Lecture 8: Linear Transformations

CEDC102: Linear Algebra

Manara University

2023-2024

Introduction to Linear Transformations

The Kernel and Range of a Linear Transformation

Matrices for Linear Transformations

Transition Matrices and Similarity

Applications of Linear Transformations

- **Images And Preimages of Functions:**

Function T that maps a vector space V into a vector space W

$T: V \xrightarrow{\text{Mapping}} W,\qquad V, W$: vector spaces

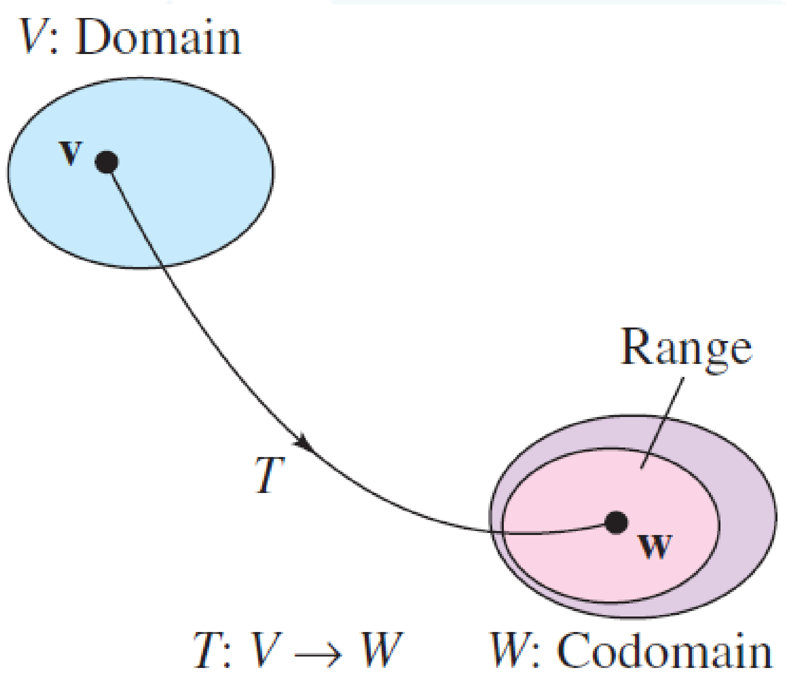
V: the domain of T

W: the codomain of T

- **Image of $\boldsymbol{v}$ under T:**

If $\boldsymbol{v}$ is in V and $\boldsymbol{w}$ is in W such that: $T(\boldsymbol{v}) = \boldsymbol{w}$

Then $\boldsymbol{w}$ is called the image of $\boldsymbol{v}$ under T



- Images And Preimages of Functions:

- The range of T: The set of all images of vectors in V.

- The preimage of $\boldsymbol{w}$: The set of all $\boldsymbol{v}$ in V such that $T(\boldsymbol{v}) = \boldsymbol{w}$.

- Ex : (A function from R^2 into R^2)

$$T: R^2 \to R^2 \qquad \boldsymbol{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of $\boldsymbol{v} = (-1, 2)$. (b) Find the preimage of $\boldsymbol{w} = (-1, 11)$

Sol:

$$(a)\ \boldsymbol{v} = (-1, 2) \Rightarrow T(\boldsymbol{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3, 3)$$

$$(b)\ T(\boldsymbol{v}) = \boldsymbol{w} = (-1, 11) \Rightarrow T(v_1, v_2) = (v_1 - v_2,\ v_1 + 2v_2) = (-1,\ 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$$\Rightarrow v_1 = 3,\ \ v_2 = 4$$

Thus $\{(3, 4)\}$ is the preimage of $\boldsymbol{w} = (-1, 11)$.

- Linear Transformation (L.T.):

V, W: vector spaces

$T: V \rightarrow W$: Linear Transformation

$$(1)\ T(\boldsymbol{u} + \boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}), \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V$$

$$(2)\ T(c\boldsymbol{u}) = cT(\boldsymbol{u}), \quad \forall\, c \in R$$

- Notes

(1)

$$T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}) \qquad T(c\boldsymbol{u}) = cT(\boldsymbol{u})$$

Addition in V	Addition in W	Scalar multiplication in V	Scalar multiplication in W

(2) A linear transformation $T: V \to V$ from a vector space into itself is called a linear operator

- Ex : (Verifying a linear transformation T from R^2 into R^2)

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

Sol:

$$\boldsymbol{u} = (u_1, u_2), \ \ \boldsymbol{v} = (v_1, v_2) \quad \text{vectors in } R^2,\ c\text{: any real}$$

$$T(\boldsymbol{u} + \boldsymbol{v}) = T(u_1 + v_1, u_2 + v_2)$$

$$= ((u_1 + v_1) - (u_2 + v_2), (u_1 + v_1) + 2(u_2 + v_2))$$

$$= ((u_1 - u_2) + (v_1 - v_2), (u_1 + 2u_2) + (v_1 + 2v_2))$$

$$= (u_1 - u_2, u_1 + 2u_2) + (v_1 - v_2, v_1 + 2v_2) = T(\boldsymbol{u}) + T(\boldsymbol{v})$$

$$T(c\boldsymbol{u}) = T(cu_1, cu_2) = (cu_1 - cu_2, cu_1 + 2cu_2)$$

$$= c(u_1 - u_2, u_1 + 2u_2) = cT(\boldsymbol{u})$$

Therefore, T is a linear transformation

- Ex : (A Linear Transformation from P_n to P_{n+1})

$$\boldsymbol{p} = p(x) = c_0 + c_1 x + \ldots + c_n x^n \in P_n$$

$$T: P_n \to P_{n+1}: \quad T(\boldsymbol{p}) = T(p(x)) = xp(x) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}$$

- Ex : (A Linear Transformation from P_n to P_{n-1} $n \geq 1$)

$$T: P_n \to P_{n-1}: \; T(\boldsymbol{p}) = T(p(x)) = p'(x) \quad \text{derivative}$$

- Ex : (A Linear Transformation from P_∞ to R)

$$T: P_\infty \to R: \quad T(\boldsymbol{p}) = T(p(x)) = \int_a^b p(x)\,dx$$

- Ex : (Functions that are not linear transformations)

$$(a)\ T(x) = \sin x \quad T: R \to R$$

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

$$(b)\ T(x) = x^2 \quad T: R \to R$$

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2$$

$$(c)\ T(x) = x + 1 \quad T: R \to R$$

$$(d)\ T(x, y) = x + y + 1 \quad T: R^2 \to R$$

$$(e)\ T(A) = \det(A) \quad T: M_n(R) \to R$$

- Notes: Two uses of the term "linear"

(1) $f(x) = x + 1$ is called a linear function because its graph is a line.

(2) $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication

- Zero transformation:

$$T: V \rightarrow W \qquad T(\mathbf{v}) = \mathbf{0}, \quad \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \rightarrow V \qquad T(\mathbf{v}) = \mathbf{v}, \quad \forall \mathbf{v} \in V$$

- **Theorem : (Properties of linear transformations)**

$$T : V \to W, \qquad \boldsymbol{u},\ \boldsymbol{v} \in V$$

(1) $T(\mathbf{0}) = \mathbf{0}$

(2) $T(-\boldsymbol{v}) = -T(\boldsymbol{v})$

(3) $T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$

(4) If $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$ then

$$\begin{aligned} T(\boldsymbol{v}) &= T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n) \\ &= c_1T(\boldsymbol{v}_1) + c_2T(v_2) + \cdots + c_nT(\boldsymbol{v}_n) \end{aligned}$$

- Ex : (Linear transformations and bases)

Let $T: R^3 \to R^3$ be a linear transformation such that

$$T(1,0,0) = (2,-1,4), \quad T(0,1,0) = (1,5,-2), \quad T(0,0,1) = (0,3,1)$$

Find $T(2, 3, -2)$

Sol:

$$(2,3,-2) = 2(1,0,0) + 3(0,1,0) - 2(0,0,1)$$

$$\begin{aligned} T(2,3,-2) &= 2T(1,0,0) + 3T(0,1,0) - 2T(0,0,1) \\ &= 2(2,-1,4) + 3(1,5,-2) - 2T(0,3,1) \\ &= (7,7,0) \end{aligned}$$

- Ex : (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$

(b) Show that T is a linear transformation from R^2 into R^3

Sol:

$$(a)\ \mathbf{v} = (2, -1) \qquad T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{R^2 \text{ vector}} = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}_{R^3 \text{ vector}} \Rightarrow T(2,-1) = (6, 3, 0)$$

$$(b)\ T(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$ (vector addition)

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$$ (scalar multiplication)

- Theorem: (The linear transformation given by a matrix)

  Let A be an $m\times n$ matrix. The function T defined by $T(\mathbf{v}) = A\mathbf{v}$ is a linear transformation from R^n into R^m.

- Note:

$$A\mathbf{v} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{R^n \text{ vector}} = \underbrace{\begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}}_{R^m \text{ vector}}$$

$$T(\mathbf{v}) = A\mathbf{v}$$

$$T: R^n \rightarrow R^m$$

- Ex : (Rotation in the plane)

Show that the L.T. $T: R^2 \rightarrow R^2$ given by the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

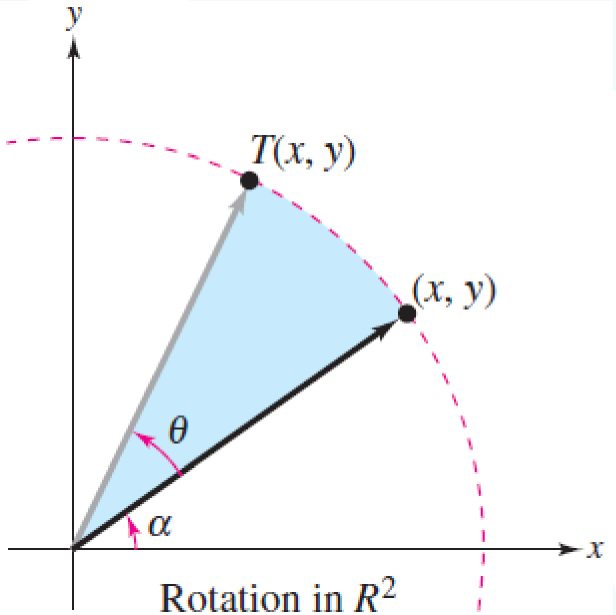
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

Sol:

$$\boldsymbol{v} = (x, y) = (r\cos\alpha, r\sin\alpha) \quad \text{(polar coordinates)}$$

r: the length of $\boldsymbol{v}$

α: the angle from the positive x-axis counterclockwise to the vector $\boldsymbol{v}$



Rotation in R^2

$$T(\boldsymbol{v}) = A\boldsymbol{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta + \alpha) \\ r\sin(\theta + \alpha) \end{bmatrix}$$

r: the length of $T(\boldsymbol{v})$

$\theta + \alpha$: the angle from the positive x-axis counterclockwise to the vector $T(\boldsymbol{v})$

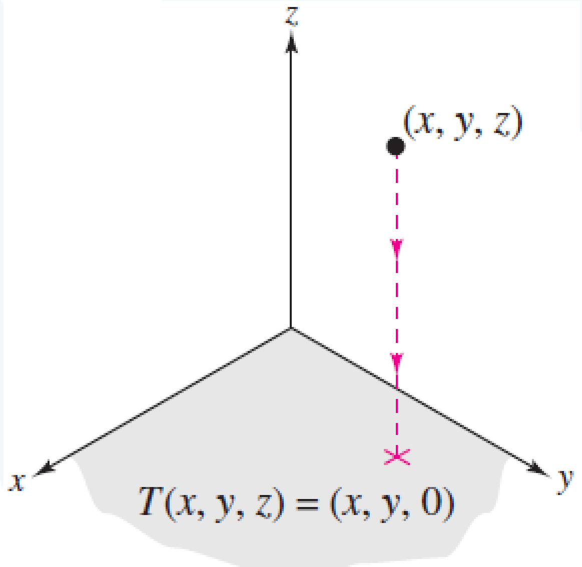
Thus, $T(\boldsymbol{v})$ is the vector that results from rotating the vector $\boldsymbol{v}$ counterclockwise through the angle θ.

- Ex : (A projection in R^3)

The linear transformation $T: R^3 \to R^3$ is given by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

is called a projection in R^3.

If $\boldsymbol{v} = (x, y, z)$ is a vector in R^3, then
$T(\mathbf{v}) = (x, y, 0)$.



Projection onto xy-plane

- Ex : (A linear transformation from $M_{m\times n}$ into $M_{n\times m}$)

$$T(A) = A^T \quad (T: M_{m\times n} \rightarrow M_{n\times m})$$

Show that T is a linear transformation.

Sol:

$$A, B \in M_{m\times n}$$

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$$

$$T(cA) = (cA)^T = cA^T = cT(A)$$

Therefore, T is a linear transformation from $M_{m\times n}$ into $M_{n\times m}$.

The Kernel and Range of a Linear Transformation

- **Kernel of a linear transformation T:**

Let $T: V \rightarrow W$ be a linear transformation. Then the set of all vectors $\mathbf{v}$ in V that satisfy $T(\mathbf{v}) = \mathbf{0}$ is called the kernel of T and is denoted by ker(T).

$$\ker(T) = \{\mathbf{v} \,|\, T(\mathbf{v}) = \mathbf{0}, \ \forall \mathbf{v} \in V\}$$

- **Ex 1: (Finding the kernel of a linear transformation)**

$$T(A) = A^T \quad (T: M_{3\times 2} \rightarrow M_{2\times 3})$$

Sol:

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

- Ex : (The kernel of the zero and identity transformations)

$(a)\ T(\mathbf{v}) = \mathbf{0}$ (the zero transformation $T: V \rightarrow W$)

$\ker(T) = V$

$(b)\ T(\mathbf{v}) = \mathbf{v}$ (the identity transformation $T: V \rightarrow V$)

$\ker(T) = \{\mathbf{0}\}$

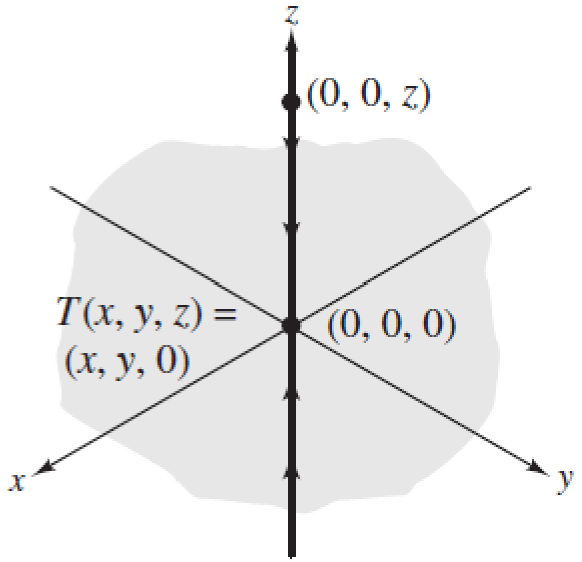
- Ex : (Finding the kernel of a L.T.)

$T(\mathbf{v}) = (x, y, 0) \qquad T: R^3 \rightarrow R^3$

$\ker(T) = ?$

Sol:

$\ker(T) = \{(0, 0, z) \mid z \text{ is a real number}\}$



- Ex : (Finding the kernel of a linear transformation)

$$T(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T: R^3 \to R^2)$$

$\ker(T) = ?$

Sol:

$$\ker(T) = \{(x_1, x_2, x_3) | T(x_1, x_2, x_3) = (0,0),\ \ \boldsymbol{x} = (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1, x_2, x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) = \{t(1, -1, 1) \mid t \text{ is a real number}\} = \operatorname{span}\{(1, -1, 1)\}$$

- **Theorem : (The kernel is a subspace of V)**

  The kernel of a linear transformation $T: V \to W$ is a subspace of the domain V

- **Note:** The kernel of T is sometimes called the nullspace of T

- Ex : (Finding the kernel of a linear transformation)

$T: P_n \rightarrow P_{n+1}: \quad T(\mathbf{p}) = T(p(x)) = xp(x) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}$

Sol:

$$T(p(x)) = xp(x) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1} = \mathbf{0} \Rightarrow c_i = 0,\ 0 \le i \le n$$

$$\ker(T) = \{\mathbf{0}\}$$

- Ex : (Finding the kernel of a linear transformation $n \ge 1$)

$T: P_n \rightarrow P_{n-1}: T(\mathbf{p}) = T(p(x)) = p'(x)$

Sol:

$$\ker(T) = \operatorname{span}\{1\}$$

- Ex : (Finding a basis for the kernel)

Let $T: R^5 \rightarrow R^4$ be defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$, where $\boldsymbol{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for ker(T) as a subspace of R^5

Sol:

$$\left[A \mid 0\right] = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 8 & 0 \end{bmatrix} \xrightarrow{\text{G. J. Elimination}} \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccccc} & & s & & t & \end{array} \end{array}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s+2t \\ s \\ -4t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

$B=\{(-2, 1, 1, 0, 0), (1, 2, 0, -4, 1)\}$: one basis for the kernel of T

- Corollary :

Let $T: R^n \to R^m$ be the L.T. given by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Then the kernel of T is equal to the solution space of $A\boldsymbol{x} = \mathbf{0}$

$T(\boldsymbol{x}) = A\boldsymbol{x}$ (a linear transformation $T: R^n \to R^m$)

$\Rightarrow \ker(T) = NS(A) = \left\{\boldsymbol{x} | A\boldsymbol{x} = \mathbf{0}, \forall \boldsymbol{x} \in R^n\right\}$ (Subspace of R^n)

- Range of a linear transformation T:

  Let $T: V \to W$ be a L.T.

  Then the set of all vectors $\boldsymbol{w}$ in W that are images of vectors in V is called the range of T and is denoted by range(T)

$$\text{range}(T) = R(T) = \{T(\mathbf{v}) | \forall \mathbf{v} \in V\}$$

- Theorem : (The range of T is a subspace of W)

  The range of a linear transformation $T: V \to W$ is a subspace of the W

- Ex : (The range of the zero and identity transformations)

  (a) $T(\mathbf{v}) = \mathbf{0}$ (the zero transformation $T: V \to W$) range(T) = $\{\mathbf{0}\}$

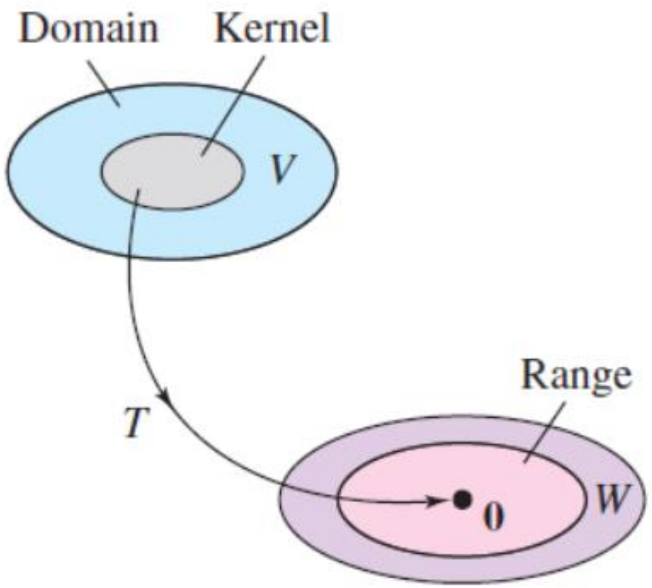
  (b) $T(\mathbf{v}) = \mathbf{v}$ (the identity transformation $T: V \to V$) range(T) = V

- **Notes:**

$T: V \to W$: is Linear Transformation

(1) $\ker(T)$ is a subspace of V

(2) Range(T) is a subspace of W



- **Corollary :**

Let $T: R^n \to R^m$ be the L.T. given by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Then the range of T is equal to the columns space of A.

$\Rightarrow \text{range}(T) = CS(A)$

- Ex: (Finding a basis for the range of a linear transformation)

Let $T: R^5 \rightarrow R^4$ be defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$, where $\boldsymbol{x}$ is in R^5 and

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

Find a basis for the range of T

Sol:

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 1 & 3 & 1 & 0 \\ -1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 8 \end{bmatrix} \\ c_1,\ c_2,\ c_3,\ c_4,\ c_5 \end{array} \xrightarrow{\text{G. J. Elimination}} \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ w_1, w_2, w_3, w_4, w_5 \end{array}$$

$\Rightarrow \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_4\}$ is a basis for $CS(B)$

$\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_4\}$ is a basis for $CS(A)$

$\Rightarrow \{(1, 2, -1, 0), (2, 1, 0, 0), (1, 1, 0, 2)\}$ is a basis for the range of T

- Ex : (range of a linear transformation)

$T: P_n \rightarrow P_{n+1}: \qquad T(\boldsymbol{p}) = T(p(x)) = xp(x) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}$

$\text{range}(T) = \text{span}\{x, x^2, \ldots, x^{n+1}\}$

- Ex : (range of a linear transformation $n \geq 1$)

$T: P_n \rightarrow P_{n-1}: \ T(\boldsymbol{p}) = T(p(x)) = p'(x)$

$\text{range}(T) = \text{span}\{1, x, \ldots, x^{n-1}\}$

- Rank of a linear transformation $T: V \to W$:

  $\text{rank}(T)$ = the dimension of the range of T

- Nullity of a linear transformation $T: V \to W$:

  $\text{nullity}(T)$ = the dimension of the kernel of T

- Note:

  Let $T: R^n \to R^m$ be the L.T. given by $T(\mathbf{x}) = A\mathbf{x}$. Then

  $\Rightarrow \text{rank}(T) = \text{rank}(A), \qquad \text{nullity}(T) = \text{nullity}(A)$

- Theorem : (Sum of rank and nullity)

Let $T: V \rightarrow W$ be a L.T. from an n-dimensional vector space V into a vector space W. Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$$

- Ex : (Finding rank and nullity of a linear transformation)

Find the rank and nullity of the L.T. $T: R^3 \rightarrow R^3$ defined by $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Sol:

$$\text{rank}(T) = \text{rank}(A) = 2$$

$$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

- Ex : (Finding rank and nullity of a linear transformation)

Let $T: R^5 \rightarrow R^7$ be a linear transformation

(a) Find the dimension of the kernel of T if the dimension of the range is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $\ker(T) = \{\mathbf{0}\}$

Sol:

(a) dim(domain of T) = 5

$$\dim(\text{ker of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$$

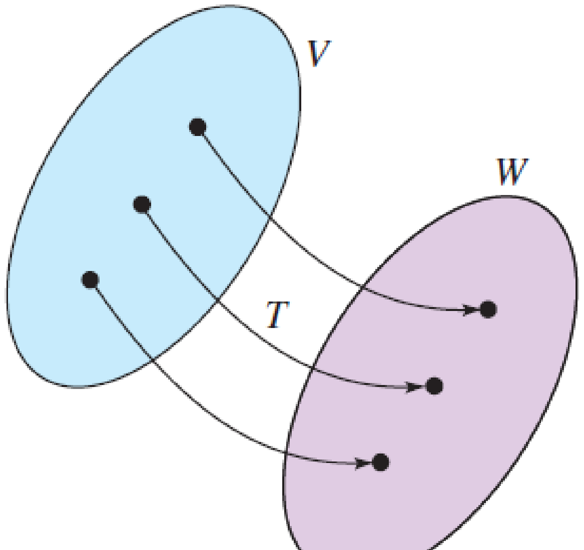
(b) $\operatorname{rank}(T) = n - \operatorname{nullity}(T) = 5 - 4 = 1$

(c) $\operatorname{rank}(T) = n - \operatorname{nullity}(T) = 5 - 0 = 5$

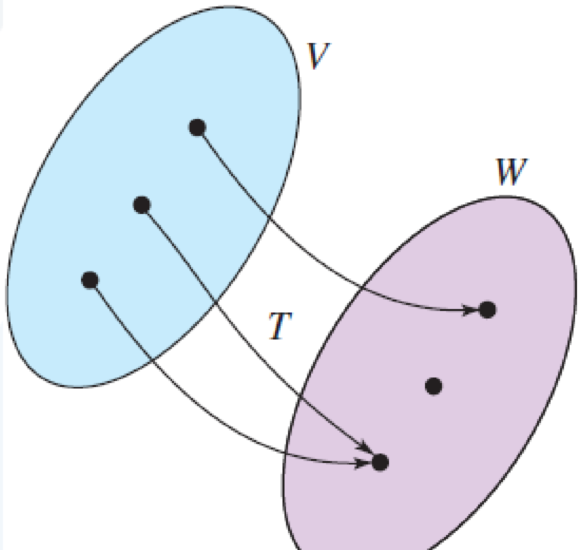
- One-to-one:

A function $T: V \to W$ is one-to-one when the preimage of every $\boldsymbol{w}$ in the range consists of a single vector

T is one-to-one if and only if, for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in V, $T(\boldsymbol{u}) = T(\boldsymbol{v})$ implies $\boldsymbol{u} = \boldsymbol{v}$.



One-to-one



Not one-to-one

- Onto:

A function $T: V \to W$ is onto when every element in W has a preimage in V. (T is onto W when W is equal to the range of T)

- Theorem : (One-to-one linear transformation)

Let $T: V \to W$ be a linear transformation. Then T is one-to-one iff $\ker(T) = \{\mathbf{0}\}$

- Ex : (One-to-one and not one-to-one linear transformation)

(a) The linear transformation $T: M_{3x2}(R) \to M_{2x3}(R)$ given by $T(A) = A^T$ is one-to-one because its kernel consists of only the mxn zero matrix

(b) The zero transformation $T: R^3 \to R^3$ is not one-to-one because its kernel is all of R^3

- Ex : (One-to-one and onto linear transformation)

(a) The L. T. $T: P_3 \to R^4$ given by $T(a + bx + cx^2 + dx^3) = (a, b, c, d)$

(b) The L. T. $T: M_{2x2}(R) \to R^4$ given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d)$$

- Ex : (One-to-one and not onto linear transformation)

$T: P_n \to P_{n+1}: T(\boldsymbol{p}) = T(p(x)) = xp(x)$

- Theorem : (Onto linear transformation)

Let $T\colon V \to W$ be a linear transformation, where W is finite dimensional Then T is onto iff the rank of T is equal to the dimension of W.

- Theorem : (One-to-one and onto linear transformation)

Let $T\colon V \to W$ be a linear transformation, with vector space V and W both of dimension n. Then T is one-to-one iff it is onto.

- Ex :

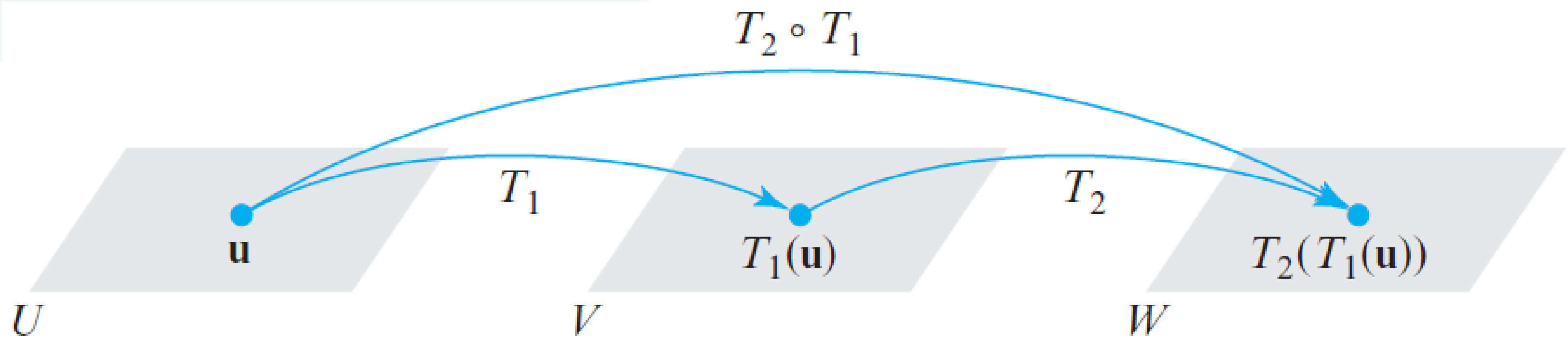
Let $T: R^n \to R^m$ be a L.T. given by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither

$$(a)\ A=\begin{bmatrix}1&2&0\\0&1&1\\0&0&1\end{bmatrix},\ (b)\ A=\begin{bmatrix}1&2\\0&1\\0&0\end{bmatrix},\ (c)\ A=\begin{bmatrix}1&2&0\\0&1&-1\end{bmatrix},\ (b)\ A=\begin{bmatrix}1&2&0\\0&1&1\\0&0&0\end{bmatrix}$$

Sol:

$T: R^n \to R^m$	dim(domain of T)	rank(T)	nullity(T)	one-to-one	onto
$(a)\ T: R^3 \to R^3$	3	3	0	Yes	Yes
$(b)\ T: R^2 \to R^3$	2	2	0	Yes	No
$(c)\ T: R^3 \to R^2$	3	2	1	No	Yes
$(d)\ T: R^3 \to R^3$	3	2	1	No	No

- Composition of linear transformations :



If $T_1: U \to V$ and $T_2: V \to W$ are L. T., then the composition of T_2 with T_1, denoted by $T_2 \circ T_1$, is the function defined by the formula

$$(T_2 \circ T_1)(\boldsymbol{u}) = T_2(T_1(\boldsymbol{u}))$$

where $\boldsymbol{u}$ is a vector in U

- Note:

This definition requires that the domain of T_2 (which is V) contain the range of T_1

- **Theorem : (Composition of linear transformations)**

If $T_1: U \to V$ and $T_2: V \to W$ are L. T., then $(T_2 \circ T_1): U \to W$ is also a linear transformation

- **Ex : (Composition of linear transformations)**

Let $T_1: P_2 \to P_3$ and $T_2: P_3 \to P_2$ be the linear transformations given by

$T_1(p(x)) = xp(x)$ and $T_2(p(x)) = p'(x)$

$(T_2 \circ T_1): P_2 \to P_2$

$(T_2 \circ T_1)(p(x)) = (T_2(T_1(p(x)) = T_2(ax + bx^2 + cx^3) = a + 2bx + 3cx^2$

$(T_1 \circ T_2): P_2 \to P_2$

$(T_1 \circ T_2)(p(x)) = (T_1(T_2(p(x)) = T_1(b + 2cx) = bx + 2cx^2$ $\qquad T_2 \circ T_1 \neq T_1 \circ T_2$

- Note: $T_2 \circ T_1 \neq T_1 \circ T_2$

- Composition with the Identity Operator

  If $T: V \to V$ is any linear operator, and if $I: V \to V$ is the identity, then for all vectors $\boldsymbol{v}$ in V, we have

  $$(T \circ I)(\boldsymbol{v}) = T(I(\boldsymbol{v})) = T(\boldsymbol{v})$$
  $$(I \circ T)(\boldsymbol{v}) = I(T(\boldsymbol{v})) = T(\boldsymbol{v})$$

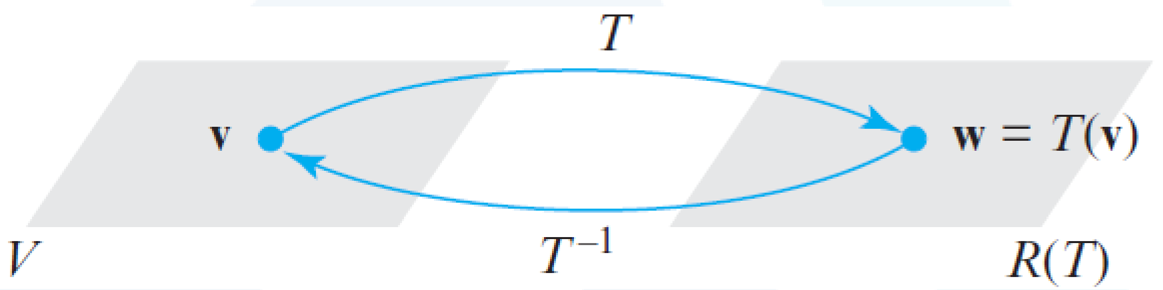
  $T \circ I = T$ and $I \circ T = T$

- Inverse Linear Transformations

  If $T: V \to W$ is a one-to-one L.T, then

  $$T^{-1}: R(T) \to V$$

  $$T^{-1}(T(\boldsymbol{v})) = \boldsymbol{v} \text{ and } T(T^{-1}(\boldsymbol{w})) = \boldsymbol{w}$$



$T \circ T^{-1} = T^{-1} \circ T = I$

- Ex : (An Inverse Transformation)

$T: P_n \to P_{n+1}: T(\mathbf{p}) = T(p(x)) = xp(x) = c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}$

is a one-to-one L.T $\Rightarrow T^{-1}(c_0 x + c_1 x^2 + \ldots + c_n x^{n+1}) = c_0 + c_1 x + \ldots + c_n x^n$

- Ex : (An Inverse Transformation)

Let $T: R^2 \to R^2$ be the linear operator defined by $T(x, y) = (2x + 3y, x + y)$

Determine whether T is one-to-one; if so, find $T^{-1}(x, y)$

Sol:

$$2x + 3y = 0,\ x + y = 0 \Rightarrow x = y = 0 \Rightarrow \ker(T) = \{\mathbf{0}\} \Rightarrow T \text{ is one-to-one}$$

$$T(x, y) = (x', y') = (2x + 3y, x + y) \Rightarrow (x, y) = (-x' + 3y', x' - 2y')$$

$$T^{-1}(x, y) = (-x + 3y, x - 2y)$$

- **Theorem : (Composition of One-to-One Linear Transformations)**

If $T_1 : U \to V$ and $T_2 : V \to W$ are one-to-one L. T., then

(a) $(T_2 \circ T_1)$ is one-to-one

(b) $(T_2 \circ T_1)^{-1} = T_1^{-1} \circ T_2^{-1}$

- **Isomorphism:**

A linear transformation $T: V \to W$ that is one to one and onto is called an isomorphism. Moreover, if V and W are vector spaces such that there exists an isomorphism from V to W, then V and W are said to be isomorphic to each other

- **Theorem : (Isomorphic spaces and dimension)**

Two finite-dimensional vector space V and W are isomorphic if and only if they are of the same dimension

- **Ex : (Isomorphic vector spaces)**

The following vector spaces are isomorphic to each other

$(a)\ R^4 = 4-\text{space}$

$(b)\ M_{4\times1} = \text{ space of all } 4\times 1 \text{ matrices}$

$(c)\ M_{2\times2} = \text{ space of all } 2\times 2 \text{ matrices}$

$(d)\ P_3(x) = \text{ space of all polynomials of degree 3 or less}$

$(e)\ V = \{(x_1, x_2, x_3, x_4, 0),\ x_i \text{ is a real number}\} \quad (\text{subspace of } R^5)$