Matrices for Linear Transformations

- Two representations of the linear transformation $T: R^3 \rightarrow R^3$

$$(1)\ T(x_1,x_2,x_3) = (2x_1 + x_2 - x_3, -x_1 + 3x_2 - 2x_3, 3x_2 + 4x_3)$$

$$(2)\ T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 3 & -2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Three reasons for matrix representation of a linear transformation:
  - It is simpler to write.
  - It is simpler to read.
  - It is more easily adapted for computer use.

- **Theorem : (Standard matrix for a linear transformation)**

Let $T: R^n \to R^m$ be a linear transformation such that

$$T(\boldsymbol{e}_1) = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad T(\boldsymbol{e}_2) = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, \quad T(\boldsymbol{e}_n) = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix},$$

then the mxn matrix whose n columns correspond to $T(\boldsymbol{e}_i)$

$$A = \left[T(\boldsymbol{e}_1) \,\middle|\, T(\boldsymbol{e}_2) \,\middle|\, \cdots \,\middle|\, T(\boldsymbol{e}_n) \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is such that $T(\boldsymbol{v}) = A\boldsymbol{v}$ for every $\boldsymbol{v}$ in R^n. A is called the standard matrix for T

- Ex : (Finding the standard matrix of a linear transformation)

Find the standard matrix for the L.T. $T: R^3 \rightarrow R^2$ defined by $T(x,y,z) = (x-2y,\ 2x+y)$

Sol:

Vector Notation	Matrix Notation
$T(\boldsymbol{e}_1) = T(1,\ 0,\ 0) = (1,\ 2)$	$T(\boldsymbol{e}_1) = T(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}1\\2\end{bmatrix}$
$T(\boldsymbol{e}_2) = T(0,\ 1,\ 0) = (-2,\ 1)$	$T(\boldsymbol{e}_2) = T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}-2\\1\end{bmatrix}$
$T(\boldsymbol{e}_3) = T(0,\ 0,\ 1) = (0,\ 0)$	$T(e_3) = T(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}0\\0\end{bmatrix}$

$$A=\left[T(\boldsymbol{e}_1)\,\middle|\,T(\boldsymbol{e}_2)\,\middle|\,T(\boldsymbol{e}_3)\right]=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}$$

- Check:

$$A\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}x-2y\\ 2x+y\end{bmatrix}$$

i.e. $T(x,y,z)=(x-2y,2x+y)$

- Notes:

(1) The standard matrix for the zero transformation from R^n into R^m is the $m\times n$ zero matrix.

(2) The standard matrix for the identity transformation from R^n into R^n is the $n\times n$ identity matrix I_n

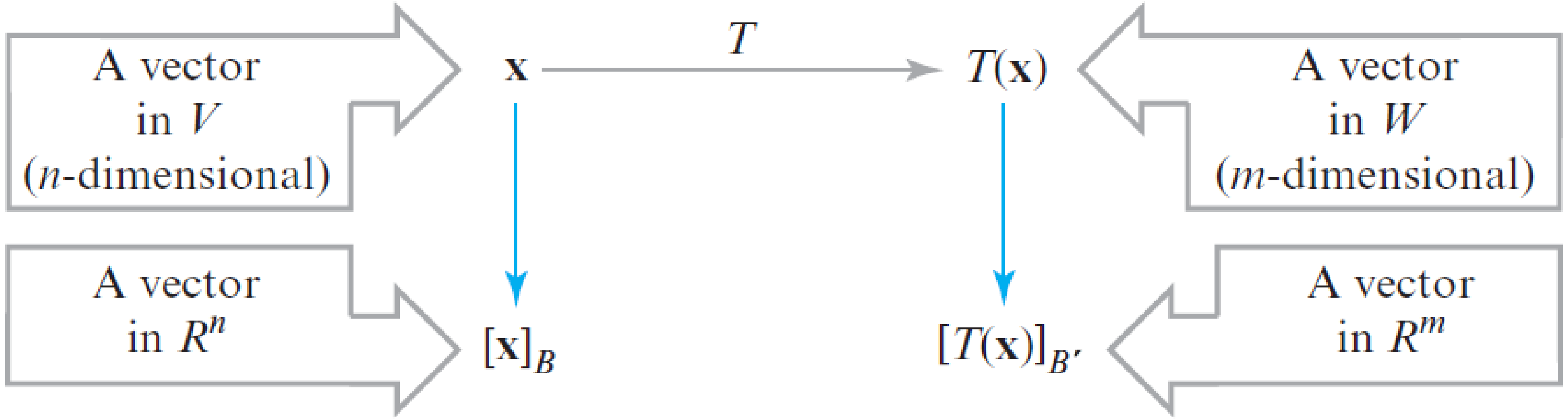
▪ Matrices for General Linear Transformations:

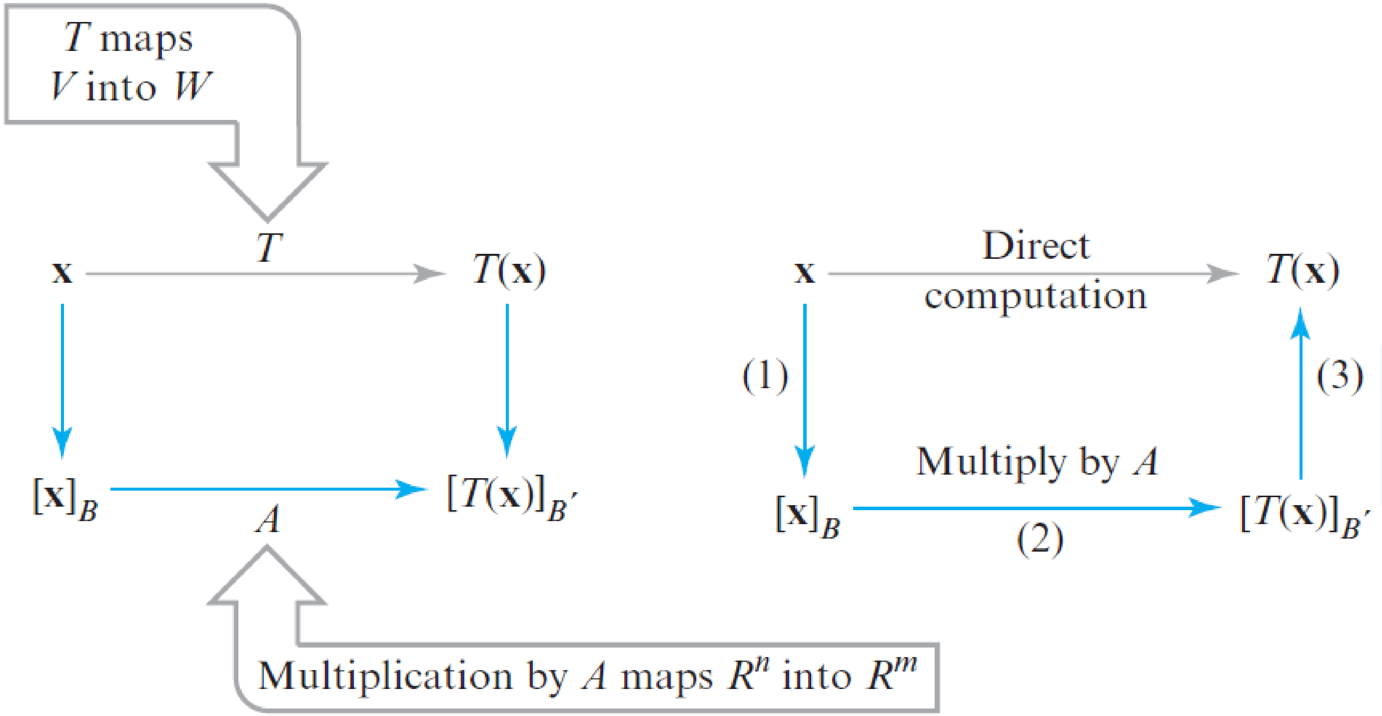
$T: V \to W$ a LT $B=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\}$ a basis for V and $B' = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_m\}$ a basis for W

$[\boldsymbol{x}]_B$ is the coordinate matrices for $\boldsymbol{x}$ in V

$[T(\boldsymbol{x})]_{B'}$ is the coordinate matrices for $T(\boldsymbol{x})$ in W

Goal: find an mxn matrix A such that multiplication by A maps the vector $[\boldsymbol{x}]_B$ into the vector $[T(\boldsymbol{x})]_{B'}$ for each $\boldsymbol{x}$ in V





Finding $T(\boldsymbol{x})$ Indirectly

Step 1. Compute the coordinate vector $[\boldsymbol{x}]_B$

Step 2. Multiply $[\boldsymbol{x}]_B$ on the left by A to produce $[T(\boldsymbol{x})]_{B'}$

Step 3. Reconstruct $T(\boldsymbol{x})$ from its coordinate vector $[T(\boldsymbol{x})]_{B'}$

$$[T(\mathbf{v}_1)]_{B'} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad [T(\mathbf{v}_2)]_{B'} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \cdots, [T(\mathbf{v}_n)]_{B'} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Then the mxn matrix whose n columns correspond to $[T(\mathbf{v}_i)]_{B'}$

$$A = \begin{array}{c} \begin{array}{cccc} T(\mathbf{v}_1) & T(\mathbf{v}_2) & \ldots & T(\mathbf{v}_n) \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{c} \\ \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_m \end{array}$$

is such that $[T(\mathbf{v})]_{B'} = A[(\mathbf{v})]_B$

for every $\mathbf{v}$ in V

We call A the matrix for T relative to the bases B and B' and will denote it by the symbol $[T]_{B', B}$

- Ex : (Finding a matrix relative to nonstandard bases)

Let the L. T. $T: R^2 \to R^2$ defined by $T(x_1, x_2) = (x_1 + x_2,\ 2x_1 - x_2)$

Find the matrix of T relative to the basis $B = \{(1,\ 2),\ (-1,\ 1)\}$ and $B' = \{(1,\ 0),\ (0,\ 1)\}$

Sol:

$$T(1,\ 2) = (3,\ 0) = 3(1,\ 0) + 0(0,\ 1),\ T(-1,\ 1) = (0,\ -3) = 0(1,\ 0) - 3(0,\ 1)$$

$$\left[T(1,\ 2)\right]_{B'} = \begin{bmatrix} 3 \\ 0 \end{bmatrix},\quad \left[T(-1,\ 1)\right]_{B'} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

The matrix of T relative to the bases B and B':

$$A = \left[\left[T(1,\ 2)\right]_{B'}\ \left[T(-1,\ 1)\right]_{B'}\right] = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$$

- Ex : (Matrix for a Linear Transformation)

Let the L. T. $T: P_1 \to P_2$ defined by $T(p(x)) = xp(x)$

Find the matrix of T relative to the standard bases $B = \{1, x\}$ and $B' = \{1, x, x^2\}$

Sol:

$$T(\boldsymbol{v}_1) = T(1) = x = 0\boldsymbol{w}_1 + 1\boldsymbol{w}_2 + 0\boldsymbol{w}_3$$

$$T(\boldsymbol{v}_2) = T(x) = x^2 = 0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 1\boldsymbol{w}_3$$

$$[T(1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad [T(x)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow A = \Big[[T(1)]_{B'} \ \ [T(\boldsymbol{x})]_{B'} \Big] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix of T relative to the bases B and B'

The Three-Step Procedure

Step 1. The coordinate matrix for $\boldsymbol{x} = a + bx$ relative to the basis $B = \{1, x\}$ is

$$[\boldsymbol{x}]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

Step 2. Multiply $[\boldsymbol{x}]_B$ on the left by A

$$A[\boldsymbol{x}]_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = [T(\boldsymbol{x})]_{B'}$$

Step 3. Reconstruction $T(\boldsymbol{x}) = T(a + bx) = 0 + ax + bx^2 = ax + bx^2$

- Ex :

For the L. T. $T: R^2 \to R^2$ given in example 2, use the matrix A to find $T(\boldsymbol{v})$, where $\boldsymbol{v} = (2, 1)$

Sol:

$$\boldsymbol{v} = (2, 1) = 1(1, 2) - 1(-1, 1) \qquad B = \{(1, 2), (-1, 1)\}$$

$$\Rightarrow [\boldsymbol{v}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow [T(\boldsymbol{v})]_{B'} = A[\boldsymbol{v}]_B = \begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\Rightarrow T(\boldsymbol{v}) = 3(1, 0) + 3(0, 1) = (3, 3) \qquad B' = \{(1, 0), (0, 1)\}$$

- Check:

$$T(2, 1) = (2 + 1, 2(2) - 1) = (3, 3)$$

- Notes:

(1) For the special case where $V = W$ and $B = B'$, the matrix A is called the matrix of T relative to the basis B

(2) If $T: V \to V$ is the identity transformation, then the matrix of T relative to the basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is the identity matrix I_n

- Theorem : (Matrices of Compositions transformations)

If $T_1: U \to V$ and $T_2: V \to W$ are L. T., and if B, B'', and B' are bases for U, V, and W, respectively, and if A_1 is the matrix of T_1 relative to the basis B, B'', A_2 is the matrix of T_2 relative to the basis B'', B', then

(1) The composition $T = T_2 \circ T_1 : U \to V$, is a L. T.

(2) The matrix A of T relative to the basis B, B' is given by $A = A_2 A_1$

- Ex : (The standard matrix of a composition)

Let T_1 and T_2 be L. T. from R^3 into R^3 such that

$$T_1(x,y,z) = (2x+y,\ 0,\ x+z), \quad T_2(x,y,z) = (x-y,\ z,\ y)$$

Find the matrices relative to the standard basis (standard matrices) for the compositions

$$T = T_2 \circ T_1 \text{ and } T' = T_1 \circ T_2$$

Sol:

$$A_1 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

standard matrices for T_1 standard matrices for T_2

The standard matrix for $T = T_2 \circ T_1$

$$A = A_2A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The standard matrix for $T' = T_1 \circ T_2$

$$A' = A_1A_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- **Theorem : (Matrices of inverse transformations)**

If $T: V \to V$ is a linear operator, and if B is a bases for V, and if A is the matrix of T relative to the basis B, then following are equivalent

(a) T is one-to-one

(b) T is invertible

(c) T is isomorphism

(d) A is invertible

- **Note:**

If T is invertible with matrix A, then the standard matrix for T^{-1} is A^{-1}

- **Ex : (Finding the inverse of a linear transformation)**

The L. T. $T: R^3 \to R^3$ defined by

$$T(x_1,x_2,x_3) = (2x_1 + 3x_2 + x_3,\ 3x_1 + 3x_2 + x_3,\ 2x_1 + 4x_2 + x_3)$$

Show that T is invertible, and find its inverse

Sol:

The standard matrix for T

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{matrix} \leftarrow 2x_1 + 3x_2 + x_3 \\ \leftarrow 3x_1 + 3x_2 + x_3 \\ \leftarrow 2x_1 + 4x_2 + x_3 \end{matrix}$$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{G.J. Elimination}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right] = \left[I \mid A^{-1} \right]$$

Therefore T is invertible and the standard matrix for T^{-1} is A^{-1}.

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$T^{-1}(\mathbf{v}) = A^{-1}\mathbf{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ -x_1 + x_3 \\ 6x_1 - 2x_2 - 3x_3 \end{bmatrix}$$

In other words $T^{-1}(x_1, x_2, x_3) = (-x_1 + x_2, -x_1 + x_3, 6x_1 - 2x_2 - 3x_3)$

Transition Matrices and Similarity

$T: V \rightarrow V$ a linear transformation

$B = \{v_1, v_2, \cdots, v_n\}$ a basis for V

$B' = \{w_1, w_2, \cdots, w_m\}$ a basis for V

$$A = \left[[T(v_1)]_B, [T(v_2)]_B, \cdots, [T(v_n)]_B \right]$$

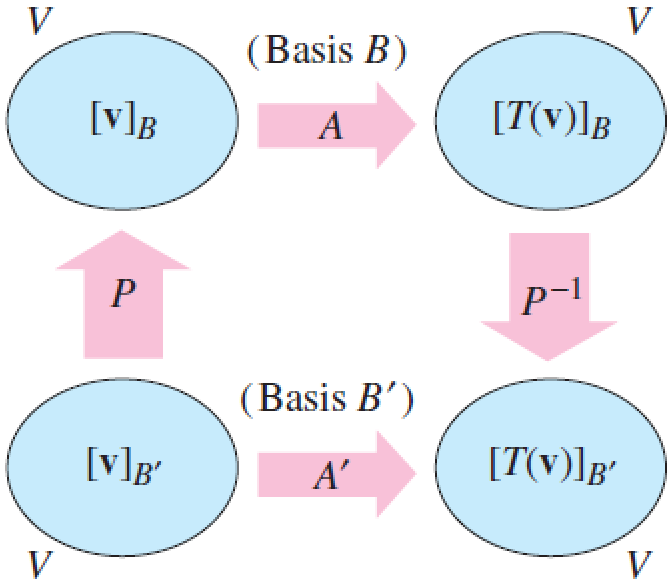
(matrix of T relative to B)

$$A' = \left[[T(w_1)]_{B'}, [T(w_2)]_{B'}, \cdots, [T(w_n)]_{B'} \right]$$

(matrix of T relative to B')

$P = \left[[w_1]_B, [w_2]_B, \cdots, [w_n]_B \right]$ (transition matrix from B' to B)

$P^{-1} = \left[[v_1]_{B'}, [v_2]_{B'}, \cdots, [v_n]_{B'} \right]$ (transition matrix from B to B')



$$\Rightarrow [\mathbf{v}]_B = P[\mathbf{v}]_{B'}, \quad [\mathbf{v}]_{B'} = P^{-1}[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B$$

$$[T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'}$$

- Two ways to get from $[\mathbf{v}]_{B'}$ to $[T(\mathbf{v})]_{B'}$:

(1) (direct) $A'[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$

(2) (indirect) $P^{-1}AP[\mathbf{v}]_{B'} = [T(\mathbf{v})]_{B'}$

$$\Rightarrow A' = P^{-1}AP$$

- Ex : (Finding a matrix for a linear transformation)

Find the matrix A for $T: R^2 \to R^2 \quad T(x_1, x_2) = (2x_1 - 2x_2,\ -x_1 + 3x_2)$

relative to the basis $B' = \{(1, 0), (1, 1)\}$

Sol:

$$\text{(I) } A' = \Big[[T(1, 0)]_{B'} \ \ [T(1, 1)]_{B'} \Big]$$

$$T(1, 0) = (2, -1) = 3(1, 0) - 1(1, 1) \Rightarrow [T(1, 0)]_{B'} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$T(1, 1) = (0, 2) = -2(1, 0) + 2(1, 1) \Rightarrow [T(1, 1)]_{B'} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\Rightarrow A' = \Big[[T(1, 0)]_{B'} \ \ [T(1, 1)]_{B'} \Big] = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

(II) Standard matrix for T (matrix of T relative to the basis $B = \{(1, 0), (0, 1)\}$)

$$A = \begin{bmatrix} T(1, 0) & T(0, 1) \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

Transition matrix from B' to B $\quad P = \begin{bmatrix} [(1, 0)]_B & [(1, 1)]_B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Transition matrix from B to B' $\quad P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Matrix of T relative to B'

$$A' = P^{-1}AP = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$$

- Ex : (Finding a matrix for a linear transformation)

Let $B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$ be basis R^2, and let $A = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}$ be the matrix for $T: R^2 \to R^2$ relative to B

Find the matrix of T relative to B'

Sol:

Transition matrix from B' to B: $P = \left[[(-1, 2)]_B \quad [(2, -2)]_B \right] = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

T.M. from B to B': $P^{-1} = \left[[(-3, 2)]_{B'} \quad [(4, -2)]_{B'} \right] = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

Matrix of T relative to B': $A' = P^{-1}AP = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$

- Ex : (Finding a matrix for a linear transformation)

  For the linear transformation $T: R^2 \to R^2$ from Ex 2, find $[\mathbf{v}]_B$, $[T(\mathbf{v})]_B$, and $[T(\mathbf{v})]_{B'}$ for the vector $\mathbf{v}$ whose coordinate matrix is $[\mathbf{v}]_{B'} = [-3\ -1]^T$

  Sol:

$$[\mathbf{v}]_B = P[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$$

$$[T(\mathbf{v})]_B = A[\mathbf{v}]_B = \begin{bmatrix} -2 & 7 \\ -3 & 7 \end{bmatrix}\begin{bmatrix} -7 \\ -5 \end{bmatrix} = \begin{bmatrix} -21 \\ -14 \end{bmatrix}$$

$$[T(\mathbf{v})]_{B'} = P^{-1}[T(\mathbf{v})]_B = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -21 \\ -14 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

$$\text{or} \quad [T(\mathbf{v})]_{B'} = A'[\mathbf{v}]_{B'} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 0 \end{bmatrix}$$

- Similar matrix

For square matrices A and A' of order n, A' is said to be similar to A if there exist an invertible matrix P such that $A' = P^{-1}AP$

- Theorem 6.15: (Properties of similar matrices)

Let A, B, and C be square matrices of order n. Then the following properties are true.

(1) A is similar to A.

(2) If A is similar to B, then B is similar to A.

(3) If A is similar to B and B is similar to C, then A is similar to C.

- Ex : (Similar matrices)

$$(a)\ A=\begin{bmatrix}2 & -2\\ -1 & 3\end{bmatrix} \text{ and } A'=\begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$(b)\ A=\begin{bmatrix}-2 & 7\\ -3 & 7\end{bmatrix} \text{ and } A'=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix} \text{ are similar}$$

$$\text{because } A'=P^{-1}AP, \text{ where } P=\begin{bmatrix}3 & -2\\ 2 & -1\end{bmatrix}$$

- Ex : (A comparison of two matrices for a linear transformation)

Let $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ be the matrix for $T: R^3 \rightarrow R^3$ relative to the standard basis. Find the matrix for T relative to the basis $B' = \{(1, 1, 0), (1, -1, 0), (0, 0, 1)\}$.

Sol:

The transition matrix from B' to the standard basis

$$P = \Big[[(1, 1, 0)]_B \;\; [(1, -1, 0)]_B \;\; [(0, 0, 1)]_B \Big] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow P^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

matrix of T relative to B':

$$A' = P^{-1}AP = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Notes: Computational advantages of diagonal matrices:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$(1)\ D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

$$(2)\ D^T = D$$

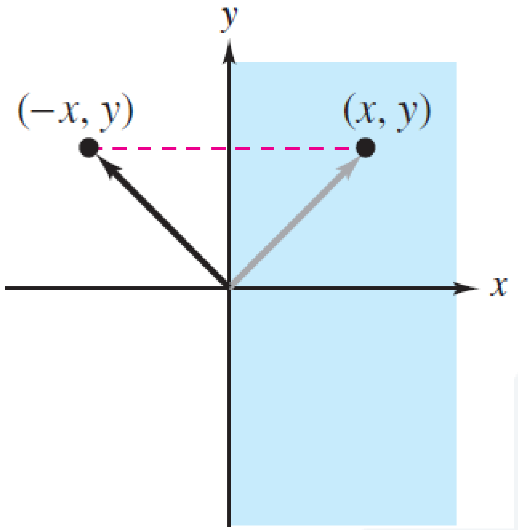
$$(3)\ D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}, \quad d_i \neq 0$$

Applications of Linear Transformations

- The Geometry of Linear Transformations In R^2

(a) Reflection in y-axis

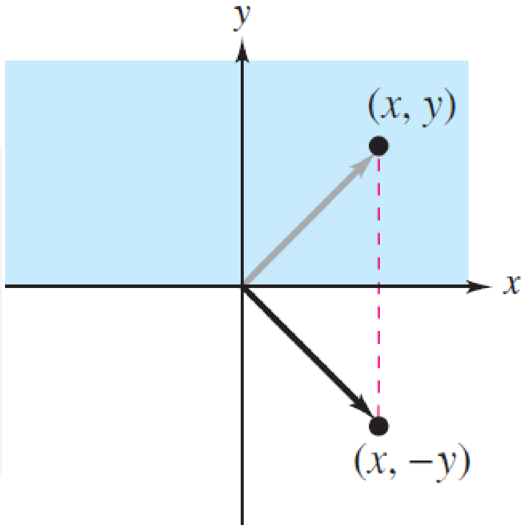
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$



$$T(x, y) = (-x, y)$$

(b) Reflection in x-axis

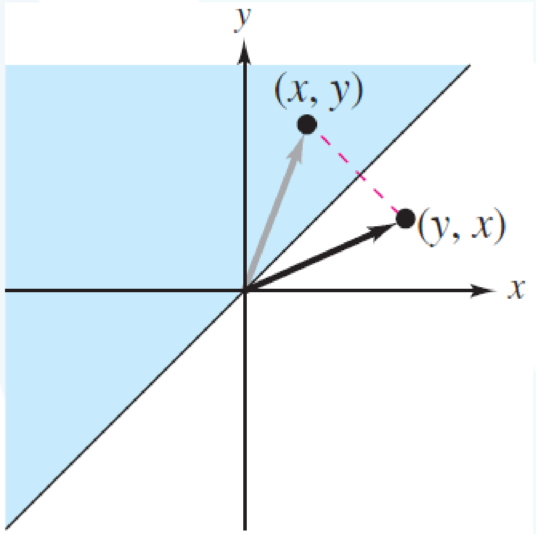
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}$$



$$T(x, y) = (x, -y)$$

(c) Reflection in line $y = x$

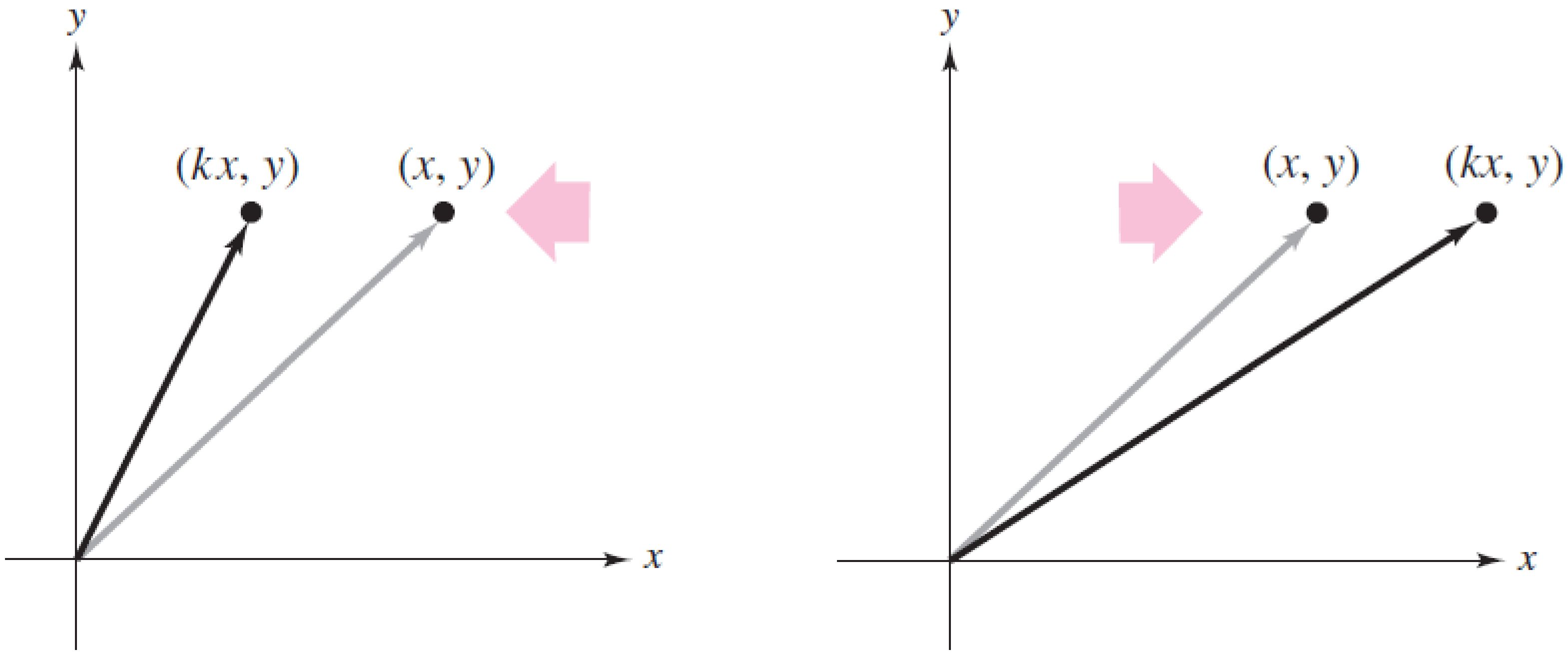
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$



$$T(x, y) = (y, x)$$

(*d*) Horizontal expansions and contractions $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ y \end{bmatrix}$

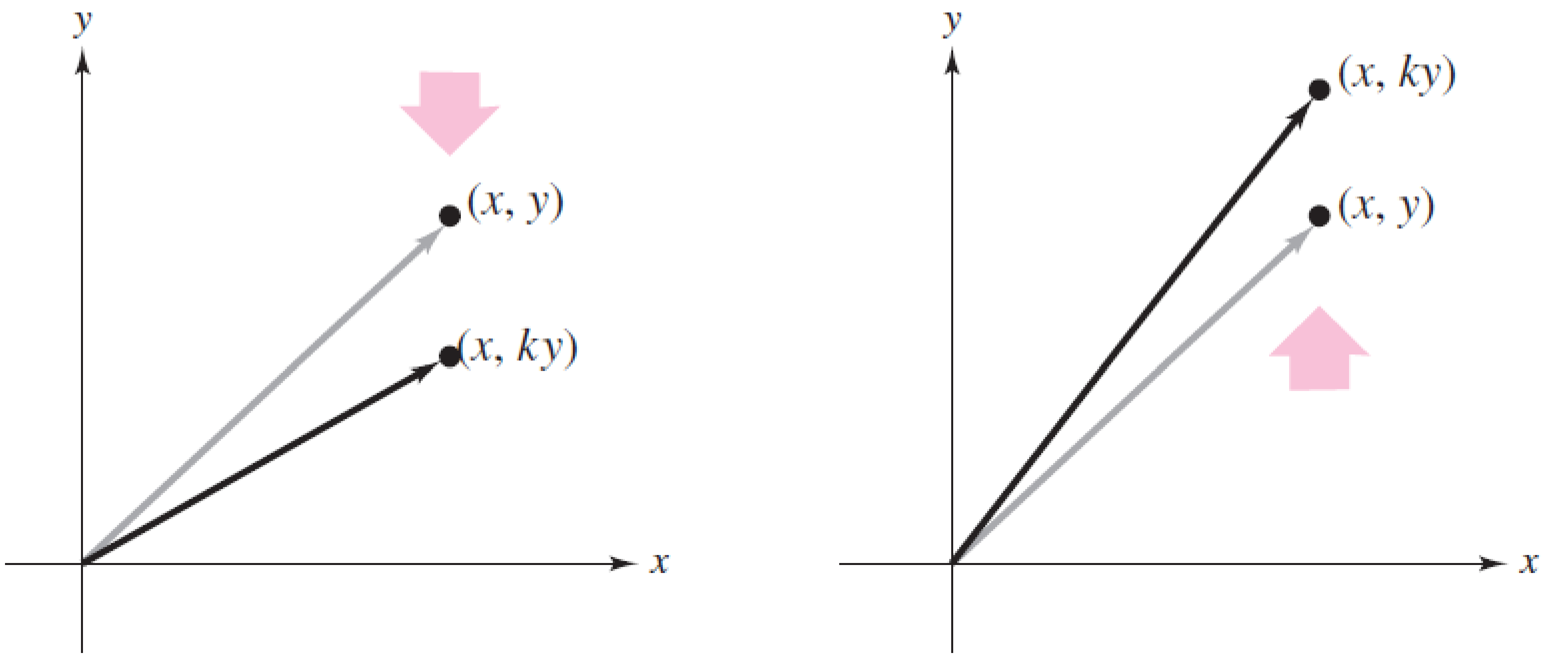
$T(x, y) = (kx, y)$



Contraction $(0 < k < 1)$ Expansion $(k > 1)$

(*d*) Vertical expansions and contractions $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ ky \end{bmatrix}$

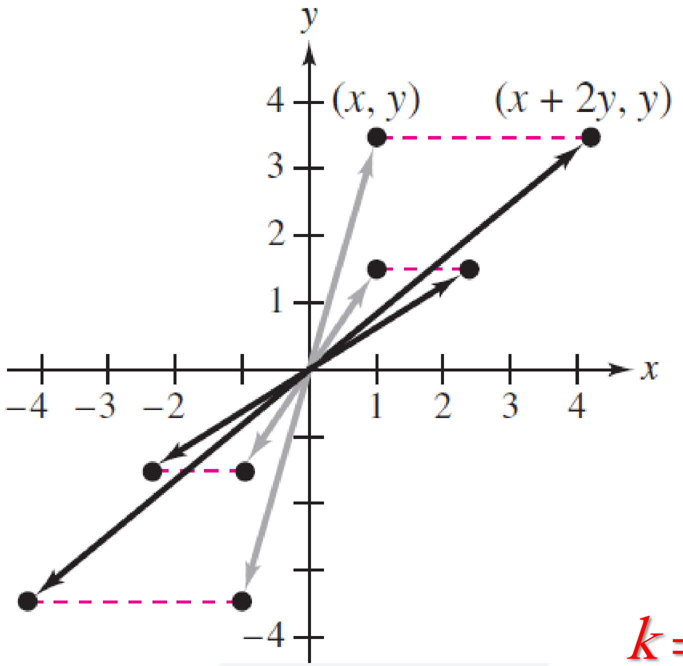
$$T(x, y) = (x, ky)$$



Contraction $(0 < k < 1)$ Expansion $(k > 1)$

(e) Horizontal shear

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ky \\ y \end{bmatrix}$$

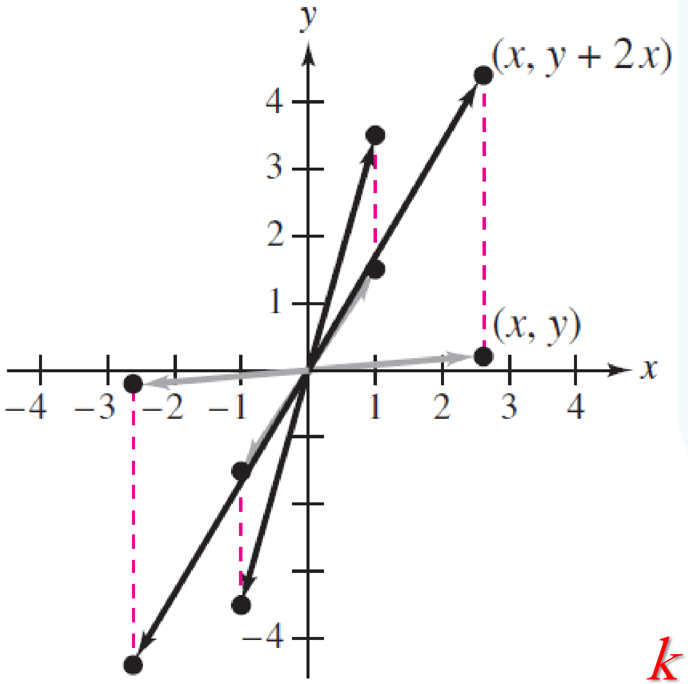


$k = 2$

$T(x, y) = (x + ky, y)$

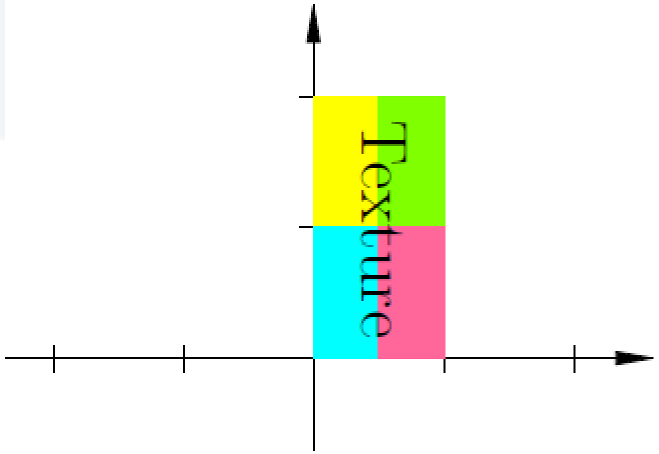
(f) Vertical shear

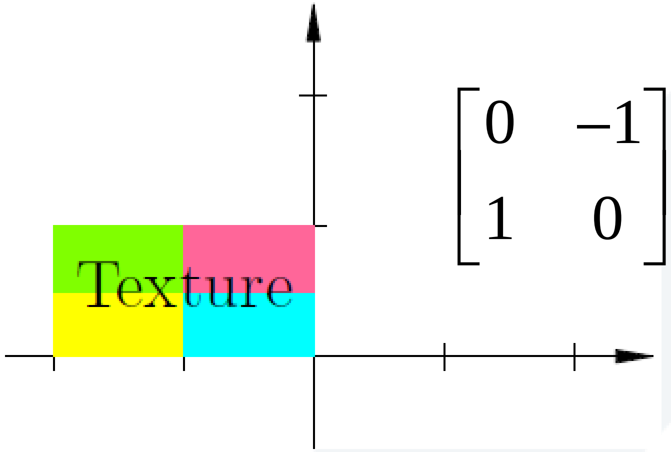
$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ kx + y \end{bmatrix}$$



$k = 2$

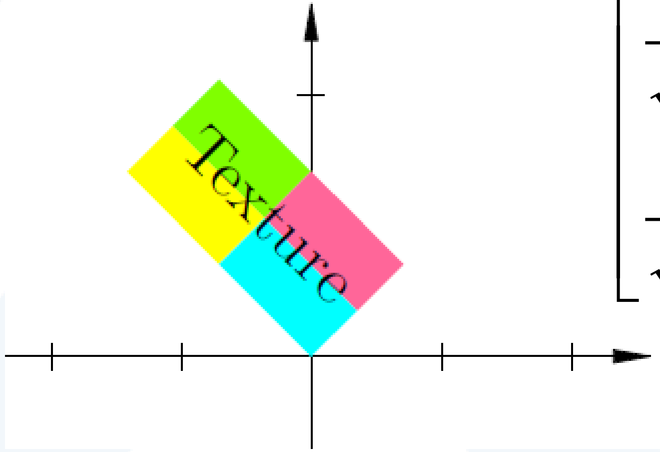
$T(x, y) = (x, kx + y)$





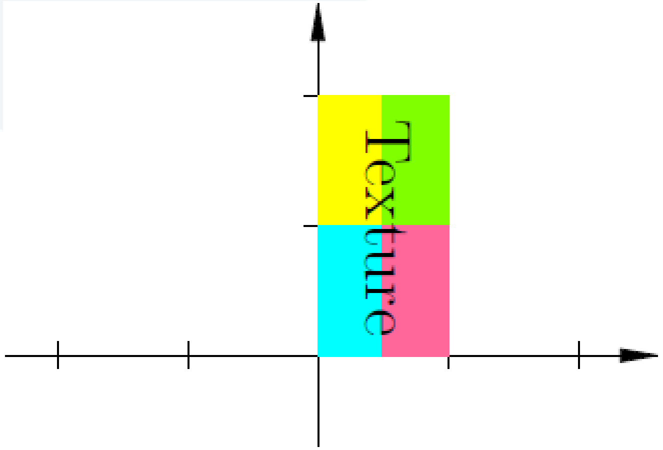
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

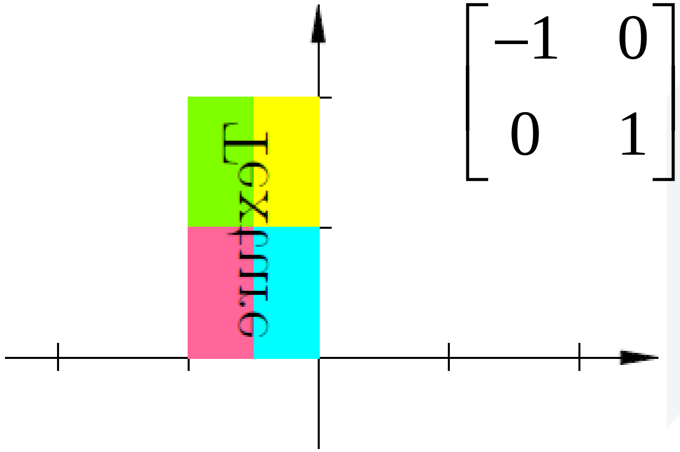
Rotation by 90°



$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

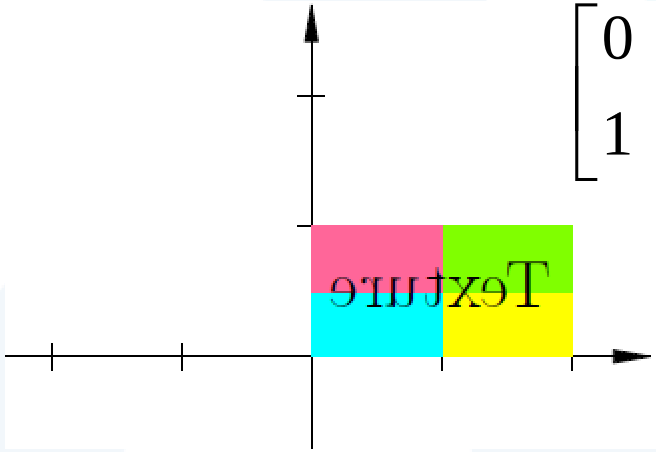
Rotation by 45°





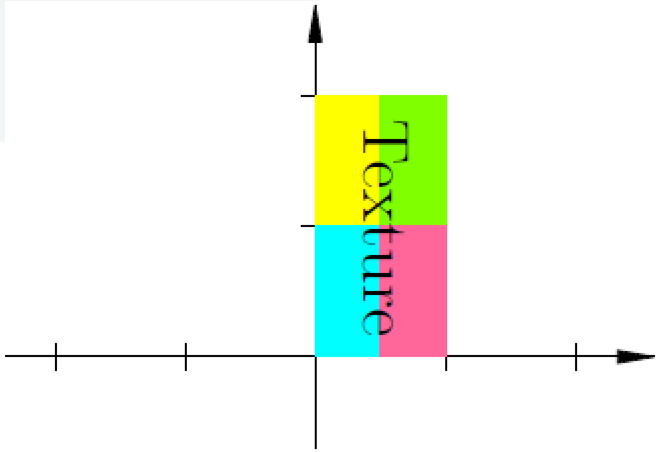
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

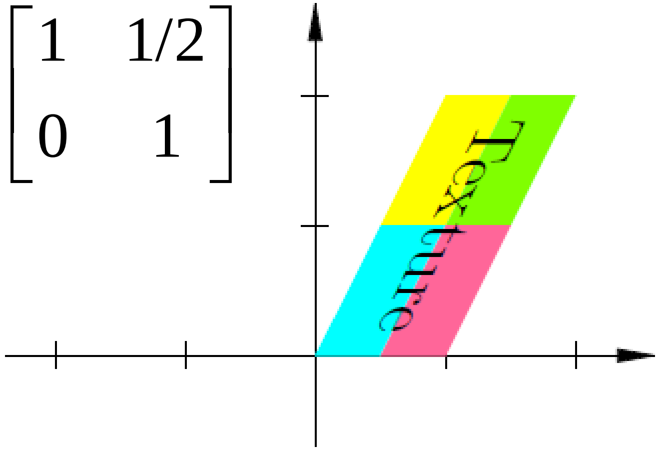
Reflection about x-axis



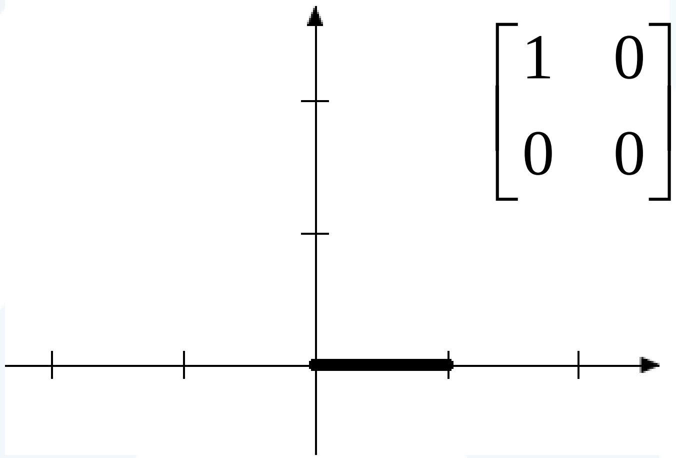
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Reflection about y = x



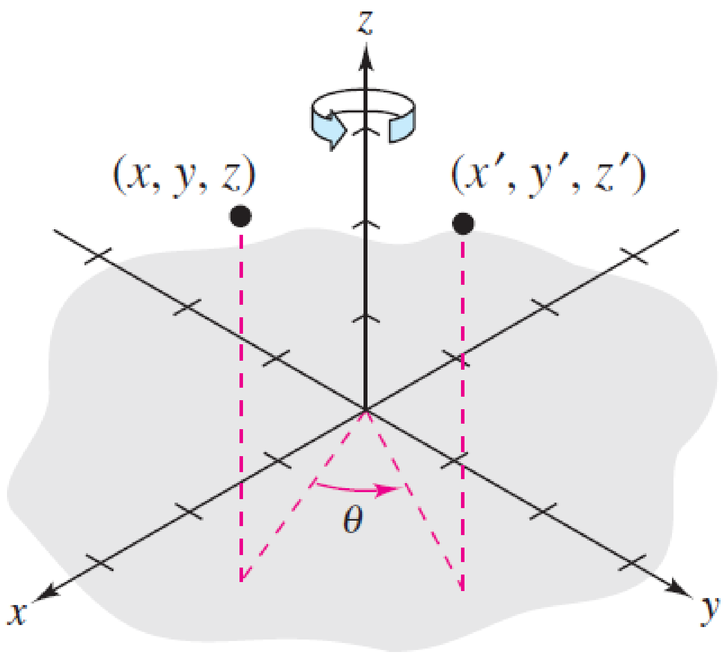


Horizontal shear



Vertical projection on x-axis

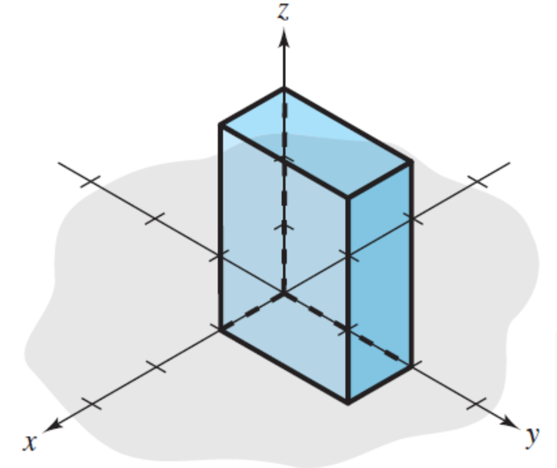
- Rotation In R^3

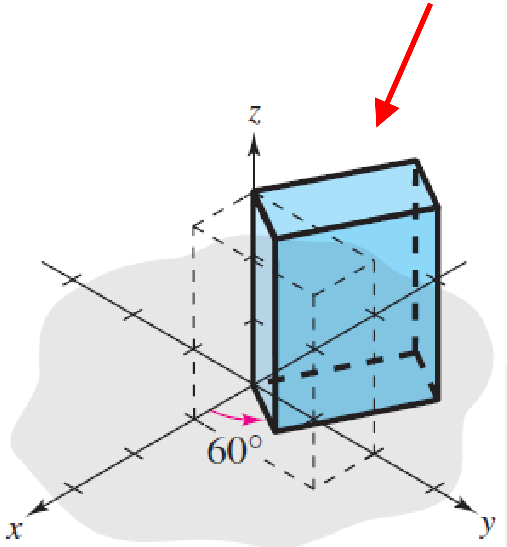


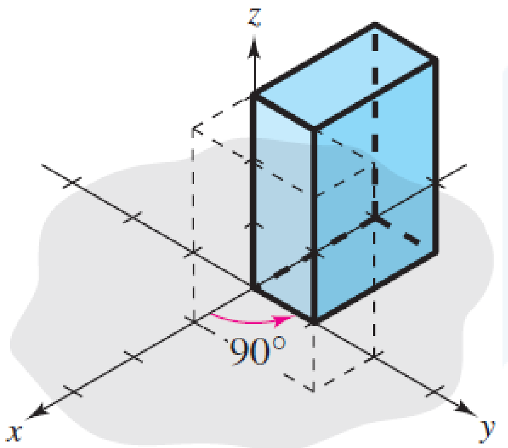
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$$

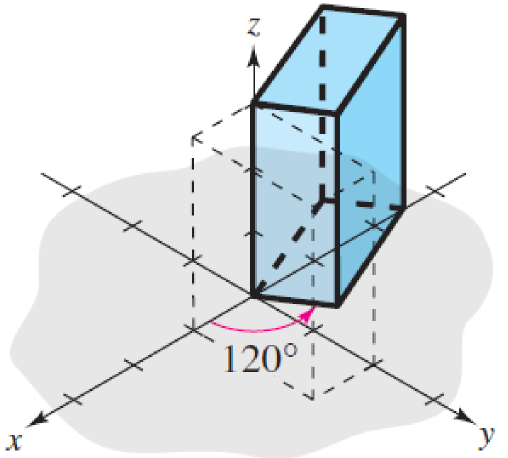
Rotation about the z-axis

$$A = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



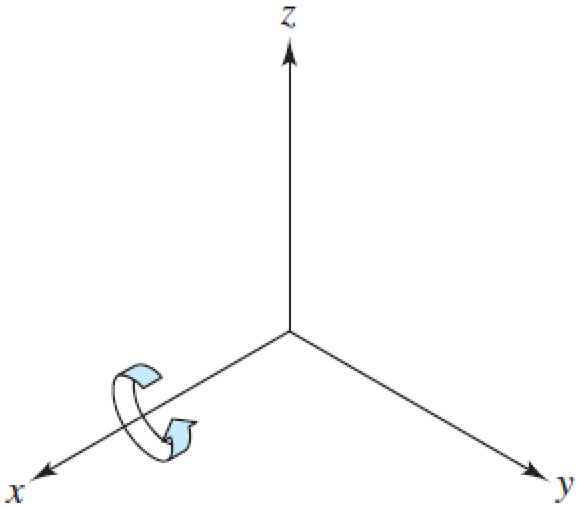






Rotation about the *x*-axis

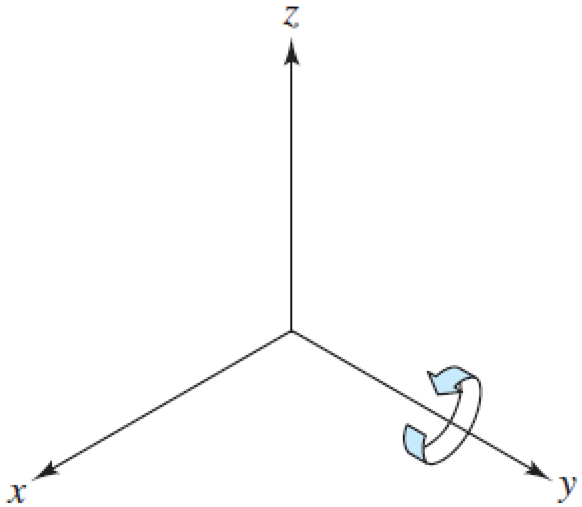
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$



Rotation about *x*-axis

Rotation about the *y*-axis

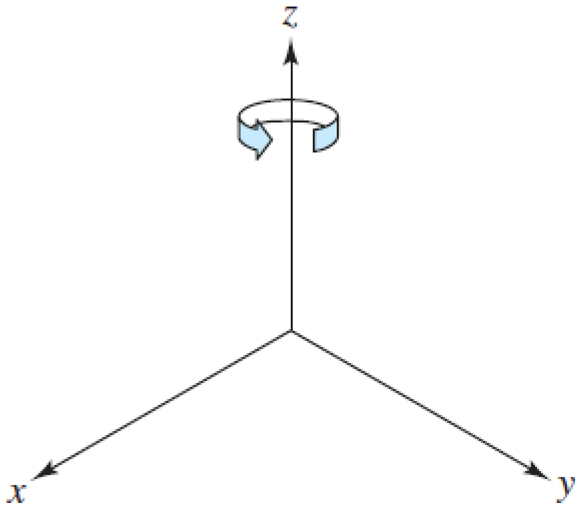
$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



Rotation about *y*-axis

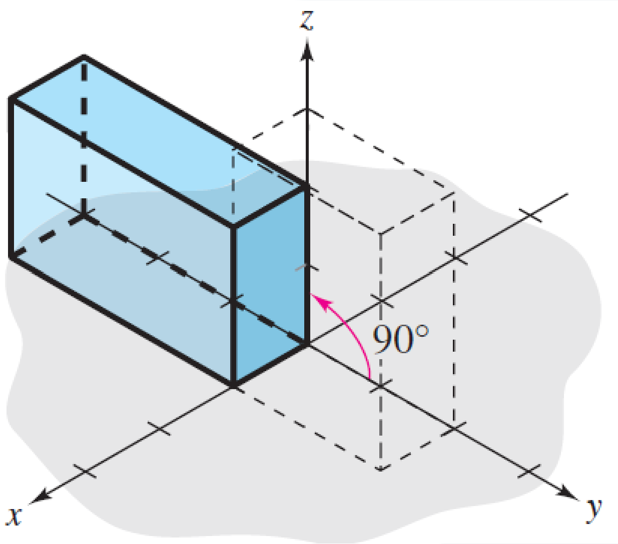
Rotation about the *z*-axis

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



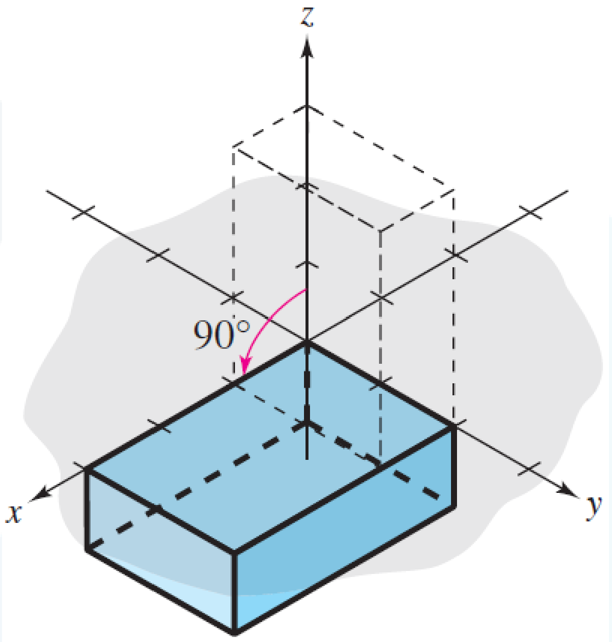
Rotation about *z*-axis

Rotation of 90° about the *x*-axis



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Rotation of 90° about the *y*-axis



$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$