MATHEMATICAL ANALAYSIS 1

Lecture 5

Prepared by

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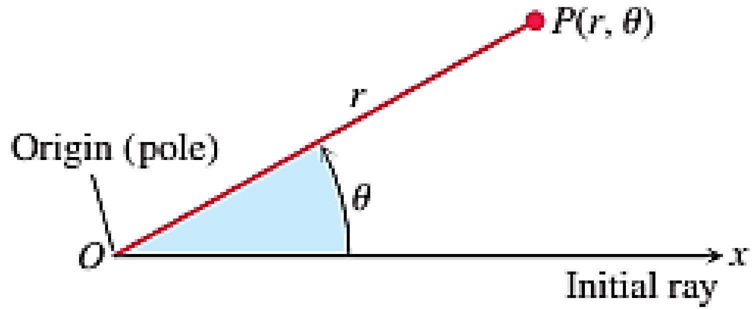
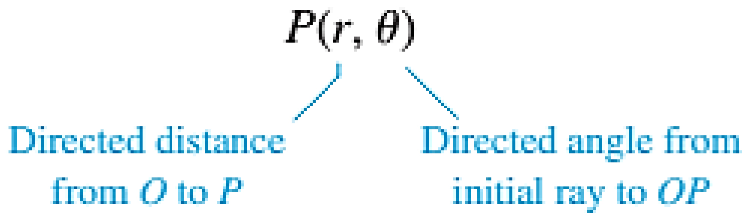
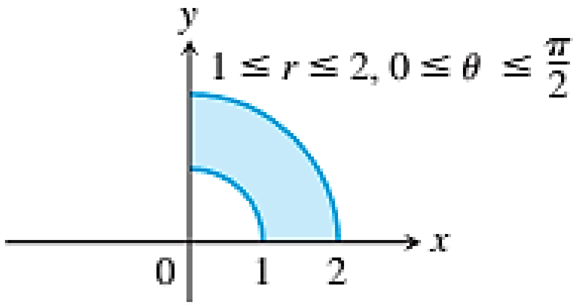
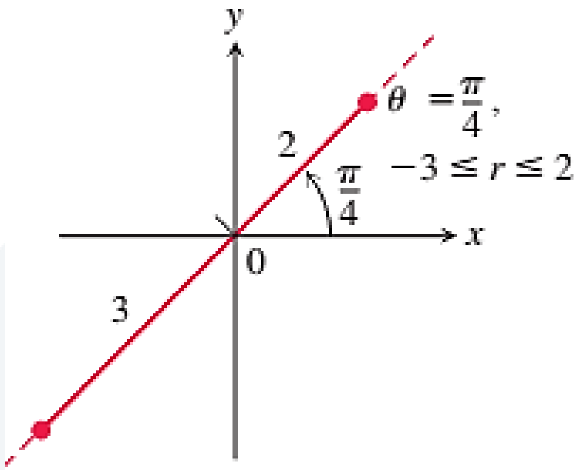
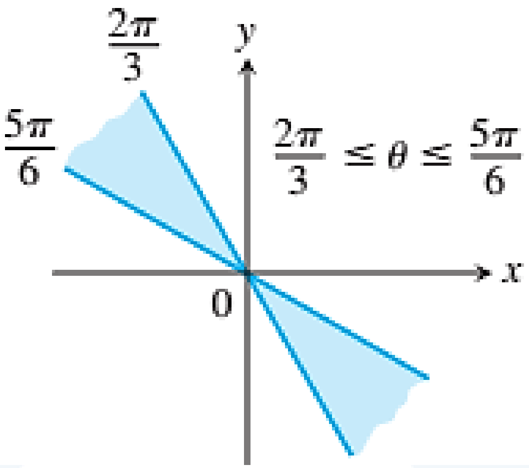
Polar Coordinates







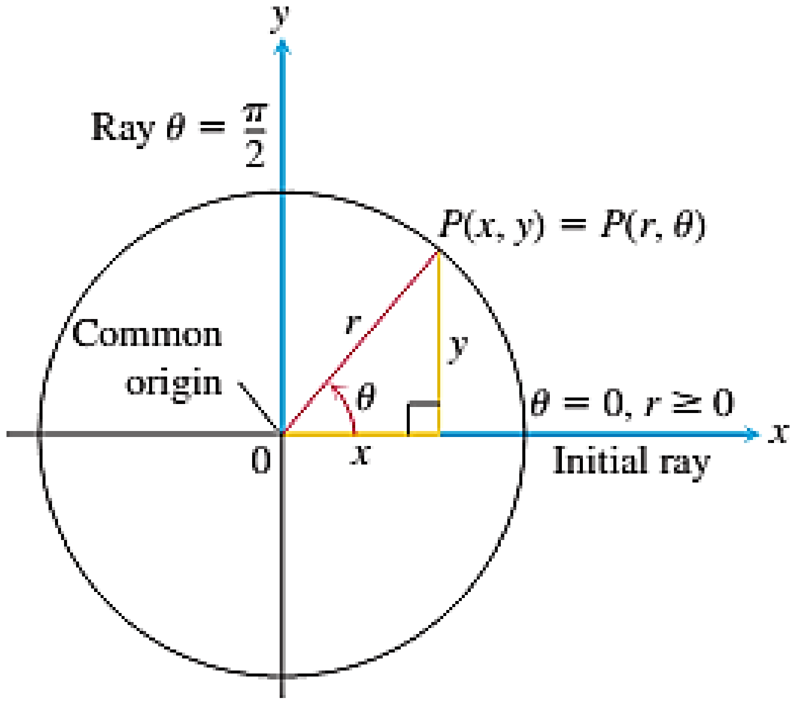


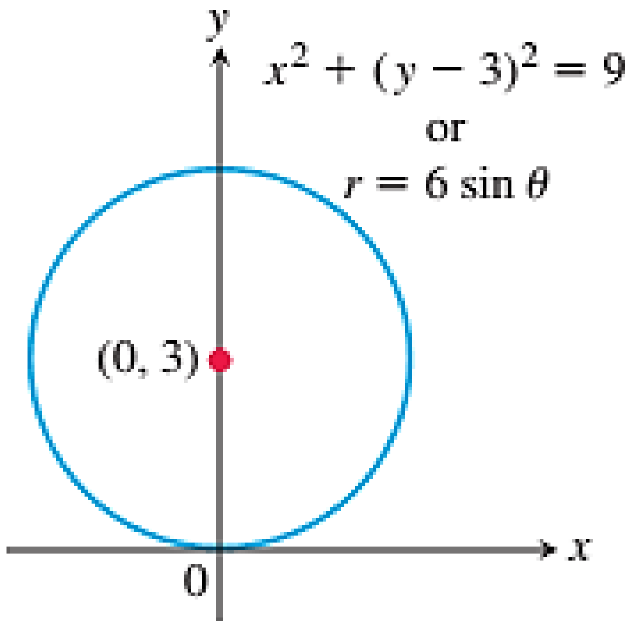


Polar Coordinates

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$





Polar Coordinates

EXAMPLE 6 Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r \cos \theta = -4$ $\qquad x = -4$

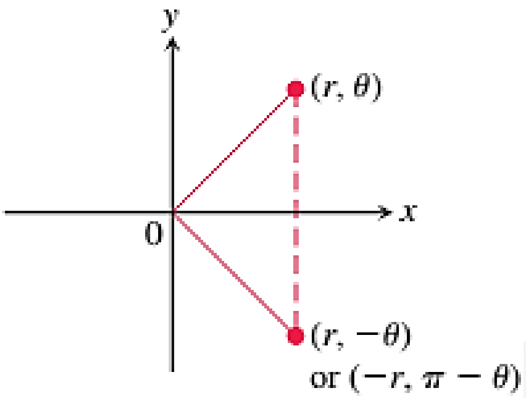
(b) $r^2 = 4r \cos \theta$ $\qquad (x - 2)^2 + y^2 = 4$

(c) $r = \dfrac{4}{2 \cos \theta - \sin \theta}$ $\qquad y = 2x - 4$

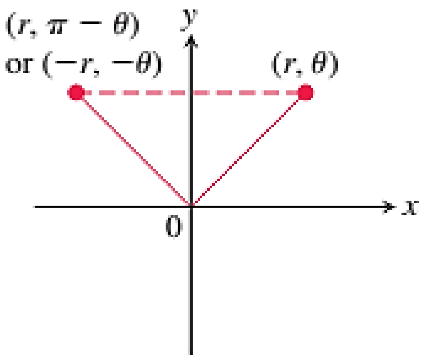
Graphing Polar Coordinate Equations

Symmetry Tests for Polar Graphs in the Cartesian *xy*-Plane

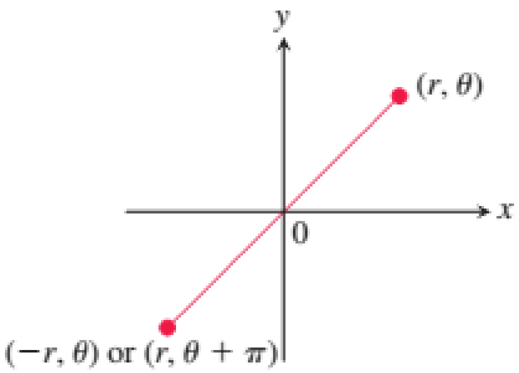
1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 11.28a).
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 11.28b).
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 11.28c).



(a) About the *x*-axis



(b) About the *y*-axis



(c) About the origin

Slope

The slope of a polar curve $r = f(\theta)$ in the xy-plane is dy/dx, but this is **not** given by the formula $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Slope of the Curve $r = f(\theta)$ in the Cartesian xy-Plane

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} \tag{1}$$

provided $dx/d\theta \neq 0$ at (r, θ).

Graphing Polar Coordinate Equations

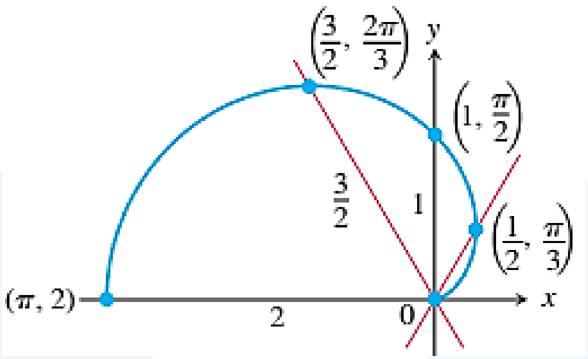
EXAMPLE 1 Graph the curve $r = 1 - \cos\theta$ in the Cartesian xy-plane.

Symmetry Tests

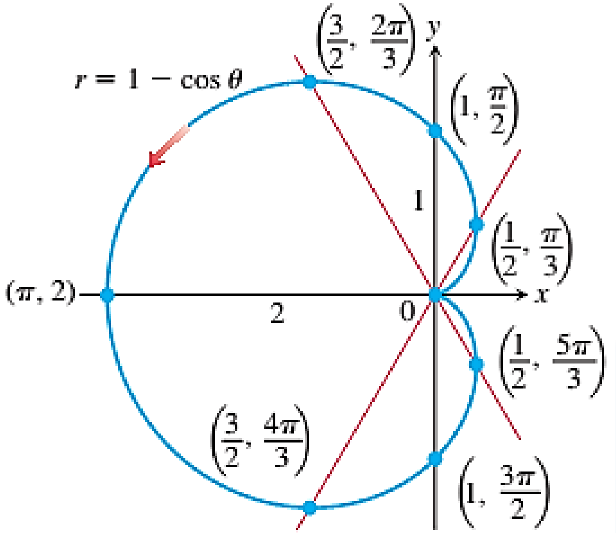
$$
\begin{aligned}
(r, \theta) \text{ on the graph} &\Rightarrow r = 1 - \cos\theta \\
&\Rightarrow r = 1 - \cos(-\theta) \\
&\Rightarrow (r, -\theta) \text{ on the graph.}
\end{aligned}
$$

The curve is symmetric about the x-axis

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2



Cardioid



Areas and Lengths in Polar Coordinates

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$ when $\alpha \le \theta \le \beta$, $r \ge 0$, and $\beta - \alpha \le 2\pi$.

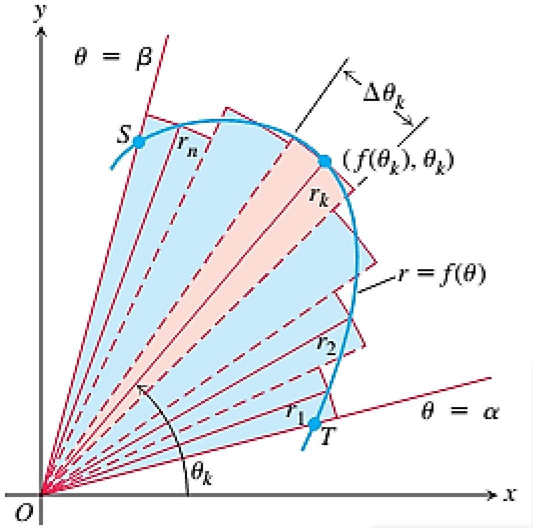
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

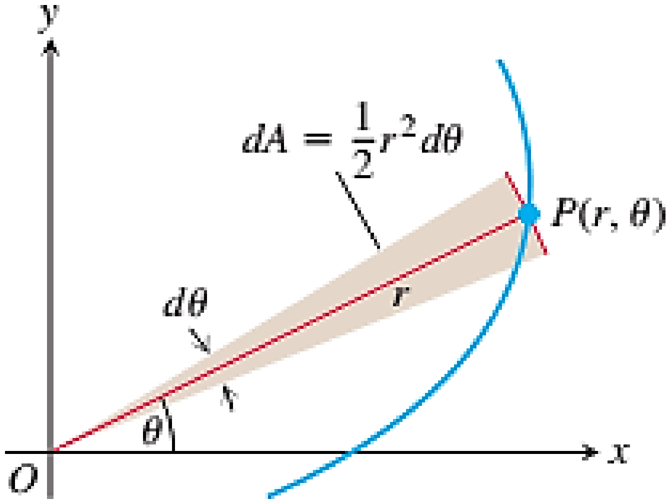
This is the integral of the **area differential** (Figure 11.33)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left(f(\theta)\right)^2 d\theta.$$

$$A_k = \frac{1}{2} {r_k}^2 \, \Delta\theta_k = \frac{1}{2} \left(f(\theta_k)\right)^2 \Delta\theta_k.$$

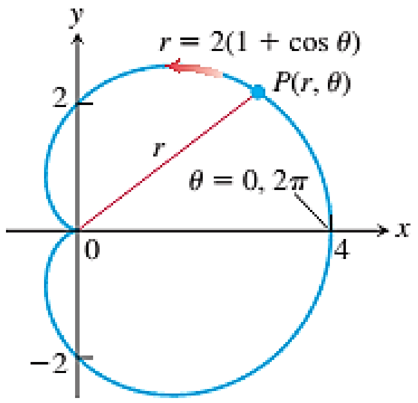
$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2} \left(f(\theta_k)\right)^2 \Delta\theta_k.$$





Areas and Lengths in Polar Coordinates

EXAMPLE 1 Find the area of the region in the xy-plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

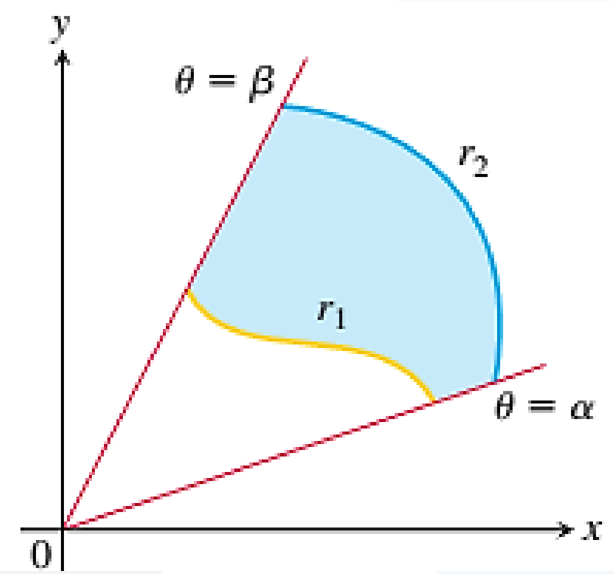


$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta = 6\pi.$$

Areas and Lengths in Polar Coordinates

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, and $\beta - \alpha \le 2\pi$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad (1)$$



Areas and Lengths in Polar Coordinates

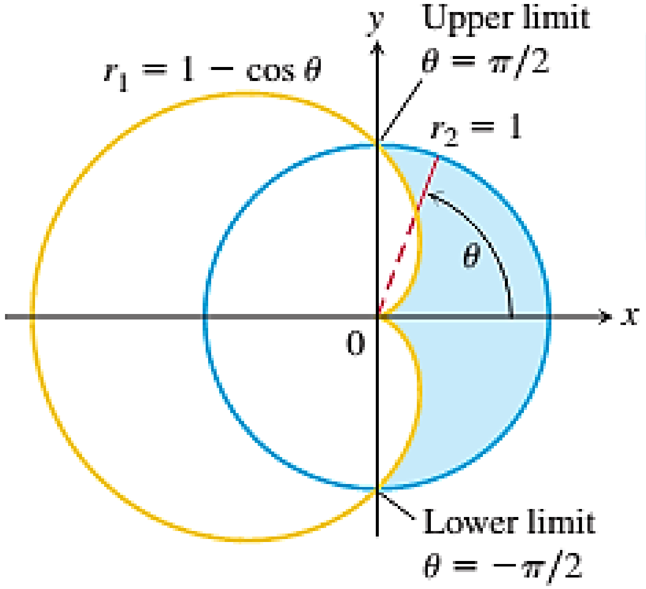
EXAMPLE 2 Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\int_{0}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= \int_{0}^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta$$

$$= 2 - \frac{\pi}{4}.$$



Length of a Polar Curve

Length of a Polar Curve
If $r = f(\theta)$ has a continuous first derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \qquad (3)$$

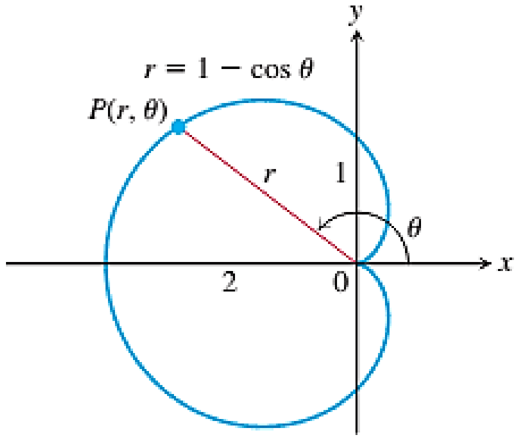
$$x = r\cos\theta = f(\theta)\cos\theta,$$
$$y = r\sin\theta = f(\theta)\sin\theta,$$

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE 4 Find the length of the cardioid $r = 1 - \cos\theta$.

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = 2 - 2\cos\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta = 8.$$

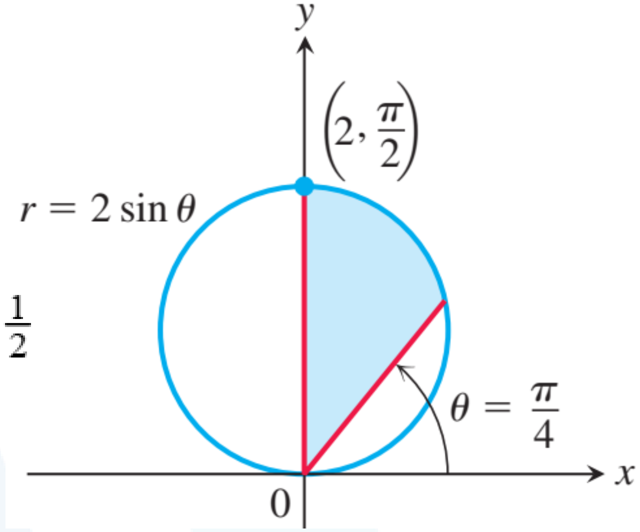


Exercises

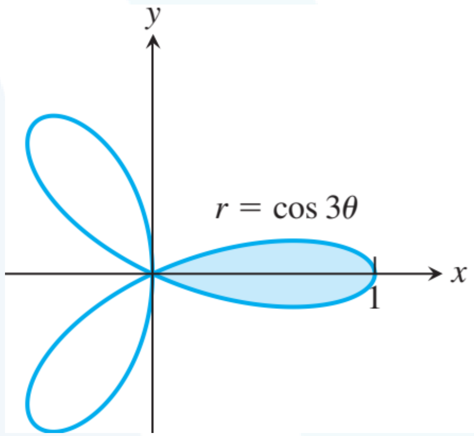
- Find the areas of the regions

  Bounded by the circle $r = 2\sin\theta$ for $\pi/4 \le \theta \le \pi/2$

  $\frac{\pi}{4} + \frac{1}{2}$



- Inside one leaf of the three-leaved rose $r = \cos 3\theta$



- Find the lengths of the curves

  The spiral $r = \theta^2, \quad 0 \le \theta \le \sqrt{5}$ $\qquad \frac{19}{3}$

  The curve $r = \sqrt{1 + \sin 2\theta}, \quad 0 \le \theta \le \pi\sqrt{2}$ $\qquad 2\pi$

Work Done by a Variable Force Along a Line

DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \qquad (2)$$

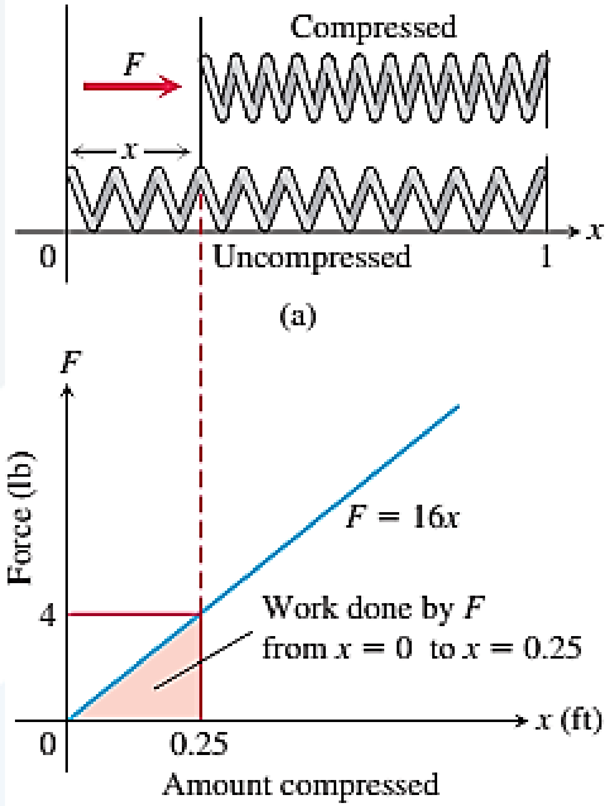
$$\text{Work} \approx \sum_{k=1}^{n} F(c_k)\,\Delta x_k .$$

Hooke's Law for Springs: $F = kx$ k is force constant

EXAMPLE 2 Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

$$F = 16x$$

$$W = \int_0^{0.25} 16x\,dx = 8x^2\Big]_0^{0.25} = 0.5 \text{ ft-lb}.$$

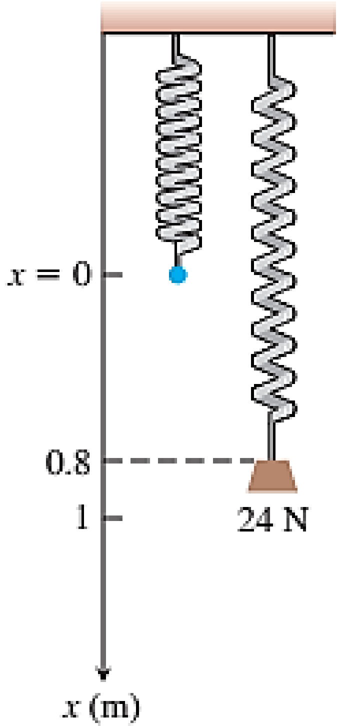


EXAMPLE 3 A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 0.8 m.

(a) Find the force constant k.

(b) How much work will it take to stretch the spring 2 m beyond its natural length?

(c) How far will a 45-N force stretch the spring?

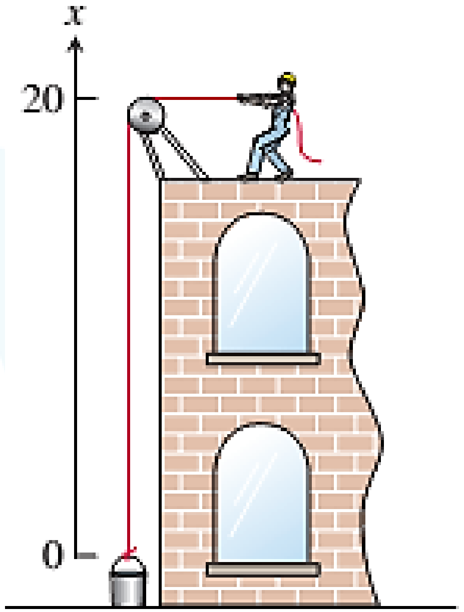


a k ? $24 = k(0.8)$ → $k = 24/0.8 = 30$ N/m.

b $$W = \int_0^2 30x\,dx = 15x^2\Big]_0^2 = 60 \text{ J.}$$

c $45 = 30x,$ → $x = 1.5$ m.

EXAMPLE 4 A 5-kg bucket is lifted from the ground into the air by pulling in 20 m of rope at a constant speed (Figure 6.38). The rope weighs 0.08 kg/m. How much work was spent lifting the bucket and rope?



Weight of Bucket $= 5 \times 9.8 = 49\,N$

Total work=work on the bucket + work on the rope

work on the bucket = weight . distance

$$= 49 \times 20 = 980\,J$$

$$\text{Work on rope} = \int_0^{20} (0.08)(20 - x)(9.8)\,dx = 156.8\text{ J.}$$

Total work= $980 + 156.8 = 1136.8$ J.

Exercises

7. **Subway car springs** It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.

   a. What is the assembly's force constant?

   b. How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in.-lb.

$$k = 7238\ \frac{\text{lb}}{\text{in}}$$

≈ 905 in-lb. ≈ 2714 in-lb

Exercises

11. Lifting an elevator cable An electric elevator with a motor at the top has a multistrand cable weighing 4.5 lb/ft. When the car is at the first floor, 180 ft of cable are paid out, and effectively 0 ft are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

72,900 ft-lb

Moments and Centers of Mass

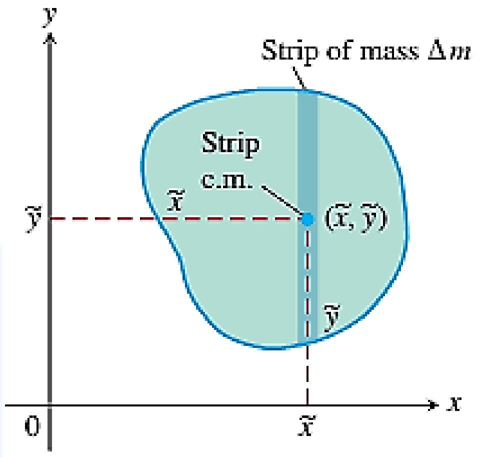
Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane

Moment about the x-axis: $$M_x = \int \tilde{y}\, dm$$

Moment about the y-axis: $$M_y = \int \tilde{x}\, dm$$

Mass: $$M = \int dm \qquad (5)$$

Center of mass: $$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$



dm Mass of strip

If the density of the plat is a constant $\Rightarrow$ $dm = \delta dA$

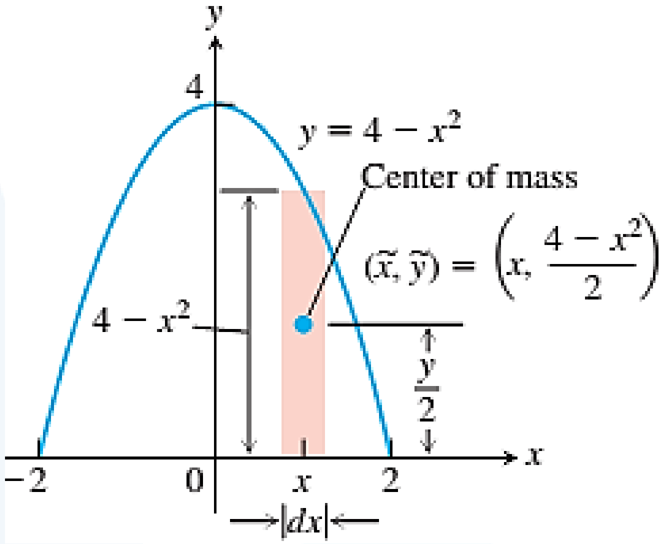
EXAMPLE 3 Find the center of mass of a thin plate covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis (Figure 6.52). Assume the density of the plate at the point (x, y) is $\delta = 2x^2$, which is twice the square of the distance from the point to the y-axis.

Symmetry about y-axis $M_y = 0, \bar{x} = 0$

$$dA = \left(4 - x^2\right)dx$$

$$dm = \delta dA = 2x^2\left(4 - x^2\right)dx$$

$\tilde{y} = y/2 = \left(4 - x^2\right)/2$ The distance of strip's center mass from x-axis



$$M_x = \int \tilde{y}\, dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\, dx = \int_{-2}^{2} x^2(4 - x^2)^2\, dx = \frac{2048}{105}.$$

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\, dx = \int_{-2}^{2} 2x^2(4 - x^2)\, dx = \frac{256}{15}.$$

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right)$$

Plates Bounded by Two Curves

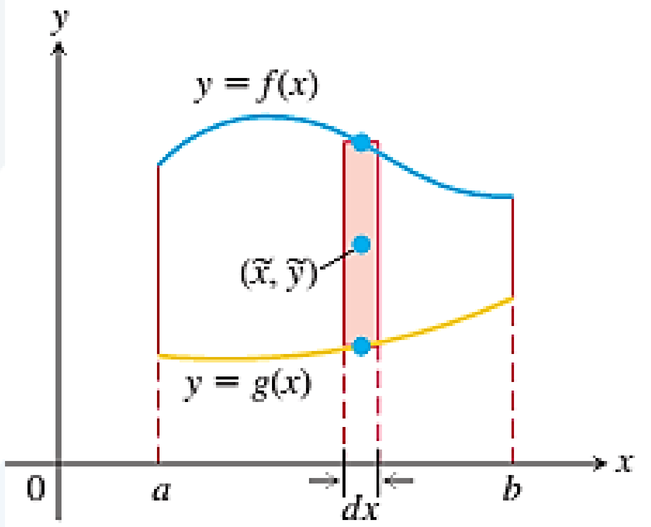
center of mass (c.m.): $(\tilde{x}, \tilde{y}) = \left(x, \frac{1}{2}[f(x) + g(x)]\right)$

length: $f(x) - g(x)$

width: dx

area: $dA = [f(x) - g(x)]\, dx$

mass: $dm = \delta\, dA = \delta[f(x) - g(x)]\, dx.$



$$\bar{x} = \frac{1}{M}\int_a^b \delta x\,[f(x) - g(x)]\,dx$$

$$\bar{y} = \frac{1}{M}\int_a^b \frac{\delta}{2}\left[f^2(x) - g^2(x)\right]dx$$

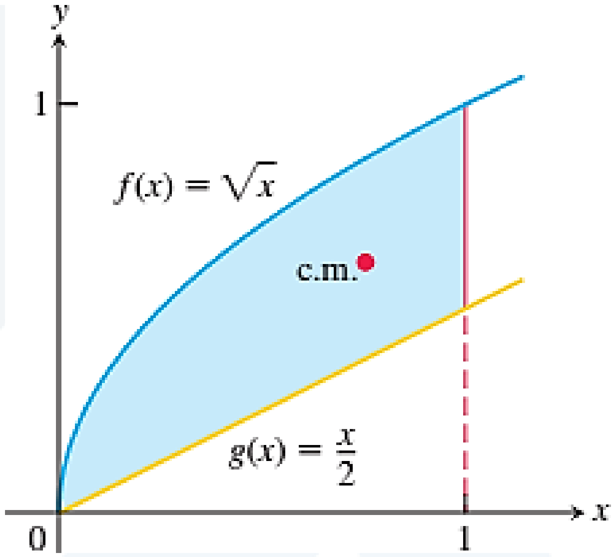
EXAMPLE 4 Find the center of mass for the thin plate bounded by the curves $g(x) = x/2$ and $f(x) = \sqrt{x}$, $0 \le x \le 1$ (Figure 6.54), using Equations (6) and (7) with the density function $\delta(x) = x^2$.

$$M = \int_a^b dm = \int_a^b \delta[f(x) - g(x)]dx$$

$$M = \int_0^1 x^2\left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{9}{56}.$$

$$\bar{x} = \frac{56}{9}\int_0^1 x^2 \cdot x\left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{308}{405},$$

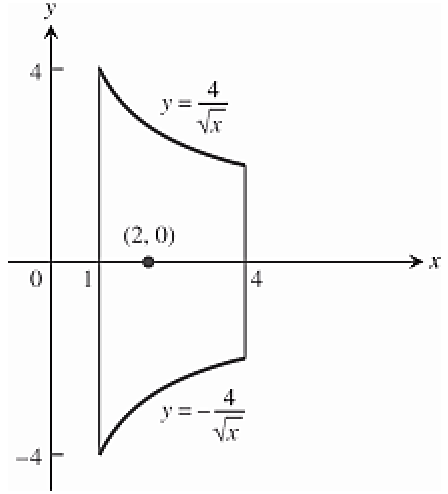
$$\bar{y} = \frac{56}{9}\int_0^1 \frac{x^2}{2}\left(x - \frac{x^2}{4}\right) dx = \frac{252}{405}.$$



Exercises

- The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y-axis to generate a solid.
  - **a.** Find the volume of the solid.
  - **b.** Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.

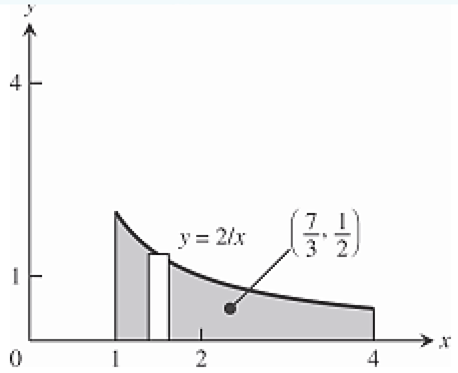
$$\frac{224\pi}{3}$$



$$(\bar{x}, \bar{y}) = (2, 0)$$

- The region between the curve $y = 2/x$ and the x-axis from $x = 1$ to $x = 4$ is revolved about the x-axis to generate a solid.
  - **a.** Find the volume of the solid.
  - **b.** Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.

$$3\pi$$



$$(\bar{x}, \bar{y}) = \left(\frac{7}{3}, \frac{1}{2}\right)$$

Numerical Integration

Trapezoidal Approximations

The Trapezoidal Rule

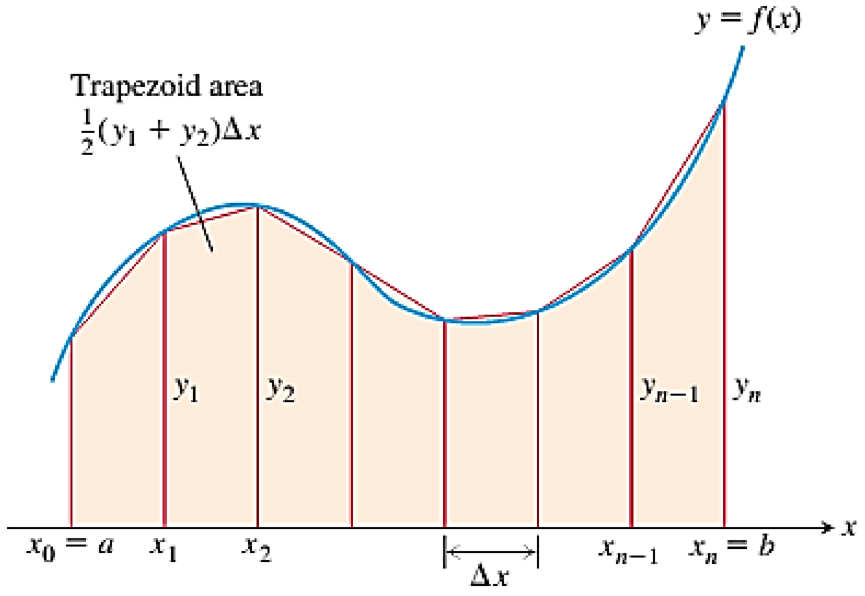
To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.



EXAMPLE 1 Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

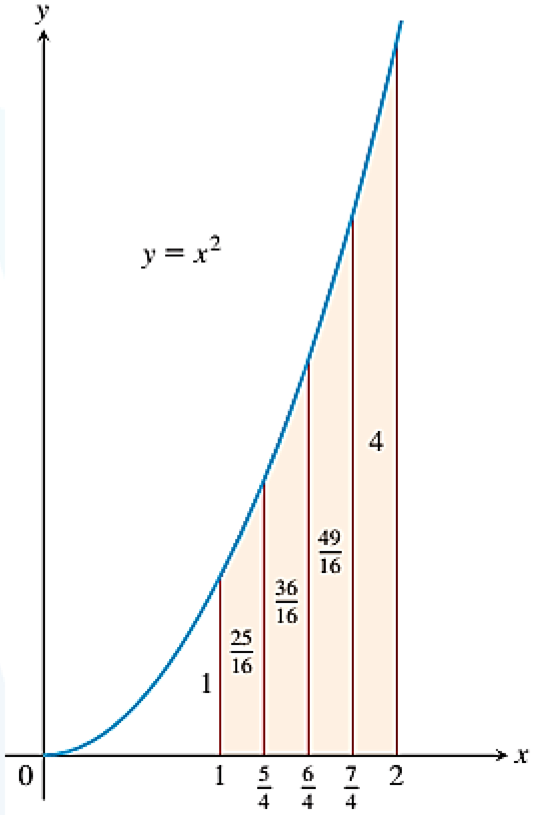
$$\Delta x = \frac{2-1}{4} = \frac{1}{4} \qquad y = x^2$$

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right)$$

$$= \frac{75}{32} = 2.34375.$$

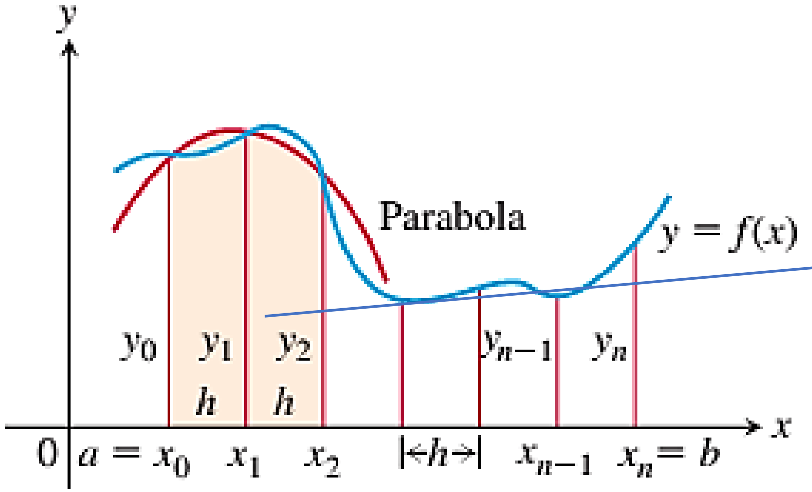
x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4



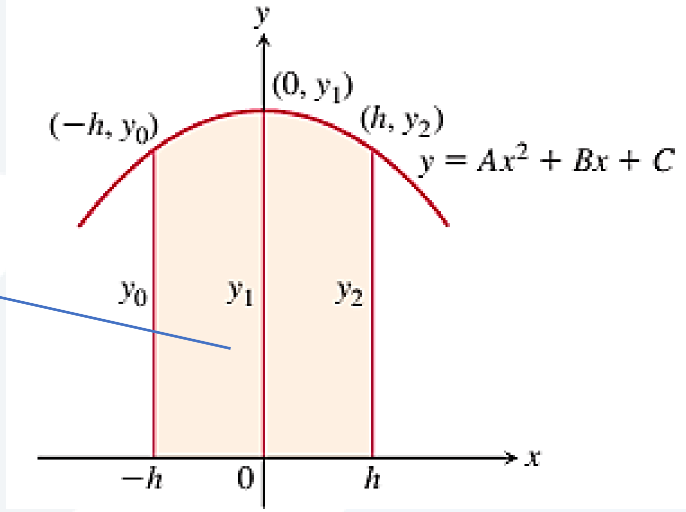
$$\int_1^2 x^2 dx = 2.33333$$

$$E_{\%} = \frac{\left|T - I_{exact}\right|}{I_{exact}} \times 100\% = \frac{\left|2.34375 - 7/3\right|}{7/3} \times 100\% = 0.446\%$$

Simpson’s Rule: Approximations Using Parabolas



Equal Area



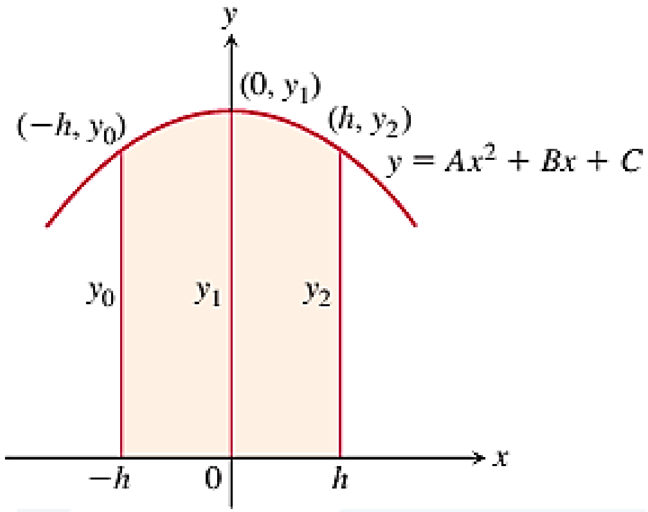
Approximating the curve $y = f(x) \geq 0$ by a parabola $y = Ax^2 + Bx + C$

A typical parabola passes through three consecutive points on the curve.

$$(x_{i-1}, y_{i-1}), (x_i, y_i), (x_{i+1}, y_{i+1}) \qquad h = \Delta x = \frac{b-a}{n}$$

Simpson's Rule: Approximations Using Parabolas

$$A_p = \int_{-h}^{h} (Ax^2 + Bx + C)\, dx = \frac{h}{3}(2Ah^2 + 6C).$$



Finding A , C

Since the curve passes through the three points

$$(-h, y_0), (0, y_1), (h, y_2)$$

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

$$2Ah^2 = y_0 - 2y_1 + y_2 \ , \ C = y_1$$

$$A_p = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Thus the area under the parabola through $(x_0, y_0), (x_1, y_1), (x_2, y_2)$

$$\frac{h}{3}(y_0 + 4y_1 + y_2)$$

Simpson’s Rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[y_0 + 2\sum_{j=1}^{\frac{n}{2}-1} y_{2j} + 4\sum_{j=1}^{\frac{n}{2}} y_{2j-1} + y_n \right]$$

$$y_i = f(x_i)\ ;\ i = 0,1,2,\ldots,n$$

$$x_0 = a, x_n = b, x_i = a + i\,\Delta x\ ; i = 1,2,\ldots,n-1$$

EXAMPLE 2 Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) = 32\frac{1}{12}.$$

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

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Error Analysis

THEOREM 1—Error Estimates in the Trapezoidal and Simpson's Rules
If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}. \qquad \text{Trapezoidal Rule}$$

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \le \frac{M(b-a)^5}{180n^4}. \qquad \text{Simpson's Rule}$$

$$\int_a^b f(x)\,dx = T - \frac{b-a}{12}\cdot f''(c)(\Delta x)^2$$

$$\int_a^b f(x)\,dx = S - \frac{b-a}{180}\cdot f^{(4)}(c)(\Delta x)^4$$

EXAMPLE 3 Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 2).

$$|E_S| \le \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180\cdot 4^4} = \frac{1}{12}.$$

EXAMPLE 4 Estimate the minimum number of subintervals needed to approximate the integral in Example 3 using Simpson's Rule with an error of magnitude less than 10^{-4}.

$$\frac{M(b-a)^5}{180n^4} < 10^{-4},$$

$$\frac{120(2)^5}{180n^4} < \frac{1}{10^4}$$

$$n > 10\left(\frac{64}{3}\right)^{1/4} \approx 21.5.$$

$$n = 22$$

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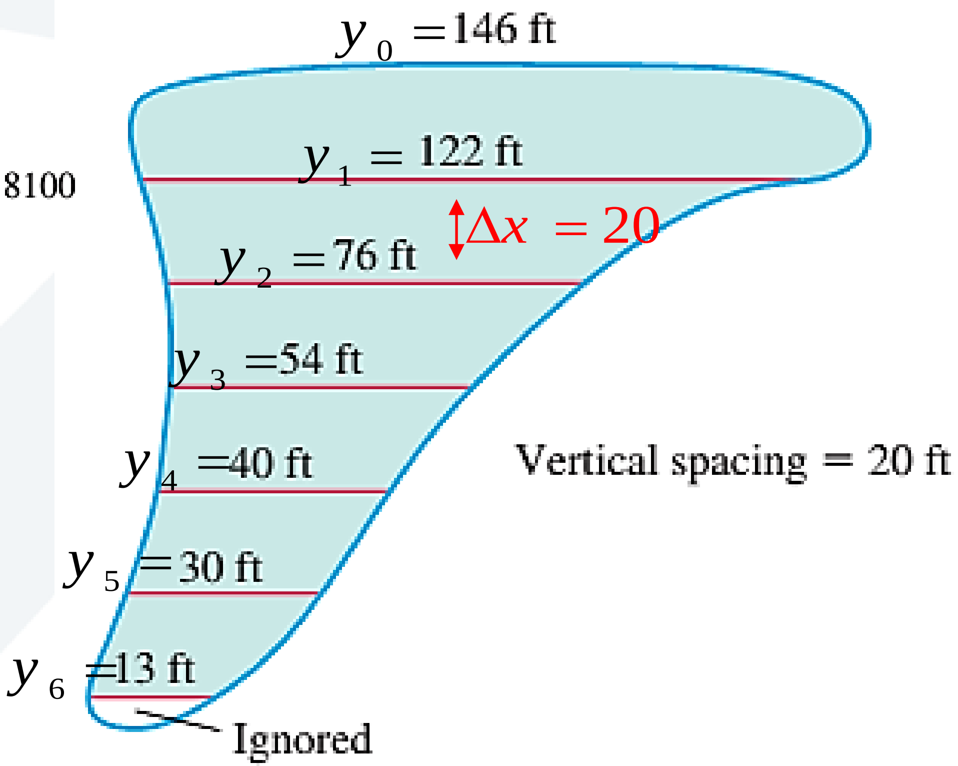
EXAMPLE 6 A town wants to drain and fill a small polluted swamp (Figure 8.11). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$

$$= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100$$

$$V = 8100 \times 5 = 40500\,\text{ft}^3 = 1500\,\text{yd}^3$$



Exercises

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The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

b. Evaluate the integral directly and find $|E_T|$.

c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

II. Using Simpson's Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

b. Evaluate the integral directly and find $|E_S|$.

c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

5. $\displaystyle\int_0^2 (t^3 + t)\,dt$

6. $\displaystyle\int_{-1}^1 (t^3 + 1)\,dt$

Exercises

In Exercises 11–22, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 11–18 are the integrals from Exercises 1–8.)

20. $\displaystyle\int_0^3 \frac{1}{\sqrt{x+1}}\,dx$

$n = 130$ $\quad$ $n = 18$

21. $\displaystyle\int_0^2 \sin(x+1)\,dx$

$n = 82$ $\quad$ $n = 8$

24. Distance traveled The accompanying table shows time-to-speed data for a sports car accelerating from rest to 130 mph. How far had the car traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

Speed change	Time (sec)
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

$5166.346 \text{ ft} \approx 0.9785 \text{ mi}$

1 mph = 1.466667 ft/s