Lecture 7: Complex Matrices, Pseudo inverses

CEDC102: Linear Algebra

Manara University

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Complex Matrices

- **Conjugate of a matrix:**

$$A \in M_{m\times n}(C) = \left[a_{ij}\right]_{m\times n} \quad \Rightarrow \quad \overline{A} \in M_{m\times n}(C) = \left[\overline{a_{ij}}\right]_{m\times n}$$

- **Ex :** $A = \begin{pmatrix} 1+i & 1 \\ i & 1-i \end{pmatrix} \Rightarrow \overline{A} = \begin{pmatrix} 1-i & 1 \\ -i & 1+i \end{pmatrix}$

- **Properties of the conjugate of a matrix:**

(1) $\overline{\overline{A}} = A$ (2) $\overline{A \pm B} = \overline{A} \pm \overline{B}$ (3) $\overline{AB} = \overline{A}\,\overline{B}$

(4) $\overline{cA} = \overline{c}\,\overline{A}, \quad c \in C$ (5) $(\overline{A})^T = \overline{A^T}$

(6) If A is invertible, then $(\overline{A})^{-1} = \overline{A^{-1}}$

- Conjugate transpose of a matrix:

$$A \in M_{m\times n}(C) \quad \Rightarrow \quad A^{*} = \overline{A^{T}} \in M_{n\times m}(C)$$

- Ex :

$$A = \begin{pmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{pmatrix} \quad \Rightarrow \quad A^{*} = \overline{A^{T}} = \begin{pmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{pmatrix}$$

- Properties of the conjugate transpose:

(1) $(A^{*})^{*} = A$

(2) $(A \pm B)^{*} = A^{*} \pm B^{*}$

(3) $(AB)^{*} = B^{*} A^{*}$

(4) $(cA)^{*} = \bar{c} A^{*}, \quad c \in C$

- **Hermitian matrix:**

A square matrix $A \in M_n(C)$ is Hermitian if $A^* = A$

- Ex :

$$A = \begin{pmatrix} 2 & 2+i & 4 \\ 2-i & 3 & i \\ 4 & -i & 1 \end{pmatrix} = A^*$$

- **Anti-Hermitian (Skew-Hermitian) matrix:**

A square matrix $A \in M_n(C)$ is skew- Hermitian if $A^* = -A$

- Ex 4:

$$A = \begin{pmatrix} -i & 2+i \\ -2+i & 0 \end{pmatrix} = -A^*$$

- Notes:

(1) Diagonal entries of an Hermitian matrix are real

(2) Diagonal entries of a Skew-Hermitian matrix are purely imaginary or zero

(3) Every square matrix $A \in M_n(C)$ can be expressed as the sum of a Hermitian matrix B and a skew-Hermitian matrix C

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2}(A - A^*)$$

Hermitian Matrices and Unitary Diagonalization

- Hermitian matrix: A square matrix $A \in M_n(C)$ is Hermitian if $A^* = A$
- Unitary matrix: A square matrix $A \in M_n(C)$ is Unitary if $A^* = A^{-1}$
- Theorem : (Properties of Hermitian matrices)

  If A is a Hermitian matrix, then:

  (a) The eigenvalues of A are all real numbers.

  (b) Eigenvalues from different eigenspaces are orthogonal

- Ex: $A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1-i \\ -1+i & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 4) = 0 \quad \Rightarrow \lambda = 1, 4 \text{ (real numbers)}$$

$$\lambda_1 = 1: \quad \boldsymbol{v}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}, \qquad \lambda_2 = 4: \quad \boldsymbol{v}_2 = \begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = (-1-i)\left(\overline{\tfrac{1}{2}(1+i)}\right) + (1)(1) = \tfrac{1}{2}(-1-i)(-1+i) + 1 = 0 \quad \Rightarrow \text{orthogonal}$$

- Theorem : (Unitary matrix)

If A is an $n \times n$ complex matrix, then the following are equivalent

(a) A is unitary

(b) $\|A\boldsymbol{x}\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x}$ in C^n

(c) $A\boldsymbol{x} \,.\, A\boldsymbol{y} = \boldsymbol{x} \,.\, \boldsymbol{y}$ for all $\boldsymbol{x}$ and y in C^n

(e) The column vectors (row) of A form an orthonormal set in C^n with respect to the complex Euclidean inner product

- Ex : (Unitary matrix)

Show that A is unitary, and then find A^{-1}

$$A=\begin{bmatrix}\frac{1}{2}(1+i) & \frac{1}{2}(1+i)\\ \frac{1}{2}(1-i) & \frac{1}{2}(-1+i)\end{bmatrix}$$

Sol:

Row vectors $\boldsymbol{r}_1=[\frac{1}{2}(1+i) \quad \frac{1}{2}(1+i)]$, $\boldsymbol{r}_2=[\frac{1}{2}(1-i) \quad \frac{1}{2}(-1+i)]$

$$\boldsymbol{r}_1\cdot\boldsymbol{r}_2=(\tfrac{1}{2}(1+i))\left(\overline{\tfrac{1}{2}(1-i)}\right)+(\tfrac{1}{2}(1+i))\left(\overline{\tfrac{1}{2}(-1+i)}\right)=0$$

$$\|\boldsymbol{r}_1\|=\sqrt{\left|\tfrac{1}{2}(1+i)\right|^2+\left|\tfrac{1}{2}(1+i)\right|^2}=1,\quad \|\boldsymbol{r}_2\|=\sqrt{\left|\tfrac{1}{2}(1-i)\right|^2+\left|\tfrac{1}{2}(-1+i)\right|^2}=1$$

are orthonormal

So A is unitary $\Rightarrow$ $A^{-1}=A^{*}=\begin{bmatrix}\frac{1}{2}(1-i) & \frac{1}{2}(1+i)\\ \frac{1}{2}(1-i) & \frac{1}{2}(-1-i)\end{bmatrix}$

- Unitary diagonalization

A square complex matrix A is said to be unitarily diagonalizable if there is a unitary matrix P such that $P^{*}AP = D$ is a complex diagonal matrix. Any such matrix P is said to unitarily diagonalize A

- Theorem : (Unitary diagonalization)

Every $n \times n$ Hermitian matrix A has an orthonormal set of n eigenvectors and is unitarily diagonalized by any $n \times n$ matrix P whose column vectors form an orthonormal set of eigenvectors of A

- Unitary diagonalization of a Hermitian matrix

Let A be an $n \times n$ Hermitian matrix.

Step 1: Find a basis for each eigenspace of A

Step 2: Apply the Gram–Schmidt process to each of these bases to obtain orthonormal bases for the eigenspaces.

Step 3: Form the matrix P whose column vectors are the basis vectors obtained in Step 2. This will be a unitary matrix and will unitarily diagonalize A

- Ex : (Unitary diagonalization of a Hermitian matrix)

Find a matrix P that unitarily diagonalizes the Hermitian matrix

$$A = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$$

Sol:

$$\lambda_1 = 1: \quad \mathbf{v}_1 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}, \qquad \lambda_2 = 4: \quad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{2}(1+i) \\ 1 \end{bmatrix}$$

Gram-Schmidt Process:

$$\mathbf{p}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \quad \mathbf{p}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \begin{bmatrix} \frac{1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad P = \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$D = P^{*}AP = \begin{bmatrix} \frac{-1+i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix} \begin{bmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

- Normal matrices

A square complex matrix A is said to be normal if it commutes with its conjugate transpose: $A^*A = AA^*$

Normal matrices include: the Hermitian, skew-Hermitian, and unitary matrices in the complex case and the symmetric, skew-symmetric, and orthogonal matrices in the real case.

- Note:

The nonzero skew-symmetric matrices are examples of real matrices that are not orthogonally diagonalizable but are unitarily diagonalizable

- Ex : (Normal matrix)

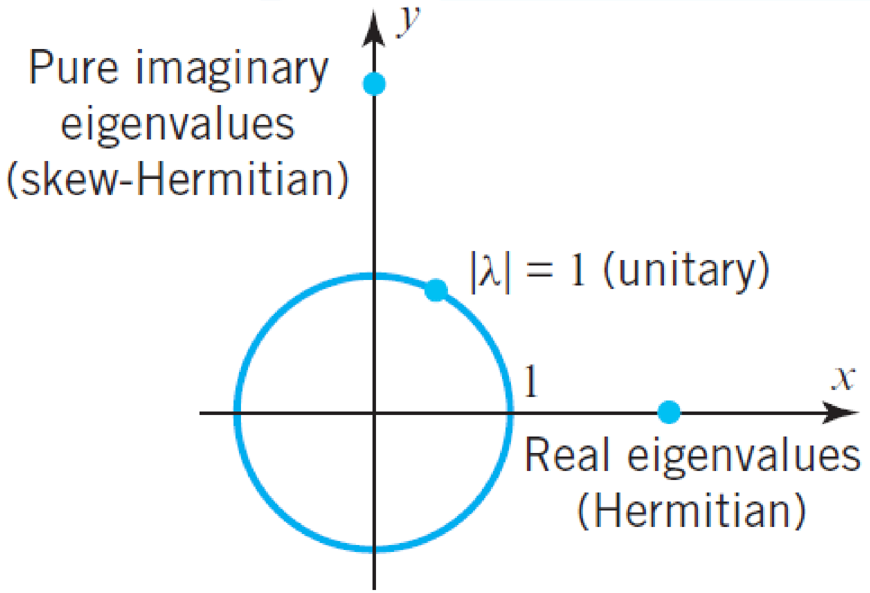
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad (A^* A = AA^*)$$

- Notes:

Hermitian matrices have real eigenvalues

Skew-Hermitian matrices have eigenvalues either zero or purely imaginary

Unitary matrices have eigenvalues of modulus 1



The Pseudoinverse A^+

By choosing good bases, A multiplies $\boldsymbol{v}_i$ in the row space to give $\sigma_i \boldsymbol{u}_i$ in the column space. A^{-1} must do the opposite! If $A\boldsymbol{v} = \sigma \boldsymbol{u}$ then $A^{-1}\boldsymbol{u} = \boldsymbol{v}/\sigma$. The singular values of A^{-1} are $1/\sigma$, just as the eigenvalues of A^{-1} are $1/\lambda$. The bases are reversed. The $\boldsymbol{u}$'s are in the row space of A^{-1}, the $\boldsymbol{v}$'s are in the column space.

Until this moment we would have added "*if* A^{-1} *exists*." Now we don't. A matrix that multiplies $\boldsymbol{u}_i$ to produce $\boldsymbol{v}_i/\sigma_i$ *does* exist. It is the pseudoinverse A^+:

Pseudoinverse of A

$$\boxed{A^+ = V\Sigma^+U^{\mathrm{T}}} = \underset{n \text{ by } n}{\begin{bmatrix} \boldsymbol{v}_1 \cdots \boldsymbol{v}_r \cdots \boldsymbol{v}_n \end{bmatrix}} \underset{n \text{ by } m}{\begin{bmatrix} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_r^{-1} \\ & & \end{bmatrix}} \underset{m \text{ by } m}{\begin{bmatrix} \boldsymbol{u}_1 \cdots \boldsymbol{u}_r \cdots \boldsymbol{u}_m \end{bmatrix}^{\mathrm{T}}}$$

The ***pseudoinverse*** $\boldsymbol{A}^{+}$ is an n by m matrix. If A^{-1} exists (we said it again), then A^{+} is the same as A^{-1}. In that case $m = n = r$ and we are inverting $U\Sigma V^{\mathrm{T}}$ to get $V\Sigma^{-1}U^{\mathrm{T}}$. The new symbol A^{+} is needed when $r < m$ or $r < n$. Then A has no two-sided inverse, but it has a *pseudo*inverse A^{+} with that same rank r:

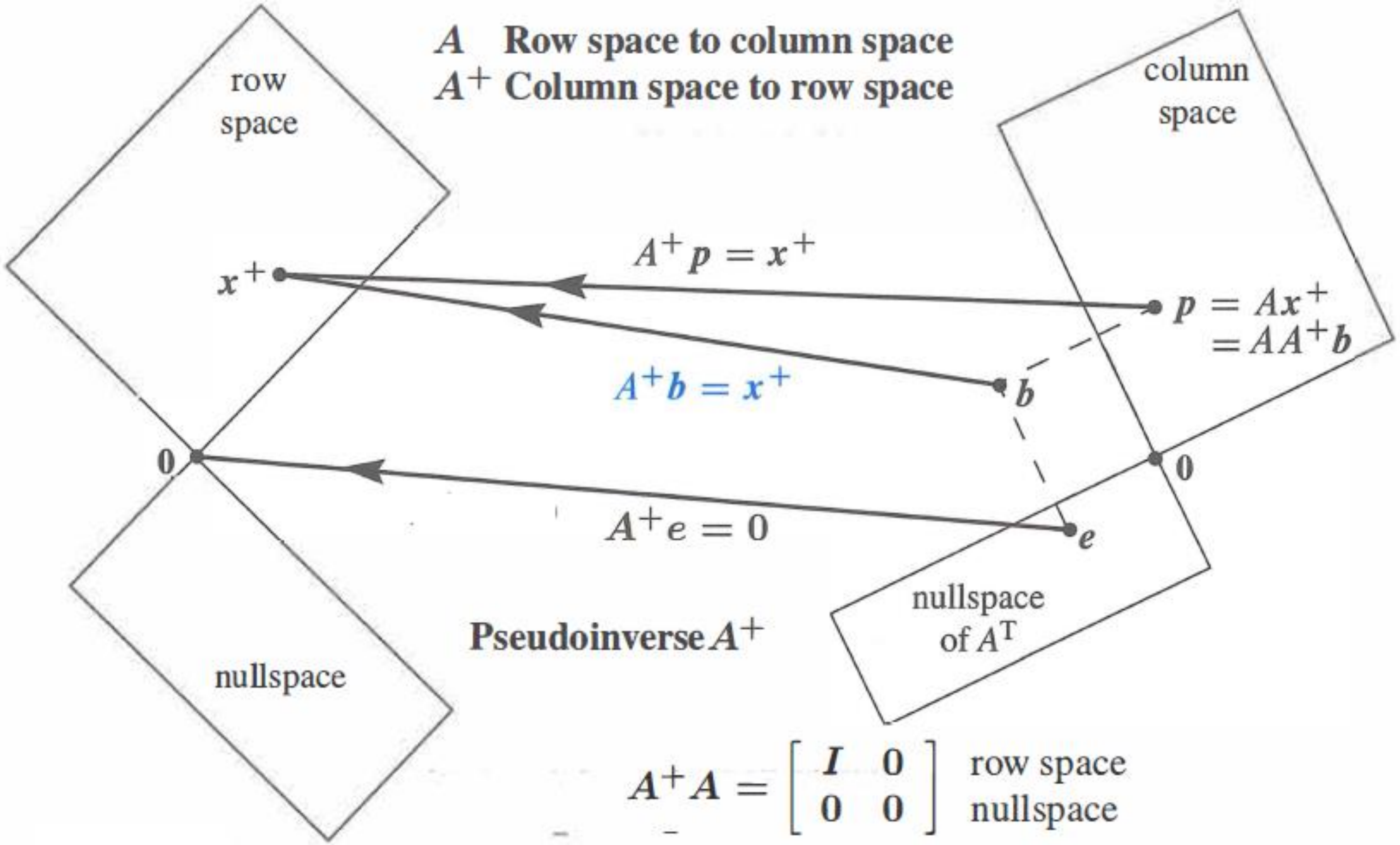
$$A^{+}\boldsymbol{u}_i = \frac{1}{\sigma_i}\boldsymbol{v}_i \quad \text{for } i \le r \qquad \text{and} \qquad A^{+}\boldsymbol{u}_i = \mathbf{0} \quad \text{for } i > r.$$

The vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r$ in the column space of A go back to $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ in the row space. The other vectors $\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m$ are in the left nullspace, and A^{+} sends them to zero. When we know what happens to all those basis vectors, we know A^{+}.

Notice the pseudoinverse of the diagonal matrix Σ. Each σ in Σ is replaced by σ^{-1} in Σ^+. The product $\Sigma^+\Sigma$ is as near to the identity as we can get. It is a projection matrix, $\Sigma^+\Sigma$ is partly I and otherwise zero. We can invert the σ's, but we can't do anything about the zero rows and columns. This example has $\sigma_1 = 2$ and $\sigma_2 = 3$:

$$\Sigma^+\Sigma = \begin{bmatrix} \mathbf{1/2} & 0 & 0 \\ 0 & \mathbf{1/3} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{2} & 0 & 0 \\ 0 & \mathbf{3} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{0} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \mathbf{0} \end{bmatrix}.$$

The pseudoinverse A^+ is the n by m matrix that makes AA^+ and A^+A into projections.



Ax^+ in the column space goes back to $A^+Ax^+ = x^+$ in the row space.

Trying for $AA^{-1} = A^{-1}A = I$	$AA^+ =$ projection matrix onto the column space of A $A^+A =$ projection matrix onto the row space of A

Example 3 Every rank one matrix is a column times a row. With unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, that is $A = \sigma \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$. Its pseudoinverse is $A^+ = \boldsymbol{v}\boldsymbol{u}^{\mathrm{T}}/\sigma$. The product AA^+ is $\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$, the projection onto the line through $\boldsymbol{u}$. The product A^+A is $\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}$.

Example 4 Find the pseudoinverse of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix is not invertible. The rank is 1. The only singular value is $\sigma_1 = 2$. That is inverted to $1/2$ in Σ^+ (also rank 1).

$$A^+ = V\Sigma^+U^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{\mathbf{1}}{\mathbf{4}}\begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}.$$

A^+ also has rank 1. Its column space is always the row space of A.