Lecture 6 : Symmetric matrices and Positive Definiteness

CEDC102: Linear Algebra

Manara University

2023-2024

- Symmetric matrices
- Positive definite matrices
- Singular Value Decomposition
- Similar Matrices and Jordan Form

Symmetric Matrices and Orthogonal Diagonalization

- Symmetric matrix: A square matrix A is symmetric if it is equal to its transpose: $A = A^T$

- Ex: (Symmetric matrices and nonsymetric matrices)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

(symmetric) (symmetric) (nonsymmetric)

- Theorem : (Eigenvalues of symmetric matrices)

If A is an $n\times n$ symmetric matrix, then the following properties are true.

(1) A is diagonalizable.

(2) All eigenvalues of A are real.

(3) A has an orthonormal set of n eigenvectors

- Ex :

Prove that a symmetric matrix is diagonalizable $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - a & -c \\ -c & \lambda - b \end{vmatrix} = \lambda^2 - (a+b)\lambda + ab - c^2 = 0$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned}(a+b)^2 - 4(ab - c^2) &= a^2 + 2ab + b^2 - 4ab + 4c^2 \\ &= a^2 - 2ab + b^2 + 4c^2 \\ &= (a-b)^2 + 4c^2 \geq 0\end{aligned}$$

$$(1)\ (a-b)^2+4c^2=0$$

$$\Rightarrow a=b,\ \ c=0$$

$$A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$ A is a diagonal matrix

$$(2)\ (a-b)^2+4c^2>0$$

The characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. Thus, A is diagonalizable.

- **Orthogonal matrix:**

  A square matrix P is called orthogonal if it is invertible and $P^{-1} = P^T$

- **Ex 3: (Orthogonal matrices)**

$$\text{(a) } P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ is orthogonal because } P^{-1} = P^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\text{(b) } P = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ \frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix} \text{ is orthogonal because } P^{-1} = P^T = \begin{bmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & \frac{3}{5} \end{bmatrix}$$

- Theorem : (Properties of orthogonal matrices)

An $n \times n$ matrix P is orthogonal

(1) if and only if its column vectors form an orthonormal set in R^n

(2) if and only if its row vectors form an orthonormal set in R^n

- Ex : (An orthogonal matrix)

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

Sol:

If P is a orthogonal matrix, then $P^{-1} = P^T \Rightarrow PP^T = I$

$$PP^T=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{3\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

$$\text{Let } p_1=\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{\sqrt{5}} \\ -\frac{2}{3\sqrt{5}} \end{bmatrix},\ p_2=\begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ -\frac{4}{3\sqrt{5}} \end{bmatrix},\ p_3=\begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

$$p_1\cdot p_2=p_1\cdot p_3=p_2\cdot p_3=0$$

$$\|p_1\|=\|p_2\|=\|p_3\|=1$$

$\{p_1, p_2, p_3\}$ is an orthonormal set

- Theorem : (Properties of orthogonal matrices)

  (a) The transpose of an orthogonal matrix is orthogonal.

  (b) The inverse of an orthogonal matrix is orthogonal.

  (c) A product of orthogonal matrices is orthogonal.

  (d) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

- Theorem : (Orthogonal Matrices as Linear Operators)

  If A is an $n \times n$ matrix, then the following are equivalent

  (a) A is orthogonal

  (b) $\|A\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x}$ in R^n

  (c) $A\boldsymbol{x} \,.\, A\boldsymbol{y} = \boldsymbol{x} \,.\, \boldsymbol{y}$ for all $\boldsymbol{x}$ and y in R^n

- Theorem : (Properties of symmetric matrices)

Let A be an $n\times n$ symmetric matrix, then Eigenvectors from different eigenspaces are orthogonal.

- Ex : (Eigenvectors of a symmetric matrix)

Show that any two eigenvectors of

$$A=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$$

corresponding to distinct eigenvalues are orthogonal

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix}\lambda-3 & -1\\ -1 & \lambda-3\end{vmatrix} = (\lambda-2)(\lambda-4)=0$$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = 4$

$$(1)\ \lambda_1 = 2 \Rightarrow \lambda_1 I - A = \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \Rightarrow \boldsymbol{x}_1 = s\begin{bmatrix}-1\\ 1\end{bmatrix},\ s\neq 0$$

$$(2)\ \lambda_2 = 4 \Rightarrow \lambda_2 I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \ t \neq 0$$

$$\boldsymbol{x}_1 . \boldsymbol{x}_2 = \begin{bmatrix} -s \\ s \end{bmatrix} \cdot \begin{bmatrix} t \\ t \end{bmatrix} = st - st = 0 \Rightarrow \boldsymbol{x}_1 \text{ and } \boldsymbol{x}_2 \text{ are orthogonal}$$

- **Orthogonal Diagonalization**

  matrix A is orthogonally diagonalizable when there exists an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal

- **Theorem : (Fundamental theorem of symmetric matrices)**

  Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable (and has real eigenvalues) if and only if A is symmetric.

- Orthogonal diagonalization of a symmetric matrix:

Let A be an $n \times n$ symmetric matrix.

(1) Find all eigenvalues of A and determine the multiplicity of each.

(2) For each eigenvalue of multiplicity 1, choose a unit eigenvector.

(3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

(4) The composite of steps 2 and 3 produces <u>an orthonormal set of n eigenvectors</u>. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

- Ex : (Orthogonal diagonalization)

Find a matrix P that orthogonally diagonalizes $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$

Sol: Characteristic equation:

(1) $|\lambda I - A| = (\lambda - 3)^2(\lambda + 6) = 0$

Eigenvalues: $\lambda_1 = -6$, $\lambda_2 = 3$ (has a multiplicity of 2)

(2) $\lambda_1 = -6,\ \boldsymbol{u}_1 = (1,\ -2,\ 2) \Rightarrow \boldsymbol{v}_1 = \dfrac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|} = (\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{2}{3})$

(3) $\lambda_2 = 3,\ \boldsymbol{u}_2 = (2, 1, 0),\ \boldsymbol{u}_3 = (-2, 0, 1)$

Linear Independent

Gram-Schmidt Process:

$$\boldsymbol{w}_2 = \boldsymbol{u}_2 = (2, 1, 0), \quad \boldsymbol{w}_3 = \boldsymbol{u}_3 - \frac{\boldsymbol{u}_3 . \boldsymbol{w}_2}{\boldsymbol{w}_2 . \boldsymbol{w}_2} \boldsymbol{w}_2 = \left(-\tfrac{2}{5}, \tfrac{4}{3}, 1\right)$$

$$\boldsymbol{v}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} = \left(\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0\right), \quad \boldsymbol{v}_3 = \frac{\boldsymbol{w}_3}{\|\boldsymbol{w}_3\|} = \left(-\tfrac{2}{3\sqrt{5}}, \tfrac{4}{3\sqrt{5}}, \tfrac{5}{3\sqrt{5}}\right)$$

$$(4)\ P = \left[p_1\ p_2\ p_3\right] = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & -\frac{2}{3\sqrt{5}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} \Rightarrow P^{-1}AP = P^{T}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Spectral Decomposition

If A is a symmetric matrix that is orthogonally diagonalized by $P = [\boldsymbol{u}_1\ \boldsymbol{u}_2\ \cdots\ \boldsymbol{u}_n]$ and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A corresponding to the unit eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$, then we know that $D = P^TAP$, where D is a diagonal matrix

$$A = PDP^T = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_n^T \end{bmatrix}$$

$$A = \left(\lambda_1 \boldsymbol{u}_1 \quad \lambda_2 \boldsymbol{u}_2 \quad \cdots \quad \lambda_n \boldsymbol{u}_n \right) \begin{bmatrix} \boldsymbol{u}_1^T \\ \boldsymbol{u}_2^T \\ \vdots \\ \boldsymbol{u}_n^T \end{bmatrix} = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^T + \cdots + \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^T$$

Spectral decomposition of A

- **Note:**

The Equation $A = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^T + \cdots + \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^T$ is called the spectral decomposition of A, because it involves only the spectrum of A and the corresponding unit eigenvectors of A

- **Ex : (A Geometric Interpretation of a Spectral Decomposition)**

$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 2$ with corresponding eigenvectors:

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ \boldsymbol{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \boldsymbol{u}_1 = \frac{\boldsymbol{x}_1}{\|\boldsymbol{x}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}, \ \boldsymbol{x}_2 = \frac{\boldsymbol{x}_2}{\|\boldsymbol{x}_2\|} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \lambda_1 \boldsymbol{u}_1 \boldsymbol{u}_1^T + \lambda_2 \boldsymbol{u}_2 \boldsymbol{u}_2^T = (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix} + (2)\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

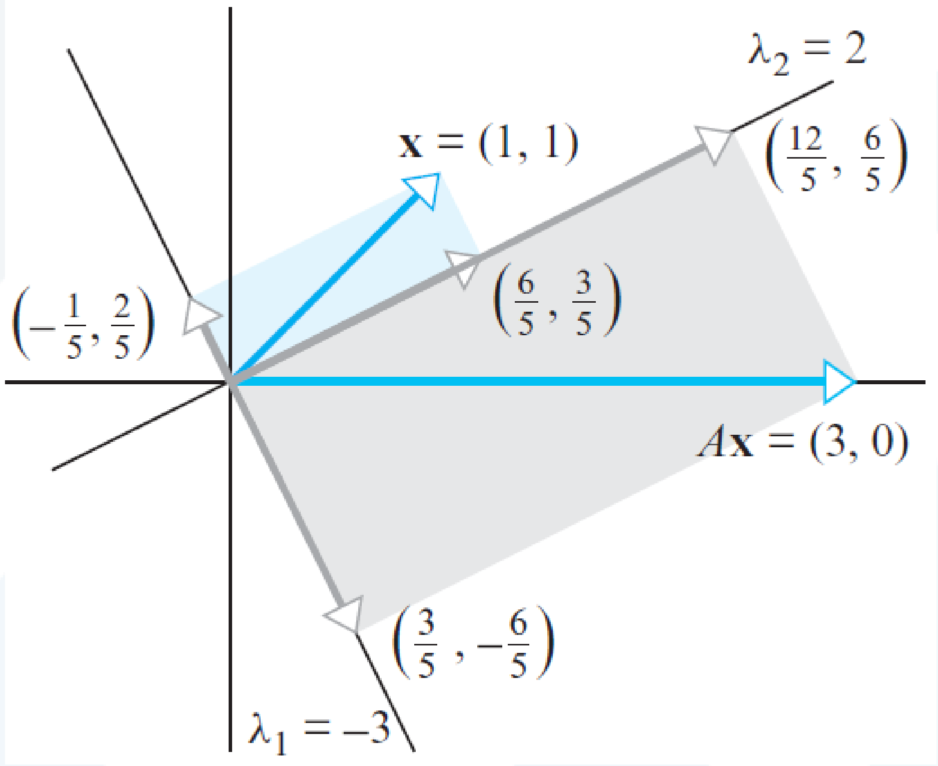
$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$A\boldsymbol{x} = (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (2)\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\boldsymbol{x} = (-3)\begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (2)\begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A\boldsymbol{x} = (-3)\begin{bmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{bmatrix} + (2)\begin{bmatrix} \frac{6}{5} \\ \frac{3}{5} \end{bmatrix}$$

$$A\boldsymbol{x} = \begin{bmatrix} \frac{3}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$



Positive Definite Matrices

1 **Symmetric** $\boldsymbol{S}$: all eigenvalues $> \mathbf{0} \Leftrightarrow$ all pivots $> \mathbf{0} \Leftrightarrow$ all upper left determinants $> \mathbf{0}$.

2 The matrix S is then **positive definite**. The energy test is $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} > 0$ for all vectors $\boldsymbol{x} \neq \mathbf{0}$.

3 One more test for positive definiteness : $S = A^{\mathrm{T}} A$ with independent columns in A.

4 **Positive semidefinite** S allows $\lambda = 0$, pivot $= 0$, determinant $= 0$, energy $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 0$.

5 The equation $\boldsymbol{x}^{\mathrm{T}} S \boldsymbol{x} = 1$ gives an ellipse in $\mathbf{R}^n$ when S is symmetric positive definite.

Singular Value Decomposition (SVD)

- Theorem : (Singular Values)

If A is an $m \times n$ matrix, then:

(a) A and A^TA have the same null space

(b) A and A^TA have the same row space

(c) A^T and A^TA have the same column space

(d) A and A^TA have the same rank

- Theorem :

If A is an $m \times n$ matrix, then:

(a) A^TA is orthogonally diagonalizable.

(b) The eigenvalues of A^TA are nonnegative

If A is an $m \times n$ matrix, and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A^TA, then the numbers

$$\sigma_1 = \sqrt{\lambda_1},\ \sigma_2 = \sqrt{\lambda_2},\ \cdots,\ \sigma_n = \sqrt{\lambda_n}$$

are called the singular values of A

- Ex : (Singular Values)

Find the singular values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

Sol:

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic polynomial of A^TA is $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$

So the eigenvalues of A^TA are: $\lambda_1 = 3$, $\lambda_2 = 1$, and the singular values of A are:

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3},\ \sigma_2 = \sqrt{\lambda_2} = 1$$

main diagonal of an $m \times n$ matrix

We define the main diagonal of an $m \times n$ matrix to be the line of entries starts at the upper left corner and extends diagonally as far as it can go

$$\begin{bmatrix} \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times \end{bmatrix} \qquad \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{bmatrix}$$

- **Theorem : (Singular Value Decomposition (Brief Form))**

If A is an $m \times n$ matrix of rank k, then A can be expressed in the form $A = U\Sigma V^T$, where Σ has size $m \times n$ and can be expressed in partitioned form as

$$\Sigma = \left[\begin{array}{c|c} D & 0_{k\times(n-k)} \\ \hline 0_{(m-k)\times k} & 0_{(m-k)\times(n-k)} \end{array}\right]$$

in which D is a diagonal $k \times k$ matrix whose successive entries are the first k singular values of A in nonincreasing order, U is an $m \times m$ orthogonal matrix, and V is an $n \times n$ orthogonal matrix

- **Note:**

$$A = U\Sigma V^T = \sigma_1 \boldsymbol{u}_1 \boldsymbol{v}_1^T + \sigma_2 \boldsymbol{u}_2 \boldsymbol{v}_2^T + \cdots + \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^T$$

Theorem : (Singular Value Decomposition (Expanded Form))

If A is an $m\times n$ matrix of rank k, then A can be factored as

$$A = U\Sigma V^T = \left[\boldsymbol{u}_1\ \boldsymbol{u}_2 \cdots \boldsymbol{u}_k \;\vdots\; \boldsymbol{u}_{k+1}\ \boldsymbol{u}_{k+2} \cdots \boldsymbol{u}_m\right] \left[\begin{array}{cccc:c} \sigma_1 & 0 & \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 & 0_{k\times(n-k)} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \sigma_k & \\ \hdashline & 0_{(m-k)\times k} & & & 0_{(m-k)\times(n-k)} \end{array}\right] \left[\begin{array}{c} \boldsymbol{v}_1^T \\ \boldsymbol{v}_2^T \\ \vdots \\ \boldsymbol{v}_k^T \\ \hdashline \boldsymbol{v}_{k+1}^T \\ \boldsymbol{v}_{k+2}^T \\ \vdots \\ \boldsymbol{v}_n^T \end{array}\right]$$

in which U, Σ, and V have sizes $m\times m$, $m\times n$, and $n\times n$, respectively, and:

(a) $V = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$ orthogonally diagonalizes $A^T A$

(b) The nonzero diagonal entries of Σ are $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \cdots, \sigma_k = \sqrt{\lambda_k}$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the nonzero eigenvalues of $A^T A$ corresponding to the column vectors of V

(c) The column vectors of V are ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$

(d) $\mathbf{u}_i = \dfrac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \dfrac{1}{\sigma_i} A\mathbf{v}_i \quad (i = 1, 2, \cdots, k)$

(e) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for $CS(A)$

(f) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \mathbf{u}_{k+2}, \ldots, \mathbf{u}_m\}$ is an extension of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ to an orthonormal basis for R^m

- Ex : (Singular Value Decomposition if A Is Not Square)

Find a singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Sol:

The eigenvalues of $A^T A$ are: $\lambda_1 = 3$, $\lambda_2 = 1$, and the singular values of A are: $\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = 1$

The unit eigenvectors corresponding to λ_1 and λ_2 are

$$\boldsymbol{v}_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \quad \Rightarrow \quad V = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

V orthogonally diagonalizes $A^T A$

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{\sqrt{3}}{3}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = (1)\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are two of the three column vectors of U

To extend the orthonormal set $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ to an orthonormal basis for R^3

the vector $\boldsymbol{u}_3$ must be a solution of $\begin{bmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{u}_3 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$A \quad = \quad U \quad \Sigma \quad V^T$$

- Ex : (Singular Value Decomposition)

Find a singular value decomposition of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Sol:

$$A^T A = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 6 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

The eigenvalues of $A^T A$ are: $\lambda_1 = 8$, $\lambda_2 = 2$, and $\lambda_3 = 0$

The singular values of A are: $\sigma_1 = \sqrt{\lambda_1} = 2\sqrt{2}, \quad \sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$

The eigenvectors corresponding to λ_1, λ_2, and λ_3 are

$$\boldsymbol{v}_1 = \left(\frac{1}{\sqrt{6}}, \frac{3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right), \ \boldsymbol{v}_2 = \left(-\frac{1}{\sqrt{3}}, 0, \frac{2}{\sqrt{6}}\right), \ \boldsymbol{v}_3 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{2}, \frac{1}{2}\right)$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} & \frac{1}{2} \end{bmatrix}$$

V orthogonally diagonalizes A^TA

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1} A\boldsymbol{v}_1 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{\sqrt{3}}{2} \\ \frac{1}{2\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{1}{\sigma_2} A\boldsymbol{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ to an orthonormal basis for $R^3 \Rightarrow \boldsymbol{u}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1 \\ \sqrt{2} & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$A \quad = \quad U \quad \Sigma \quad V^T$$

Jordan Decomposition

- Jordan Canonical Form (JCF)

Let A an $n{\times}n$ matrix, with either real (complex) entries. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the distinct eigenvalues of A ($k < n$)

A Jordan chain of length j for A is a sequence of non-zero vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_j \in K^n$ that satisfies: $A\boldsymbol{v}_1 = \lambda \boldsymbol{v}_1$, $A\boldsymbol{v}_i = \lambda \boldsymbol{v}_i + \boldsymbol{v}_{i-1}$, $i = 1, 2, \ldots, j$ where λ is an eigenvalue of A

A Jordan chain associated with a zero eigenvalue is called a null Jordan chain, and satisfies: $A\boldsymbol{v}_1 = \mathbf{0}$, $\quad A\boldsymbol{v}_i = \boldsymbol{v}_{i-1}$, $\quad i = 1, 2, \ldots, j$

- Note:

The initial vector $\boldsymbol{v}_1$ in a Jordan chain is a genuine eigenvector, and so Jordan chains exist only when λ is an eigenvalue

- Note: The rest, $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_j$, are generalized eigenvectors

A nonzero vector $\boldsymbol{v}$ such that $(A - \lambda I)^k\ \boldsymbol{v} = 0$ for some $k > 0$ and $\lambda \in K$ is called a generalized eigenvector of the matrix A

- Notes:

(1) Every ordinary eigenvector is automatically a generalized eigenvector, since we can just take $k = 1$

(2) The minimal value of k for which $(A - \lambda I)^k\ \boldsymbol{v} = 0$ is called the index of the generalized eigenvector

- Ex 12:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The only eigenvalue is $\lambda = 2$

$$\boldsymbol{v}_1 \in \ker(A-2I) \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad A-2I = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ is a genuine eigenvector

$$A\boldsymbol{v}_2 = 2\boldsymbol{v}_2 + \boldsymbol{v}_1 \Rightarrow (A-2I)\boldsymbol{v}_2 = \boldsymbol{v}_1 \Rightarrow (A-2I)^2\boldsymbol{v}_2 = (A-2I)\boldsymbol{v}_1 = \mathbf{0}$$

$$\Rightarrow \boldsymbol{v}_2 \in \ker(A-2I)^2$$

$$(A-2I)\boldsymbol{v}_2 = \boldsymbol{v}_1 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ is a generalized eigenvector of index 2

$$(A-2I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\boldsymbol{v}_3 = 2\boldsymbol{v}_3 + \boldsymbol{v}_2 \Rightarrow (A-2I)\boldsymbol{v}_3 = \boldsymbol{v}_2 \Rightarrow (A-2I)^3\boldsymbol{v}_3 = (A-2I)^2\boldsymbol{v}_2 = \mathbf{0}$$

$$(A-2I)\boldsymbol{v}_3 = \boldsymbol{v}_2 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$(A-2I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a generalized eigenvector of index 3

$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ is called a Jordan basis for the matrix A

- Theorem : (Jordan basis)

Every $n \times n$ matrix admits a Jordan basis of C^n. The first elements of the Jordan chains form a maximal set of linearly independent eigenvectors. Moreover, the number of generalized eigenvectors in the Jordan basis that belong to the Jordan chains associated with the eigenvalue λ is the same as the eigenvalue's multiplicity.

- Ex 13:

Find a Jordan basis for the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -2 & 2 & -4 & 1 & 1 \\ -1 & 0 & -3 & 0 & 0 \\ -4 & -1 & 3 & 1 & 0 \\ 4 & 0 & 2 & -1 & 0 \end{bmatrix}$$

Sol:

Characteristic equation:

$$|\lambda I - A| = (\lambda - 1)^3 (\lambda + 2)^2 = 0$$

A has two eigenvalues: $\lambda_1 = 1$ (triple eigenvalue), and $\lambda_2 = -2$ (double)

A has only two eigenvectors: $\boldsymbol{v}_1 = (0, 0, 0, -1, 1)^T$ for $\lambda_1 = 1$ and, $\boldsymbol{v}_4 = (-1, 1, 1, -2, 0)^T$ for $\lambda_2 = -2$

A has 2 linearly independent eigenvectors, the Jordan basis will contain two Jordan chains of length 3 and 2

$$A\boldsymbol{v}_2 = \boldsymbol{v}_2 + \boldsymbol{v}_1 \Rightarrow \boldsymbol{v}_2 = (0, 1, 0, 0, -1)^T$$

$$A\boldsymbol{v}_3 = \boldsymbol{v}_3 + \boldsymbol{v}_2 \Rightarrow \boldsymbol{v}_3 = (0, 0, 0, 1, 0)^T$$

$$A\boldsymbol{v}_5 = -2\boldsymbol{v}_5 + \boldsymbol{v}_4 \Rightarrow \boldsymbol{v}_5 = (-1, 0, 0, -2, 1)^T$$

$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ is a Jordan basis for the matrix A

An $k \times k$ matrix of the form

$$J_{\lambda,k} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & 0 & 0 & \lambda \end{bmatrix}$$

in which λ is a real or complex number, is known as a Jordan block

A Jordan matrix is a square matrix of block diagonal form

$$J = \operatorname{diag}(J_{\lambda_1,n_1}, J_{\lambda_2,n_2}, \cdots, J_{\lambda_k,n_k}) = \begin{bmatrix} J_{\lambda_1,n_1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_2,n_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_{\lambda_k,n_k} \end{bmatrix}$$

- Ex :

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

6 distinct 1x1 Jordan blocks

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

4x4 Jordan block followed by a 2x2 Jordan block

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Three 2x2 Jordan blocks with respective diagonal entries 0, 1, 2

$$\begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}} & \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} \boxed{-1} & 0 & 0 & 0 \\ 0 & \boxed{4} & 0 & 0 \\ 0 & 0 & \boxed{\begin{matrix} 4 & 1 \\ 0 & 4 \end{matrix}} \end{matrix} \end{bmatrix}$$

4 Jordan blocks, 3 different eigenvalues

The algebraic multiplicity for $\lambda = 2$ is 3, geometric multiplicity is 1

The algebraic multiplicity for $\lambda = -1$ is 1, geometric multiplicity is 1

The algebraic multiplicity for $\lambda = 4$ is 3, geometric multiplicity is 2

- Theorem : (Jordan canonical form)

Let A be an $n \times n$ real or complex matrix. Let $S = (\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n)$ be a matrix whose columns form a Jordan basis of A. Then S places A into the Jordan canonical form:

$$S^{-1}AS = J = \operatorname{diag}(J_{\lambda_1, n_1}, J_{\lambda_2, n_2}, \cdots, J_{\lambda_k, n_k}), \text{ or, equivalently, } A = SJS^{-1}$$

- Notes:

(1) The diagonal entries of the similar Jordan matrix J are the eigenvalues of A

(2) A is diagonalizable if and only if every Jordan block is of size 1×1

- Ex :

$$A=\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ -2 & 2 & -4 & 1 & 1 \\ -1 & 0 & -3 & 0 & 0 \\ -4 & -1 & 3 & 1 & 0 \\ 4 & 0 & 2 & -1 & 0 \end{bmatrix}$$

Sol

$$S=\begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & -2 & -2 \\ 1 & -1 & 0 & 0 & 1 \end{bmatrix}, \quad J=S^{-1}AS=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$