Lecture 5: Eigenvalues and Eigenvectors

CEDC102: Linear Algebra

Manara University

2023-2024

- Eigenvalues and Eigenvectors
- Diagonalization and Powers of A

Introduction Eigenvalues and Eigenvectors

- **Eigenvalue problem:**

  If A is an $n \times n$ matrix, do there exist <u>nonzero vectors</u> $\boldsymbol{x}$ in R^n such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$?

- **Eigenvalue and eigenvector:**

  A: an $n \times n$ matrix

  λ: a scalar

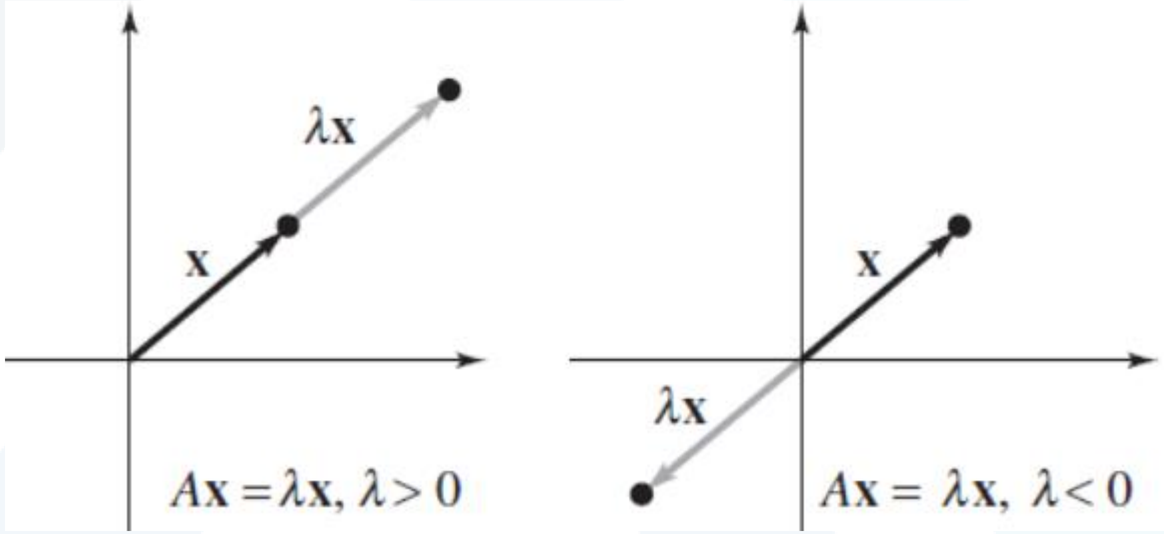
  $\boldsymbol{x}$: <u>a nonzero vector</u> in R^n

Eigenvalue

$$A\boldsymbol{x} = \lambda \boldsymbol{x}$$

Eigenvector

- **Geometrical Interpretation:**



- Ex : (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix},\qquad x_1=\begin{bmatrix}1\\0\end{bmatrix},\qquad x_2=\begin{bmatrix}0\\1\end{bmatrix}$$

$$Ax_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}=\overset{\text{Eigenvalue}}{2}\underset{\text{Eigenvector}}{\begin{bmatrix}1\\0\end{bmatrix}}=2x_1$$

$$Ax_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\-1\end{bmatrix}=\overset{\text{Eigenvalue}}{-1}\underset{\text{Eigenvector}}{\begin{bmatrix}0\\1\end{bmatrix}}=(-1)x_2$$

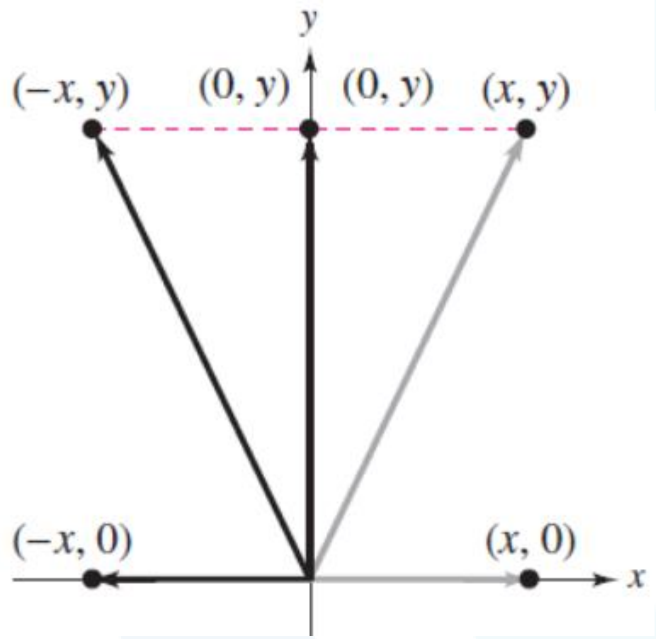
- Theorem: (The eigenspace of A corresponding to λ)

  If A is an $n \times n$ matrix with an eigenvalue λ, then the set of all eigenvectors of λ together with the zero vector is a subspace of R^n. This subspace is called the eigenspace of λ.

- Ex 2: (An example of eigenspaces in the plane)

  Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$



Sol:

If $\boldsymbol{v} = (x, y)$, then $A\boldsymbol{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$

For a vector on the x-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Eigenvalue $\lambda_1 = -1$

For a vector on the y-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$

Eigenvalue $\lambda_2 = 1$

Geometrically, multiplying a vector (x, y) in R^2 by the matrix A corresponds to a reflection in the y-axis.

The eigenspace corresponding to $\lambda_1 = -1$ is the x-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the y-axis.

- Theorem : (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

Let A is an $n\times n$ matrix.

(1) An eigenvalue of A is a scalar λ such that that $\det(\lambda I - A) = 0$

(2) The eigenvectors of A corresponding to λ are <u>the nonzero solutions</u> of $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$

- Note:

$A\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow (\lambda I - A)\boldsymbol{x} = \mathbf{0}$ (homogeneous system)

If $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ has nonzero solutions iff $\det(\lambda I - A) = 0$

- Characteristic polynomial of $A \in M_{n\times n}$:

$$\det(\lambda I - A) = |(\lambda I - A)| = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Characteristic polynomial of $A \in M_n$:

$$\det(\lambda I - A) = |(\lambda I - A)| = p_A(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

- Properties of the characteristic polynomial:

$$\lambda = 0 \quad \Rightarrow \quad \det(-A) = c_0 \quad \Rightarrow \quad c_0 = (-1)^n \det(A)$$

$$c_{n-1} = -\mathrm{tr}(A)$$

- Properties of the eigenvalues:

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$$

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

- Characteristic equation of A: $\det(\lambda I - A) = 0$
- Ex : (Finding eigenvalues and eigenvectors)

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol:

Characteristic equation:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, -2$$

Eigenvalues: $\lambda_1 = -1, \lambda_2 = -2$

$$(1)\ \lambda_1 = -1 \Rightarrow \ (\lambda_1 I - A)\boldsymbol{x} = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$$(2)\ \lambda_2 = -2 \Rightarrow \ (\lambda_2 I - A)\boldsymbol{x} = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad t \neq 0$$

Check: $A\boldsymbol{x} = \lambda \boldsymbol{x}$

- Ex : (Finding eigenvalues and eigenvectors)

Find the eigenvalues and corresponding eigenvectors for the matrix A. What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol:

Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of A corresponding to $\lambda = 2$:

$$(\lambda I - A)\mathbf{x} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

s and t not both zero

$$\left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s,t \in R \right\}$$

the eigenspace of A corresponding to $\lambda = 2$

has two linearly independent eigenvectors

Thus, the dimension of its eigenspace is 2.

- Notes:

(1) If an eigenvalue λ_1 occurs as a multiple root (k times) for the characteristic polynomial, then λ_1 has multiplicity k.

(2) The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

- Ex : Find the eigenvalues of the matrix A and find a basis for each of the corresponding eigenspaces.

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & -10\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 3\end{bmatrix}$$

Sol:

Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix}\lambda-1 & 0 & 0 & 0\\ 0 & \lambda-1 & -5 & 10\\ -1 & 0 & \lambda-2 & 0\\ -1 & 0 & 0 & \lambda-3\end{vmatrix}=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

Eigenvalues: $\lambda_1=1,\ \lambda_2=2,\ \lambda_3=3$

$$(1)\ \lambda_1 = 1 \Rightarrow \quad (\lambda_1 I - A)\boldsymbol{x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix};\quad s,t \neq 0$$

$$\Rightarrow \left\{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right\}$$ is a basis for the eigenspace of A corresponding to $\lambda = 1$

$$(2)\ \lambda_2 = 2 \Rightarrow \quad (\lambda_2 I - A)\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}; \quad t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ is a basis for the eigenspace of } A \text{ corresponding to } \lambda = 2$$

(3) $\lambda_3 = 3 \Rightarrow \quad (\lambda_3 I - A)\boldsymbol{x} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}; \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- Theorem : (Eigenvalues of triangular matrices)

If A is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex: (Finding eigenvalues for diagonal and triangular matrices)

$$(a)\ A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix}, \qquad (b)\ A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a)\ |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = -3$$

$$(b)\ \lambda_1 = -1,\ \lambda_2 = 2,\ \lambda_3 = 0,\ \lambda_4 = -4,\ \lambda_5 = 3$$

- Theorem : (Eigenvalues and Invertibility)

  A square matrix A is invertible iff $\lambda = 0$ is not an eigenvalue of A

- Eigenvalues and eigenvectors of linear transformations:

  A number λ is an eigenvalue of a linear transformation $T: V \to V$ when there is a nonzero vector $\boldsymbol{x}$ such that $T(\boldsymbol{x}) = \lambda \boldsymbol{x}$. The vector $\boldsymbol{x}$ is an eigenvector of T corresponding to λ, and the set of all eigenvectors of λ (with the zero vector) is the eigenspace of λ.

- Ex : (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

Sol:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda + 2)^2(\lambda - 4)$$

Eigenvalues: $\lambda_1 = 4, \lambda_2 = -2$

The eigenspaces for these two eigenvalues are as follows:

$B_1 = \{(1, 1, 0)\}$ Basis for $\lambda_1 = 4$

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$ Basis for $\lambda_2 = -2$

- Notes:

(1) If $T: R^3 \rightarrow R^3$ is the linear transformation whose standard matrix is A, and B' is a basis for R^3 made up of the three linearly independent eigenvectors corresponding to the eigenvalues of A, then the matrix A' for T relative to the basis B' is diagonal.

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues of A

Nonstandard basis:

$B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$

Eigenvectores of A

(2) The main diagonal entries of the matrix A' are the eigenvalues of A.

Diagonalization

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

  A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. (P diagonalizes A)

- Note:

  If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called similar. and the transformation from A to $P^{-1}AP$ is called a similarity transformation

Property that is preserved by a similarity transformation is called similarity invariant. Table below lists the most important similarity invariants

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible
Rank	A and $P^{-1}AP$ have the same rank
Nullity	A and $P^{-1}AP$ have the same nullity
Trace	A and $P^{-1}AP$ have the same trace
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues
Eigenspace dimension	A and $P^{-1}AP$ have the same dimension for the same λ

- Ex : (A diagonalizable matrix)

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -3 & 0 \\ -3 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 4)(\lambda + 2)^2 = 0$$

Eigenvalues: $\lambda_1 = 4, \lambda_2 = -2, \lambda_3 = -2$

(1) $\lambda_1 = 4 \Rightarrow$ Eigenvector: $p_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$(2)\ \lambda_2 = -2 \Rightarrow \text{Eigenvectors: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes:

$$(1)\ P = \begin{bmatrix} p_2 & p_1 & p_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(2)\ P = \begin{bmatrix} p_2 & p_3 & p_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \Rightarrow \quad P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

- Theorem : (Condition for diagonalization)

  An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

- Note:

  If n linearly independent vectors do not exist, then an $n \times n$ matrix A is not diagonalizable.

- Ex : (A matrix that is not diagonalizable)

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

Eigenvalue: $\lambda_1 = 1$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

A does not have two ($n = 2$) L. I. eigenvectors, so A is not diagonalizable

- Steps for diagonalizing an $n \times n$ square matrix:

Step 1: Find n linearly independent eigenvectors $p_1, p_2, \ldots p_n$ for A with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$

Step 2: Let $P = \left[p_1 \mid p_2 \mid \cdots \mid p_n \right]$

Step 3: Let $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$, where $Ap_i = \lambda_i p_i, \; i = 1, 2, \ldots, n$

- Note:

The order of the eigenvalues used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

- Ex : (Diagonalizing a matrix)

Find a matrix P such that is $P^{-1}AP$ diagonal

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 2)(\lambda + 2)(\lambda - 3) = 0$$

Eigenvalues: $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$

$$(1)\ \lambda_1 = 2 \qquad \Rightarrow \lambda_1 I - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$(2)\ \lambda_2 = -2 \quad \Rightarrow \lambda_2 I - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/4)t \\ -(1/4)t \\ t \end{bmatrix} = \frac{1}{4} t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$(3)\ \lambda_3 = 3 \Rightarrow \lambda_3 I - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \text{Eigenvector: } p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{Let } P = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Notes: k is a positive integer

$$(1)\ D=\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k=\begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2)\ D=P^{-1}AP \Rightarrow D^k=(P^{-1}AP)^k=P^{-1}A^kP \Rightarrow A^k=PD^kP^{-1}$$

- Ex : (Powers of a Matrix)

Find A^6, where

$$A=\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Sol:

Eigenvalues: $\lambda_1 = 1, \lambda_2 = 2$ double $\quad p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, p_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^6 = PD^6P^{-1} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^6 & 0 & 0 \\ 0 & 2^6 & 0 \\ 0 & 0 & 1^6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -62 & 0 & -126 \\ 63 & 64 & 63 \\ 63 & 0 & 127 \end{bmatrix}$$

- Theorem : (Sufficient conditions for diagonalization)

If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- Ex : (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol:

Because A is a triangular matrix, its eigenvalues are the main diagonal entries.

$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$

These three values are distinct, so A is diagonalizable.

If λ_0 is an eigenvalue of an $n x n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the geometric multiplicity of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the algebraic multiplicity of λ_0

- Theorem : (Geometric and Algebraic Multiplicity)

If A is a square matrix, then:

(a) For every eigenvalue of A, the geometric multiplicity is less than or equal to the algebraic multiplicity.

(b) A is diagonalizable if and only if the geometric multiplicity of every eigenvalue is equal to the algebraic multiplicity

- **Theorem : (Eigenvalues and Eigenvectors of Matrix Powers)**

  If k is a positive integer, λ is an eigenvalue of a matrix A, and $\boldsymbol{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\boldsymbol{x}$ is a corresponding eigenvector

- **Ex :** Find the eigenvalues and corresponding eigenvectors of A^7

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Sol:

The eigenvalues of A are $\lambda_1 = 2, \lambda_2 = -2, \lambda_3 = 3$

The eigenvalues of A^7 are $\lambda_1 = 2^7 = 128, \lambda_2 = (-2)^7 = -128, \lambda_3 = 3^7 = 2187$

The Eigenvectors of A^7 are: $p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \quad p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$