MATHEMATICAL ANALAYSIS 1

Lecture 8

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Absolute Convergence; The Ratio and Root Tests

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

THEOREM 12—The Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} \qquad \sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

The Ratio Test

THEOREM 13—The Ratio Test
Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

EXAMPLE 2 Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{2^n+5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{n!n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty}\frac{4^n n!n!}{(2n)!}$

(a) $\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} \longrightarrow \frac{2}{3}.$$

Absolutely Convergent

$$\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n + \sum_{n=0}^{\infty} \frac{5}{3^n} = \frac{1}{1-(2/3)} + \frac{5}{1-(1/3)} = \frac{21}{2}.$$

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!n!(2n+2)(2n+1)(2n)!}{(n+1)!(n+1)!(2n)!} \longrightarrow 4.$$

Absolutely Divergent

(c) $\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n!n!} = \frac{2(n+1)}{2n+1} \longrightarrow 1.$$

$$\frac{2n+2}{2n+1} > 1 \Rightarrow a_{n+1} > a_n \Rightarrow a_n > a_1 = 2 \Rightarrow a_n \nrightarrow 0$$

Absolutely Divergent

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The Root Test

THEOREM 14—The Root Test
Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

EXAMPLE 4 Which of the following series converge, and which diverge?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$

(a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ $\qquad \sqrt[n]{\frac{n^2}{2^n}} \to \frac{1^2}{2} < 1.$ Absolutely Convergent

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ $\qquad \sqrt[n]{\frac{2^n}{n^3}} \to \frac{2}{1^3} > 1$ Absolutely Divergent

(c) $\sum_{n=1}^{\infty} \left(\frac{1}{1+n}\right)^n$ $\qquad \sqrt[n]{\left(\frac{1}{1+n}\right)^n} \to 0 < 1.$ Absolutely Convergent

Exercises

use the Ratio Test to determine if each series converges absolutely or diverges

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2} \qquad \sum_{n=1}^{\infty} \frac{2^{n+1}}{n3^{n-1}} \qquad \sum_{n=1}^{\infty} (-1)^n \frac{n^2(n+2)!}{n!\, 3^{2n}} \qquad \sum_{n=1}^{\infty} \frac{n5^n}{(2n+3)\ln(n+1)}$$

Using the Root Test

$$\sum_{n=1}^{\infty} \left(-\ln\left(e^2 + \frac{1}{n}\right)\right)^{n+1} \qquad \sum_{n=1}^{\infty} \sin^n\left(\frac{1}{\sqrt{n}}\right) \qquad \sum_{n=1}^{\infty} (-1)^n \left(1 - \frac{1}{n}\right)^{n^2}$$

Recursively Defined Terms Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 47–56 converge, and which diverge? Give reasons for your answers.

$$a_1 = 1, \quad a_{n+1} = \frac{1 + \tan^{-1} n}{n} a_n \qquad a_1 = \frac{1}{2}, \quad a_{n+1} = \frac{n + \ln n}{n + 10} a_n$$

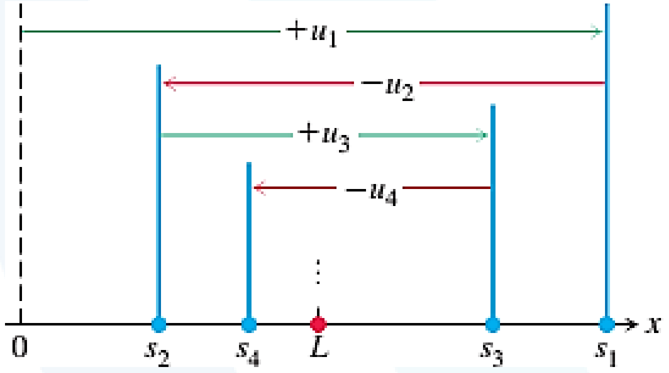
Alternating Series and Conditional Convergence

THEOREM 15—The Alternating Series Test
The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

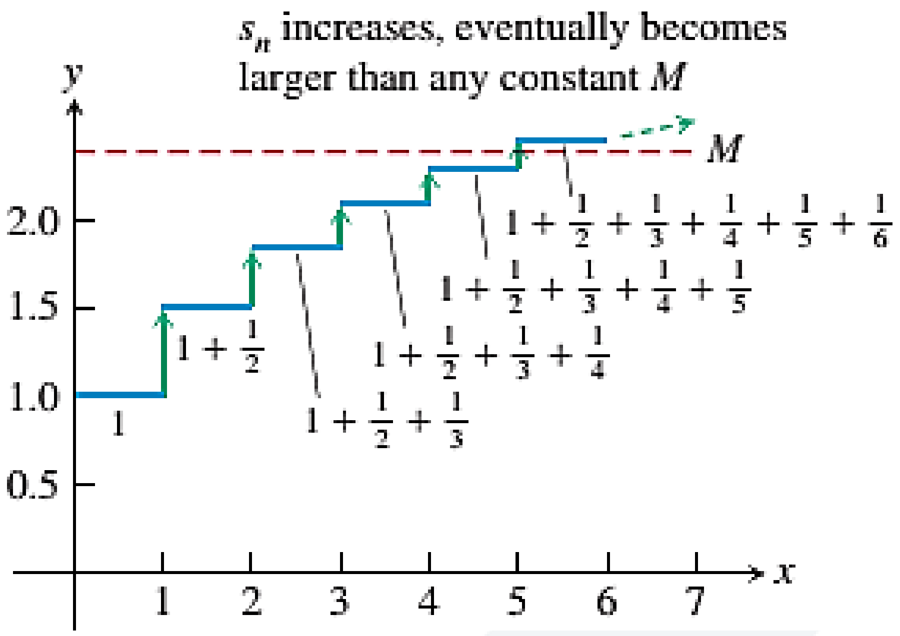
converges if the following conditions are satisfied:

1. The u_n's are all positive.
2. The u_n's are eventually nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \longrightarrow 0$.

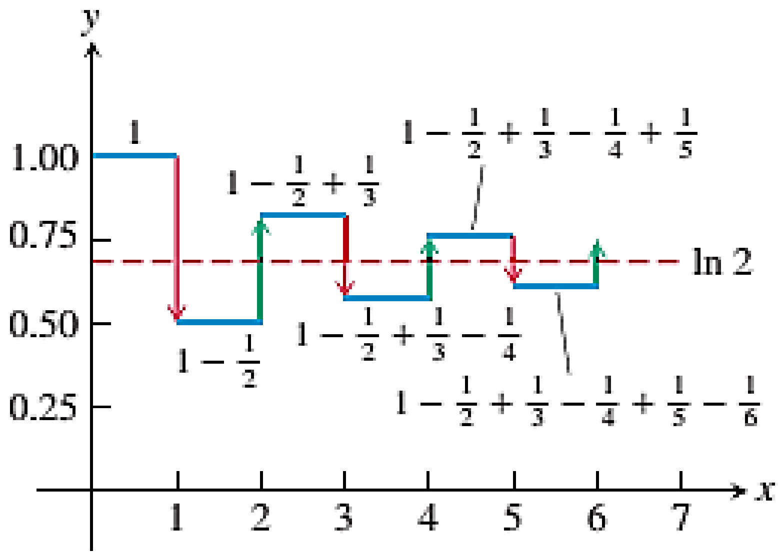


EXAMPLE 1 The alternating harmonic series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$



The harmonic series diverges,



The alternating harmonic series converges to $\ln 2 \approx .693$.

Conditional Convergence

DEFINITION A series that is convergent but not absolutely convergent is called **conditionally convergent**.

EXAMPLE 4 If p is a positive constant, the sequence $\{1/n^p\}$ is a decreasing sequence with limit zero. Therefore, the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

$$p > 1$$

Absolutely Convergent

$$0 < p \leq 1$$

Conditionally Convergent

Exercises

Determine if the alternating series converges or diverges. Some of the series do not satisfy the conditions of the Alternating Series Test.

$$\sum_{n=2}^{\infty}(-1)^n\frac{4}{(\ln n)^2}$$

converges

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2^n}{n^2}$$

diverges

$$\sum_{n=1}^{\infty}(-1)^n\frac{10^n}{(n+1)!}$$

converges absolutely

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{3\sqrt{n}+1}{\sqrt{n}+1}$$

diverges

Absolute and Conditional Convergence

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{1+\sqrt{n}}$$

converges conditionally

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n^2}$$

converges absolutely

$$\sum_{n=1}^{\infty}(-1)^n\frac{\tan^{-1} n}{n^2+1}$$

converges absolutely

$$\sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n-\ln n}$$

converges conditionally

$$\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{n}}$$

converges absolutely

$$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$$

diverges

$$\sum_{n=1}^{\infty}(-1)^n \operatorname{sech} n$$

converges absolutely

Power Series

DEFINITIONS **A power series about $x = 0$** is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad (1)$$

A power series about $x = a$ is a series of the form

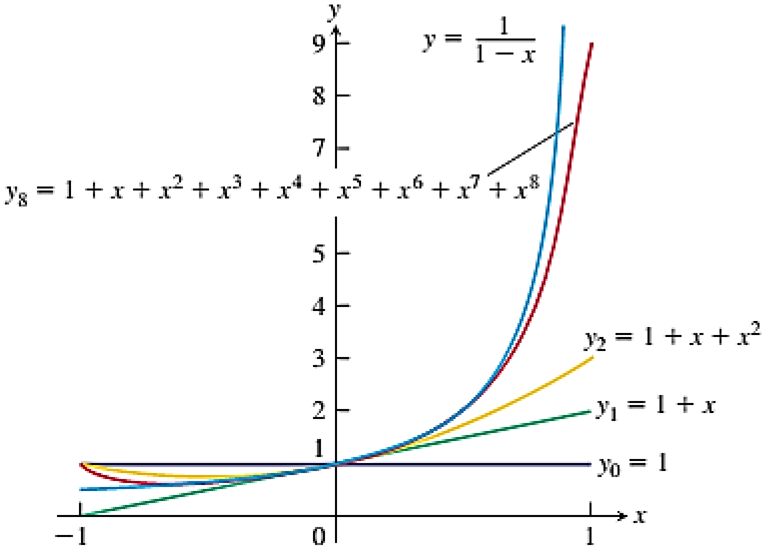
$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad (2)$$

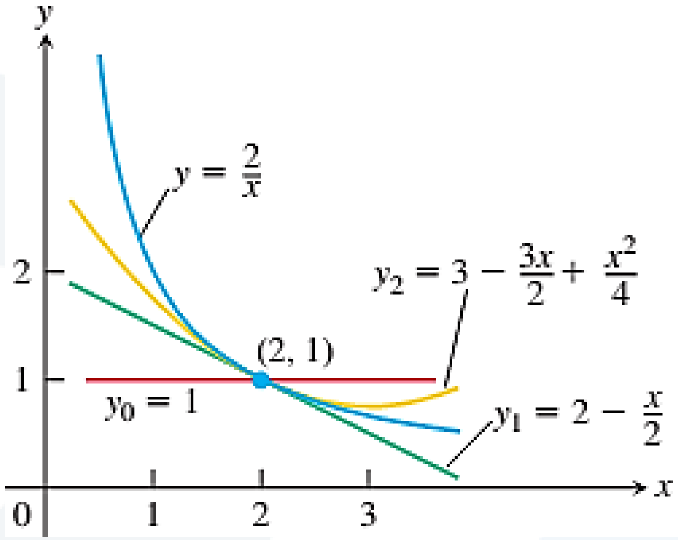
in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

EXAMPLE 1 Taking all the coefficients to be 1 in Equation (1) gives the geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \qquad -1 < x < 1.$$





EXAMPLE 2 The power series

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots \tag{4}$$

$$\frac{2}{x} = 1 - \frac{(x-2)}{2} + \frac{(x-2)^2}{4} - \cdots + \left(-\frac{1}{2}\right)^n (x-2)^n + \cdots, \qquad 0 < x < 4.$$

EXAMPLE 3 For what values of x do the following power series converge?

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

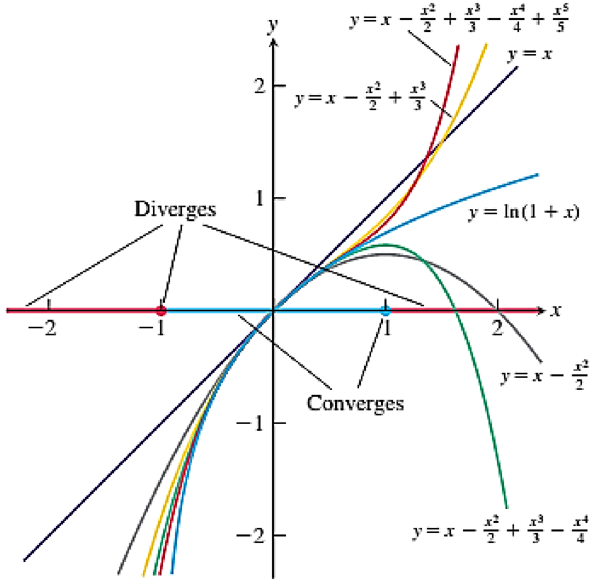
(b) $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(d) $\displaystyle\sum_{n=0}^{\infty}n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

(a) $\displaystyle\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{n+1}\cdot\frac{n}{x}\right| = \frac{n}{n+1}|x| \to |x|.$

Absolutely Convergent $|x| < 1$



$$x = 1 \Longrightarrow \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$ Convergent

$$x = -1 \Longrightarrow \sum_{n=1}^{\infty}\frac{-1}{n}$$ divergent

$$\textbf{(b)} \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$\textbf{(b)} \quad \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} \right| = \frac{2n-1}{2n+1} x^2 \to x^2.$$

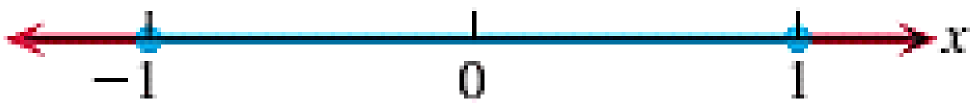
Absolutely Convergent $\quad x^2 < 1$

$$x = 1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n-1}$$

Convergent

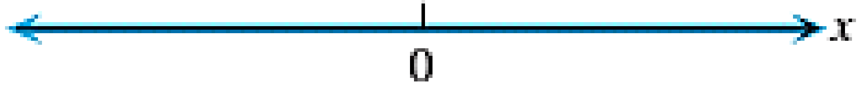
$$x = -1 \quad \Longrightarrow \quad \sum_{n=1}^{\infty} (-1)^{3n-1} \frac{1}{2n-1}$$

Convergent



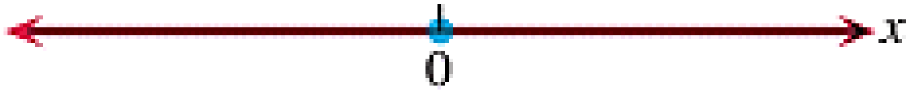
(c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(c) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n}\right| = \frac{|x|}{n+1} \longrightarrow 0$ for every x.



(d) $\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$

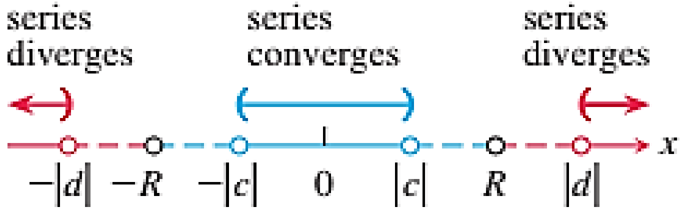
(d) $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \longrightarrow \infty$ unless $x = 0$.



THEOREM 18—The Convergence Theorem for Power Series

If the power series

$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.



Corollary to Theorem 18

The convergence of the series $\sum c_n (x - a)^n$ is described by one of the following three cases:

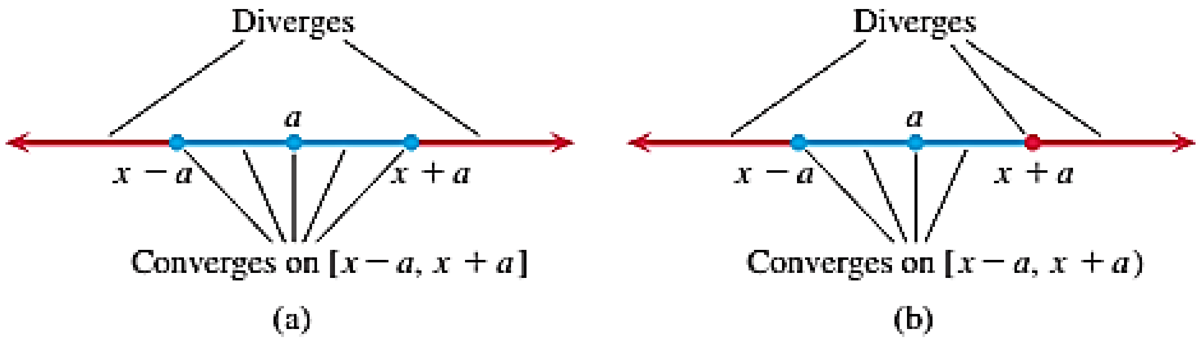
1. There is a positive number R such that the series diverges for x with $|x - a| > R$ but converges absolutely for x with $|x - a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

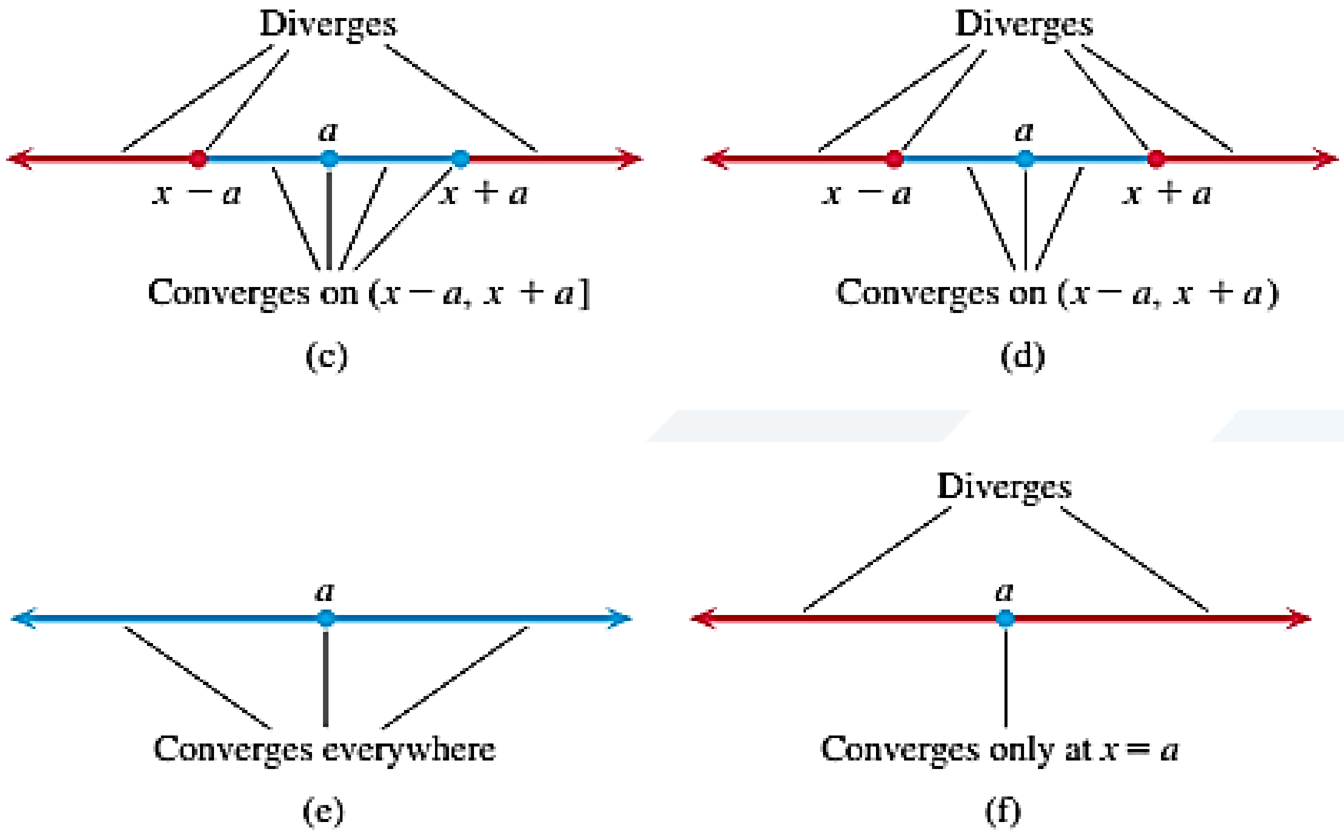
How to Test a Power Series for Convergence

1. Use the Ratio Test (or Root Test) to find the largest open interval where the series converges absolutely,

$$|x - a| < R \qquad \text{or} \qquad a - R < x < a + R.$$

2. If R is finite, test for convergence or divergence at each endpoint, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
3. If R is finite, the series diverges for $|x - a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.




Diverges
a
x − a
x + a
Converges on (x − a, x + a]
(c)
Diverges
a
x − a
x + a
Converges on (x − a, x + a)
(d)
a
Converges everywhere
(e)
Diverges
a
Converges only at x = a
(f)

Operations on Power Series

THEOREM 19—Series Multiplication for Power Series
If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

THEOREM 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$ and f is a continuous function, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely on the set of points x where $|f(x)| < R$.

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THEOREM 21—Term-by-Term Differentiation

If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \qquad \text{on the interval} \qquad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

EXAMPLE 4 Find series for $f'(x)$ and $f''(x)$ if

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$$

$$= \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1.$$

$$f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots \quad = \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1;$$

$$f''(x) = \frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \cdots \ + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \qquad -1 < x < 1.$$

THEOREM 22—Term-by-Term Integration

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

converges for $a - R < x < a + R\ (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1} + C$$

for $a - R < x < a + R$.

EXAMPLE 6 The series

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots$$

converges on the open interval $-1 < t < 1$

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots \Big]_0^x$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x < 1.$$

Exercises

(a) find the series' radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

$$\sum_{n=0}^{\infty}(-1)^n(4x+1)^n \qquad \frac{1}{4} \qquad -\frac{1}{2}<x<0$$

$$\sum_{n=1}^{\infty}\frac{3\cdot5\cdot7\cdots(2n+1)}{n^2\cdot2^n}x^{n+1} \qquad 0 \qquad x=0$$

$$\sum_{n=1}^{\infty}\frac{(3x-2)^n}{n} \qquad \frac{1}{3} \qquad \frac{1}{3}<x<1$$

conditionally at $x=\frac{1}{3}$

$$\sum_{n=1}^{\infty}\frac{x^n}{n\sqrt{n}\,3^n} \qquad 3 \qquad -3\le x\le 3$$

Use a geometric series to represent the given function as a power series about $x=0$, and find their intervals of convergence.

$$g(x)=\frac{3}{x-2} \qquad \sum_{n=0}^{\infty}-\frac{3}{2^{n+1}}x^n \qquad |x|<2$$

about $x=5$ $\qquad \sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n(x-5)^n \qquad 2<x<8.$

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Taylor and Maclaurin Series

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

EXAMPLE 1 Find the Taylor series generated by $f(x) = 1/x$ at $a = 2$. Where, if anywhere, does the series converge to $1/x$?

$$f(x) = x^{-1}, \quad f'(x) = -x^{-2}, \quad f''(x) = 2!x^{-3}, \ldots, f^{(n)}(x) = (-1)^n n! x^{-(n+1)},$$

$$f(2) = 2^{-1} = \frac{1}{2}, \quad f'(2) = -\frac{1}{2^2}, \quad \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}, \ldots, \frac{f^{(n)}(2)}{n!} = \frac{(-1)^n}{2^{n+1}}.$$

$$f(2) + f'(2)(x-2) - \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{(n)}(2)}{n!}(x-2)^n + \cdots$$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots.$$

$$\frac{1/2}{1 + (x-2)/2} = \frac{1}{2 + (x-2)} = \frac{1}{x}.$$

$$|x - 2| < 2$$

$$0 < x < 4.$$

Taylor Polynomials

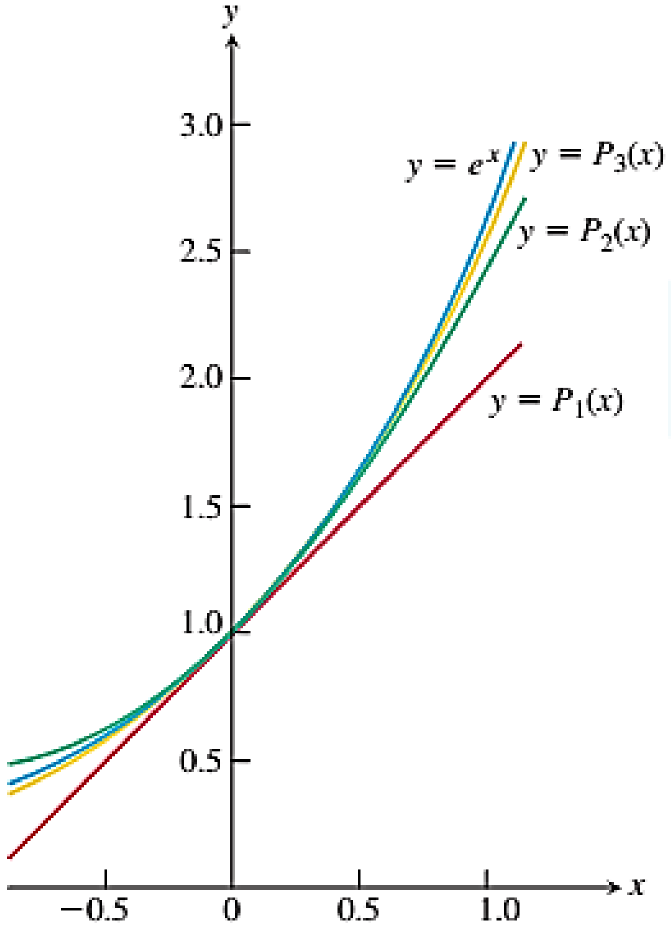
DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x - a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

EXAMPLE 2 Find the Taylor series and the Taylor polynomials generated by $f(x) = e^x$ at $x = 0$.

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

$$P_n(x) = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!}.$$



$P_1(x) = 1 + x$

$P_2(x) = 1 + x + (x^2/2!)$

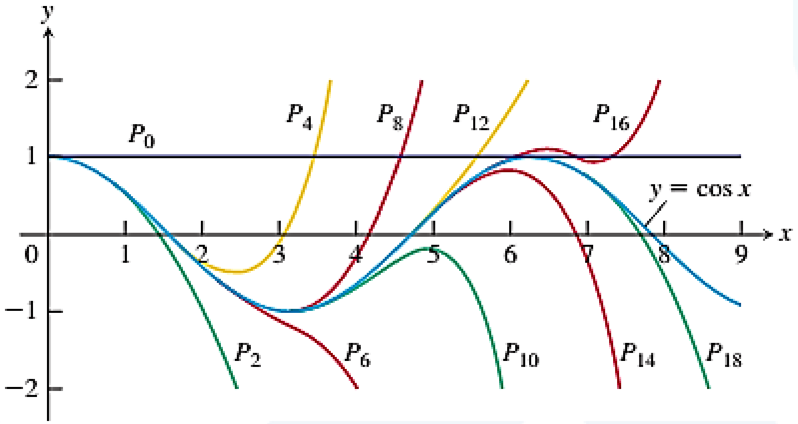
$P_3(x) = 1 + x + (x^2/2!) + (x^3/3!).$

EXAMPLE 3 Find the Taylor series and Taylor polynomials generated by $f(x) = \cos x$ at $x = 0$.

$$\begin{aligned} f(x) &= \cos x, & f'(x) &= -\sin x, \\ f''(x) &= -\cos x, & f^{(3)}(x) &= \sin x, \\ &\vdots & &\vdots \\ f^{(2n)}(x) &= (-1)^n \cos x, & f^{(2n+1)}(x) &= (-1)^{n+1} \sin x. \end{aligned}$$

$$f^{(2n)}(0) = (-1)^n, \qquad f^{(2n+1)}(0) = 0.$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$



$$P_{2n}(x) = P_{2n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!}.$$

Exercises

find the Taylor polynomials of orders 0, 1, 2, and 3 generated by f at a.

$f(x) = \ln x, \quad a = 1$

$P_0(x) = 0, P_1(x) = (x-1), P_2(x) = (x-1) - \frac{1}{2}(x-1)^2, P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

$f(x) = \sqrt{1-x}, \quad a = 0$

$P_0(x) = 1, P_1(x) = 1 - \frac{1}{2}x, P_2(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2, P_3(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3$

find the Taylor series generated by f at $x = a$.

$f(x) = 1/(1-x)^3, \quad a = 0$

$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$$

$f(x) = \cos(2x + (\pi/2)), \quad a = \pi/4$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} \left(x - \frac{\pi}{4}\right)^{2n}$$

$f(x) = 2^x, \quad a = 1$

$$\sum_{n=0}^{\infty} \frac{2(\ln 2)^n (x-1)^n}{n!}$$

Convergence of Taylor Series

THEOREM 23—Taylor's Theorem

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Convergence of Taylor Series

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \qquad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x. \qquad (2)$$

Remainder of order n or the error term

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If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k.$$

EXAMPLE 1 Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x)$$

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} \qquad c \text{ between 0 and } x.$$

$$x \le 0 \longrightarrow c < 0 \longrightarrow e^c < 1 \longrightarrow |R_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

$$x > 0 \longrightarrow c < x \longrightarrow e^c < e^x \longrightarrow |R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0 \qquad \text{for every } x,$$

Estimating the Remainder

THEOREM 24—The Remainder Estimation Theorem
If there is a positive constant M such that $|f^{(n+1)}(t)| \le M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \le M\frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Estimating the Remainder

EXAMPLE 2 Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

$$\begin{aligned} f(x) &= \sin x, & f'(x) &= \cos x, \\ f''(x) &= -\sin x, & f'''(x) &= -\cos x, \\ &\vdots & &\vdots \\ f^{(2k)}(x) &= (-1)^k \sin x, & f^{(2k+1)}(x) &= (-1)^k \cos x, \end{aligned}$$

$$f^{(2k)}(0) = 0 \quad \text{and} \quad f^{(2k+1)}(0) = (-1)^k.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + R_{2k+1}(x). \qquad M = 1$$

$$|R_{2k+1}(x)| \le 1 \cdot \frac{|x|^{2k+2}}{(2k+2)!}. \quad \left(|x|^{2k+2}/(2k+2)!\right) \to 0 \text{ as } k \to \infty. \quad R_{2k+1}(x) \to 0$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Using Taylor Series

EXAMPLE 4 Using known series, find the first few terms of the Taylor series for the given function by using power series operations.

(a) $\frac{1}{3}(2x + x\cos x)$ **(b)** $e^x \cos x$

(a) $$\frac{1}{3}(2x + x\cos x) = \frac{2}{3}x + \frac{1}{3}x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots\right)$$

$$= \frac{2}{3}x + \frac{1}{3}x - \frac{x^3}{3!} + \frac{x^5}{3 \cdot 4!} - \cdots = x - \frac{x^3}{6} + \frac{x^5}{72} - \cdots$$

(b) $$e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

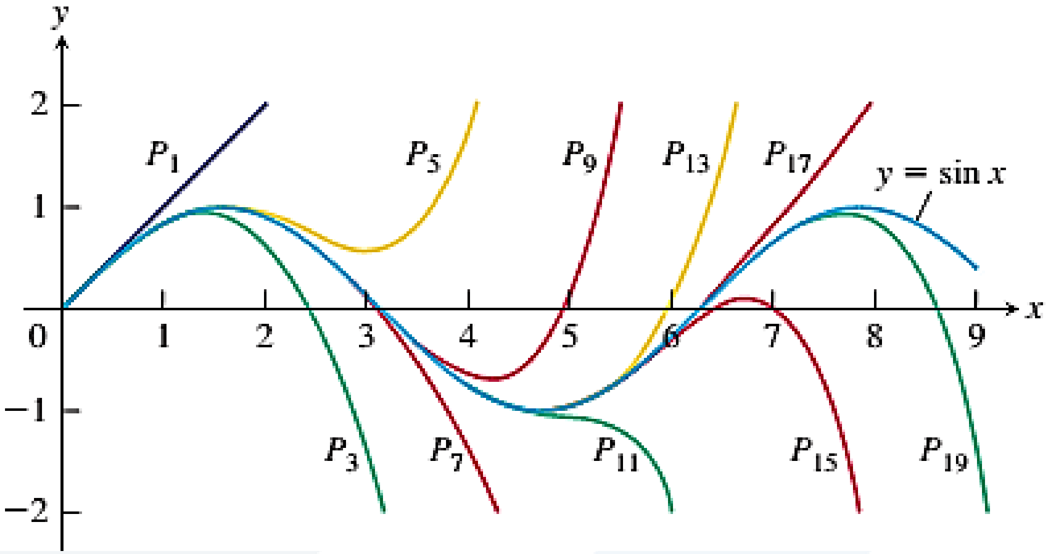
$$= 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$$

Multiply the first series by each term of the second series.

EXAMPLE 5 For what values of x can we replace $\sin x$ by $x - (x^3/3!)$ and obtain an error whose magnitude is no greater than 3×10^{-4}?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\left|\frac{x^5}{5!}\right| = \frac{|x|^5}{120}. \Rightarrow \frac{|x|^5}{120} < 3 \times 10^{-4} \Rightarrow |x| < \sqrt[5]{360 \times 10^{-4}} \approx 0.514.$$



Exercises

find the Taylor series at x = 0 of the functions

$\cos\left(x^{2/3}/\sqrt{2}\right)$	$\sin x \cdot \cos x$	$\frac{x}{3}\ln(1 + x^2)$	$\cos^2 x$	$\tan^{-1}(3x^4)$
$\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n}}{2^n (2n)!}$	$\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!}$	$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n} x^{2n+1}$	$1+\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$	$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{8n+4}}{2n+1}$

Estimate the error if $P_3(x) = x - (x^3/6)$ is used to estimate the value of $\sin x$ at $x = 0.1$.

$$\text{error} \leq 4.2 \times 10^{-6}$$

Estimate the error if $P_4(x) = 1 + x + (x^2/2) + (x^3/6) + (x^4/24)$ is used to estimate the value of e^x at $x = 1/2$.

$$\text{error} \leq 7.03 \times 10^{-4}$$