proj}_{W_1} u_2 = u_2 - \frac{< u_2, v_1 >}{< v_1, v_1 >} v_1$$

$$v_3 = u_3 - \text{proj}_{W_2} u_3 = u_3 - \frac{< u_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< u_3, v_2 >}{< v_2, v_2 >} v_2$$

$$\vdots$$

$$v_n = u_n - \text{proj}_{W_{n-1}} u_n = u_n - \sum_{i=1}^{n-1} \frac{< u_n, v_i >}{< v_i, v_i >} v_i$$

Then B' is an orthogonal basis for V

(3) Let $w_i = \dfrac{v_i}{\|v_i\|}$

Then $B'' = \{ w_1, w_2, \ldots, w_n \}$ is an orthonormal basis for V

Also, span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ = span$\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_k\}$ for $k = 1, 2, \ldots, n$

- Ex : (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt orthonormalization process to the basis B for R^2

$$B = \{\overset{\boldsymbol{u}_1}{(1, 1)}, \overset{\boldsymbol{u}_2}{(0, 1)}\}$$

Sol:

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1, 1)$$

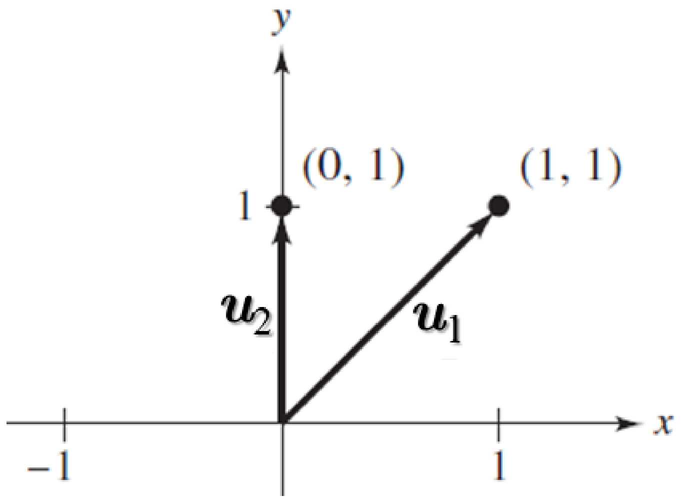
$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{<\boldsymbol{u}_2, \boldsymbol{v}_1>}{<\boldsymbol{v}_1, \boldsymbol{v}_1>}\boldsymbol{v}_1 = (0, 1) - \frac{1}{2}(1, 1) = (-\frac{1}{2}, \frac{1}{2})$$

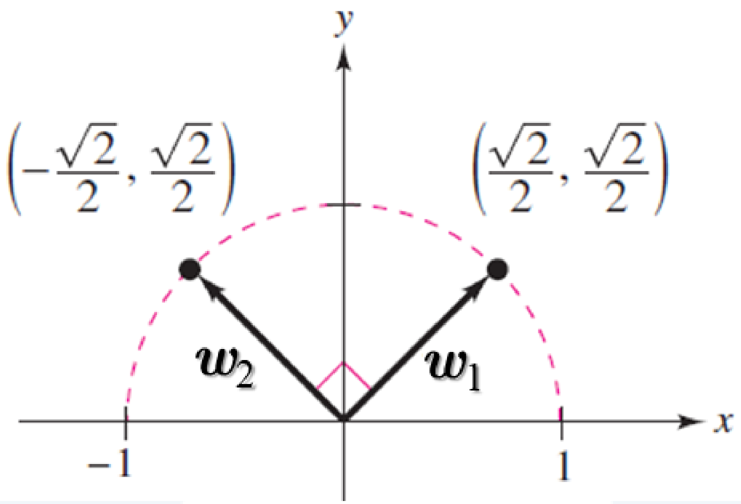
The set $B' = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ is an orthogonal basis for R^2

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1, 1) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

The set $B' = \{w_1, w_2\}$ is an orthonormal basis for R^2





- Ex : (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt orthonormalization process to the basis B for R^3

$$B = \{\overset{u_1}{(1,1,0)},\ \overset{u_2}{(1,2,0)},\ \overset{u_3}{(0,1,2)}\}$$

Sol:

$$v_1 = u_1 = (1, 1, 0)$$

$$v_2 = u_2 - \frac{< u_2, v_1 >}{< v_1, v_1 >} v_1 = (1, 2, 0) - \frac{3}{2}(1, 1, 0) = (-\frac{1}{2}, \frac{1}{2}, 0)$$

$$v_3 = u_3 - \frac{< u_3, v_1 >}{< v_1, v_1 >} v_1 - \frac{< u_3, v_2 >}{< v_2, v_2 >} v_2$$

$$= (1, 2, 0) - \frac{1}{2}(1, 1, 0) - \frac{1/2}{1/2}(-\frac{1}{2}, \frac{1}{2}, 0) = (0, 0, 2)$$

The set $B = \{v_1, v_2, v_3\}$ is an orthogonal basis for R^3

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}, 0\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1)$$

The set $B' = \{w_1, w_2, w_3\}$ is an orthonormal basis for R^3

- Ex : (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt orthonormalization process to the standard basis $B = \{1, x, x^2\}$ in P_2

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx$$

Sol:

Let $B = \{1, x, x^2\} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = 1$$

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\langle \boldsymbol{u}_2, \boldsymbol{v}_1\rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle}\boldsymbol{v}_1 = x - \frac{0}{2}(1) = x$$

$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_1\rangle}{\langle \boldsymbol{v}_1, \boldsymbol{v}_1\rangle}\boldsymbol{v}_1 - \frac{\langle \boldsymbol{u}_3, \boldsymbol{v}_2\rangle}{\langle \boldsymbol{v}_2, \boldsymbol{v}_2\rangle}\boldsymbol{v}_2 = x^2 - \frac{2/3}{2}(1) - \frac{0}{2/3}(x) = x^2 - \frac{1}{3}$$

by normalizing $B = \{v_1, v_2, v_3\}$

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}$$

$$w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2/3}}(x) = \sqrt{\frac{3}{2}}x$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{8/45}}(x^2 - \frac{1}{3}) = \frac{1}{2}\sqrt{\frac{5}{2}}(3x^2 - 1)$$

Legendre Polynomials