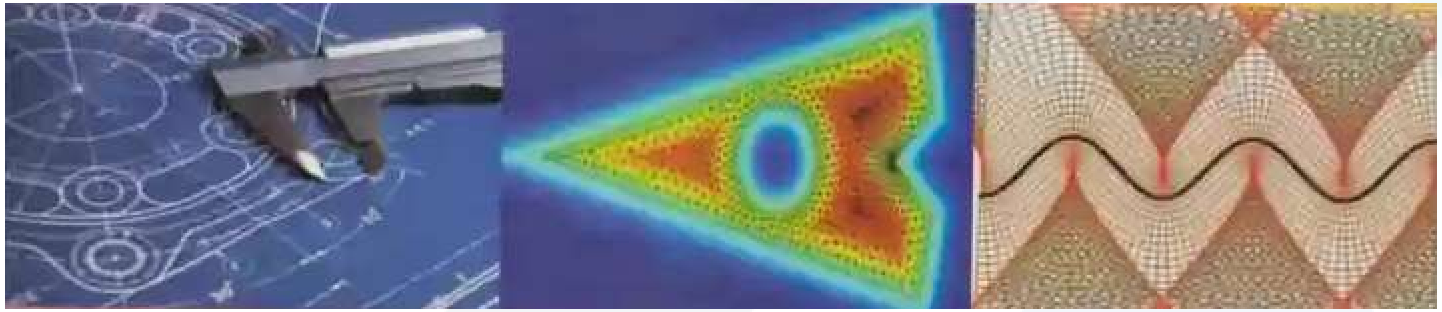
CEDC301: Engineering Mathematics

Lecture Notes 4: Series and Residues: Part A



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Chapter 3

Series and Residues

1. Sequences and Series

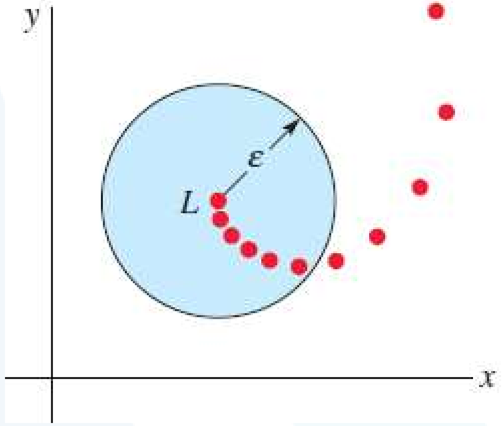
Sequences

- A sequence $\{z_n\}$ is a function whose domain is the set of positive integers; in other words, to each integer $n = 1, 2, 3, \ldots$, we assign a complex number z_n. For example, the sequence $\{1 + i^n\}$ is

$$\begin{array}{ccccc} 1+i, & 0, & 1-i, & 2, & 1+i, \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ n=1, & n=2, & n=3, & n=4, & n=5, \ldots \end{array}$$

- If $\lim_{n\to\infty} z_n = L$ we say the sequence $\{z_n\}$ is convergent.

$\{z_n\}$ converges to the number L if, for each positive number ε, an N can be found such that $|z_n - L| < \varepsilon$ whenever $n > N$.

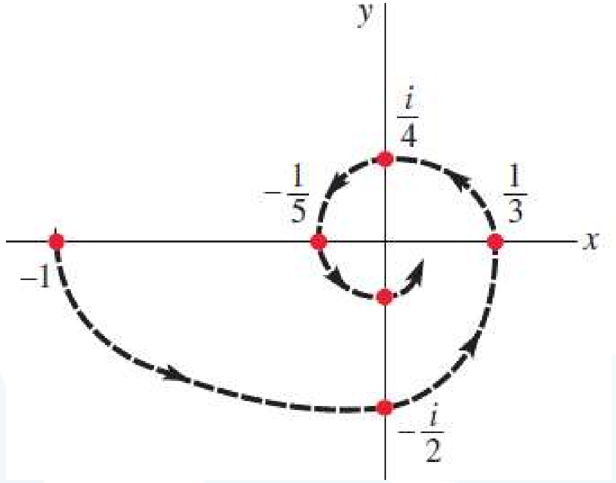


The sequence $\{1 + i^n\}$ is divergent.

- Example 1: A Convergent Sequence

  The sequence $\left\{\dfrac{i^{n+1}}{n}\right\}$ converges, since $\lim\limits_{n\to\infty} \dfrac{i^{n+1}}{n} = 0$

$$-1,\ -\frac{i}{2},\ \frac{1}{3},\ \frac{i}{4},\ -\frac{1}{5},\ \ldots$$



- Theorem 1 (Criterion for Convergence): A sequence $\{z_n\}$ converges to a complex number L if and only if $Re(z_n)$ converges to $Re(L)$ and $Im(z_n)$ converges to $Im(L)$.

- Example 2: The sequence $\left\{\dfrac{ni}{n+2i}\right\}$ converges to i. since

  $Re(z_n) = 2n/(n^2+4) \to 0$ and $Im(z_n) = n^2/(n^2+4) \to 1$ as $n \to \infty$

Series

- An infinite series of complex numbers

$$\sum_{k=1}^{\infty} z_k = z_1 + z_2 + z_3 + \cdots + z_n + \cdots$$

is convergent if the sequence of partial sums $\{S_n\}$, where

$$S_n = z_1 + z_2 + z_3 + \cdots + z_n$$

converges. If $S_n \to L$ as $n \to \infty$, we say that the sum of the series is L.

Geometric Series

$$\sum_{k=0}^{\infty} az^k = a + az + az^2 + \cdots + az^{n-1} + \cdots$$

$$S_n = a + az + az^2 + \cdots + az^{n-1} = \frac{a(1-z^n)}{1-z} \underset{n\to\infty}{\longrightarrow} \frac{a}{1-z} \text{ when } |z| < 1$$

$\sum_{k=1}^{\infty} az^k$ converges when $|z| < 1$, and diverges when $|z| > 1$.

$1/(1-z) = 1 + z + z^2 + z^3 + \cdots,$ $\quad 1/(1+z) = 1 - z + z^2 - z^3 + \cdots$ valid for $|z| < 1$

$(1-z^n)/(1-z) = 1 + z + z^2 + z^3 + \cdots + z^{n-1}$

- Example 3: Convergent Geometric Series

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$$

is a geometric series with $a = (1 + 2i)/5$ and $z = (1 + 2i)/5$.

$|z| = \sqrt{5}/5 < 1 \Rightarrow$ the series converges $\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{\frac{1+2i}{5}}{1 - \frac{1+2i}{5}} = \frac{i}{2}$

- Theorem 2 (Necessary Condition for Convergence): If $\sum_{k=1}^{\infty} z_k$ converges, then:

$$\lim_{n\to\infty} z_n = 0$$

- Theorem 3 (The nth Term Test for Divergence): If $\lim_{n\to\infty} z_n \neq 0$, then the series: $\sum_{k=1}^{\infty} z_k$ diverges.

  For example, the series $\sum_{k=1}^{\infty} \frac{k+5i}{k}$ diverges since $z_n = (n+5i)/n \to 1$ as $n \to \infty$

- Definition: An infinite series $\sum_{k=1}^{\infty} z_k$ is absolutely convergent if $\sum_{k=1}^{\infty} |z_k|$ converges.

  For example, the series $\sum_{k=1}^{\infty} (i^k)/k^2$ is absolutely convergent $\left|(i^k)/k^2\right| = 1/k^2$

- Note: Absolute convergence implies convergence.

$\sum_{k=1}^{\infty} (i^k)/k^2$ is convergent

- Theorem 4 (Ratio Test): Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = L$$

(i) If $L < 1$, then the series converges absolutely.
(ii) If $L > 1$ or $L = \infty$, then the series diverges.
(iii) If $L = 1$, the test is inconclusive.

- Theorem 5 (Root Test): Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that:

$$\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$$

(i) If $L < 1$, then the series converges absolutely.

(ii) If $L > 1$ or $L = \infty$, then the series diverges.

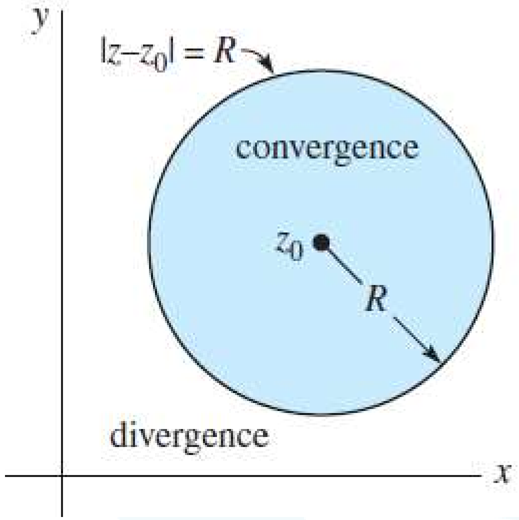
(iii) If $L = 1$, the test is inconclusive.

Power Series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$$

where the coefficients a_k are complex constants, is called a power series in $z - z_0$, centered at z_0,

Circle of Convergence



- Every complex power series has radius of convergence R, where R is a real number.
- When $0 < R < \infty$, a complex power series has a circle of convergence defined by $|z - z_0| = R$.
- The power series converges absolutely for all z satisfying $|z - z_0| < R$ and diverges for $|z - z_0| > R$. The radius R of convergence can be:

  (i) zero (the power series converges at only $z = z_0$),

  (ii) a finite number (the power series converges at all interior points of the circle $|z - z_0| = R$), or

  (iii) ∞ (the power series converges for all z).

- Example 4: Circle of Convergence

Consider the power series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$. By the ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} |z| = |z|$$

Thus the series converges absolutely for $|z| < 1$. The circle of convergence is $|z| = 1$ and the radius of convergence is $R = 1$.

On the circle of convergence, the series does not converge absolutely.

It can be shown that the series converges at all points on the circle $|z| = 1$ except at $z = 1$.

- Note: the radius of convergence is $R = 1/L$. $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ or $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

- **Example 5:** Radius of Convergence

Consider the power series $\sum_{k=1}^{\infty}\left(\frac{6k+1}{2k+5}\right)^{k}(z-2i)^{k}$

$$\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\frac{6n+1}{2n+5}=3$$

The radius of convergence of the series is $R = 1/3$. The circle of convergence is $|z-2i| = 1/3$, the series converges absolutely for $|z-2i| < 1/3$.

2. Taylor Series

- A power series defines or represents a function f; for a specified z within the circle of convergence, the number L to which the power series converges is defined to be the value of f at z; that is, $f(z) = L$.

- **Theorem 6 (Continuity):** A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ represents a continuous function f within its circle of convergence $|z - z_0| = R$, $R \neq 0$.
- **Theorem 7 (Term-by-Term Integration):** A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be integrated term by term within its circle of convergence $|z - z_0| = R$, $R \neq 0$, for every contour C lying entirely within the circle of convergence.
- **Theorem 8 (Term-by-Term Differentiation):** A power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ can be differentiated term by term within its circle of convergence $|z - z_0| = R$, $R \neq 0$.

Taylor Series

- A power series represents an analytic function within its circle of convergence.

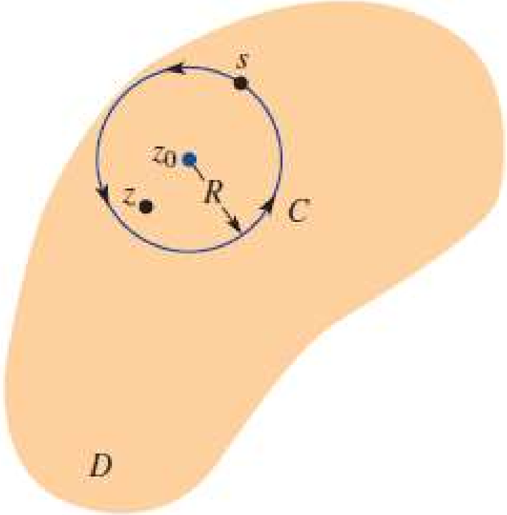
$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$ Taylor series for f centered at z_0.

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}z^k$$ Maclaurin series for f.

- Theorem 9 (Taylor's Theorem): Let f be analytic within a domain D and let z_0 be a point in D. Then f has the series representation

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

valid for the largest circle C with center at z_0 and radius R that lies entirely within D.



$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

are valid for all z

- Note: the radius of convergence R is the distance from the center z_0 of the series to the nearest isolated singularity of f. An isolated singularity is a point at which f fails to be analytic but is, nonetheless, analytic at all other points throughout some neighborhood of the point.

- Example 6: Radius of Convergence

Suppose the function $f(z)=\dfrac{3-i}{1-i+z}$ is expanded in a Taylor series with center $z_0=4-2i$. What is its radius of convergence R?

The function is analytic at every point except at $z=-1+i$, which is an isolated singularity of f. The distance from $z=-1+i$ to $z_0=4-2i$ is:

$$\left|z-z_0\right|=\sqrt{(-1-4)^2+(1-(-2))^2}=\sqrt{34}=R$$

- Example 7: Maclaurin Series

Find the Maclaurin expansion of $f(z)=\dfrac{1}{(1-z)^2}$

$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots,\quad \left|z\right|<1$$

Differentiating both sides

$$\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots = \sum_{k=1}^{\infty} kz^{k-1}, \quad |z| < 1$$

- Example 8: Taylor Series

Expand $f(z) = \dfrac{1}{1-z}$ in a Taylor series with center $z_0 = 2i$.

First Method:

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}} \Rightarrow f^{(n)}(2i) = \frac{n!}{(1-2i)^{n+1}}$$

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} \frac{1}{(1-2i)^{n+1}}(z-2i)^k$$

circle of convergence $|z-2i| = \sqrt{5}$

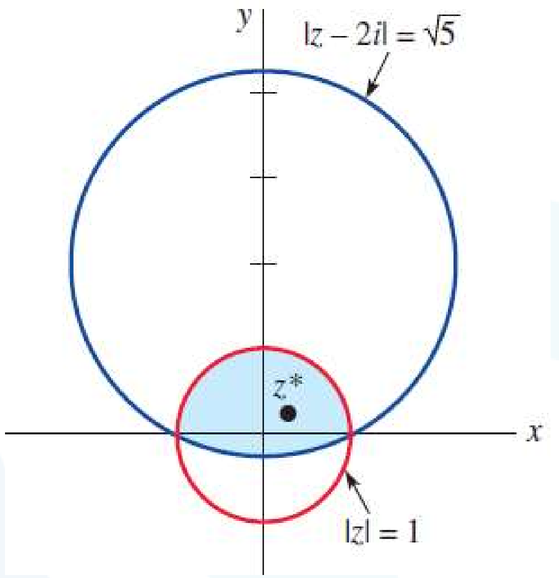
(using ratio test)

Second Method:

$$\frac{1}{1-z}=\frac{1}{1-z+2i-2i}=\frac{1}{1-2i-(z-2i)}=\frac{1}{1-2i}\frac{1}{1-\dfrac{z-2i}{1-2i}}$$

$$\frac{1}{1-z}=\frac{1}{1-2i}\left[1+\frac{z-2i}{1-2i}+\left(\frac{z-2i}{1-2i}\right)^2+\left(\frac{z-2i}{1-2i}\right)^3+\cdots\right]$$

$$\frac{1}{1-z}=\frac{1}{1-2i}+\frac{1}{(1-2i)^2}(z-2i)+\frac{1}{(1-2i)^3}(z-2i)^2+\cdots$$



- Note: we represented the same function $1/(1-z)$ by 2 different power series. The first has center 0 and radius of convergence (ROC) 1. The second has center $2i$ and ROC $\sqrt{5}$. The shaded region is where both series converge.

3. Laurent Series

- If a complex function f fails to be analytic at a point $z = z_0$, then this point is said to be a singularity or a singular point of the function.
- For example, the complex numbers $z = 2i$ and $z = -2i$ are singularities of the function $f(z) = z/(z^2 + 4)$ because f is discontinuous at each of these points.

Isolated Singularities

- Suppose that $z = z_0$ is a singularity of a complex function f. The point $z = z_0$ is said to be an isolated singularity of the function f if there exists some deleted neighborhood of z_0, $0 < |z - z_0| < R$ throughout which f is analytic.
- For example, $z = \pm 2i$ are isolated singularities of $f(z) = z/(z^2 + 4)$ since f is analytic at every point in the neighborhood $|z - 2i| < 1$ except at $z = 2i$ and at every point in the neighborhood $|z - (-2i)| < 1$ except at $z = -2i$.

- On the other hand, the branch point $z = 0$ is not an isolated singularity of $\mathrm{Log}\ z$ since every neighborhood of $z = 0$ must contain points on the negative x-axis.
- We say that a singular point $z = z_0$ of a function f is nonisolated if every neighborhood of z_0 contains at least one singularity of f other than z_0.
- For example, the branch point $z = 0$ is a nonisolated singularity of $\mathrm{Log}\ z$ since every neighborhood of $z = 0$ contains points on the negative real axis.

A New Kind of Series

- If $z = z_0$ is a singularity of a function f, then certainly f cannot be expanded in a power series with z_0 as its center. However, about an isolated singularity $z = z_0$ it is possible to represent f by a new kind of series:

$$f(z)=\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

Such series representation is called a Laurent series or a Laurent expansion of f.

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z-z_0)^k}$$

is called the principal part and will converge for $|1/(z-z_0)| < r^*$ or equivalently for $|z-z_0| > 1/r^* = r$.

$\sum_{k=0}^{\infty} a_k (z-z_0)^k$ is called the analytic part and will converge for $|z-z_0| < R$.

$f(z)=\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ will converge for $r < |z-z_0| < R$

- Example 9: A New Kind of Series

The function $f(z) = (\sin z)/z^4$ is not analytic at $z = 0$ and hence cannot be expanded in a Maclaurin series.

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

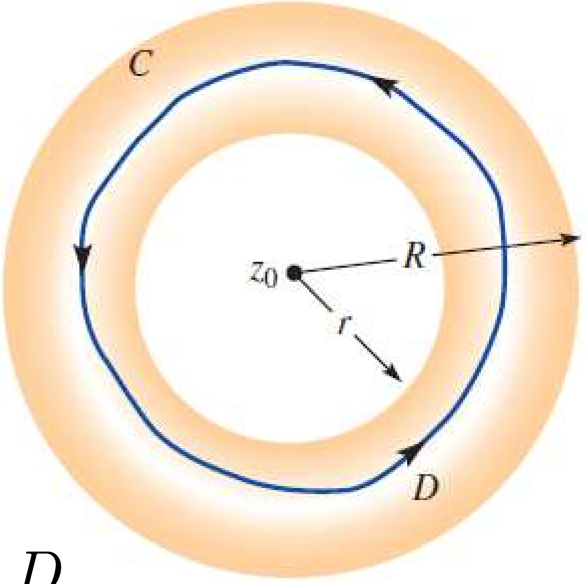
converges for all $|z| < \infty$

$$f(z) = \frac{\sin z}{z^4} = \overbrace{\frac{1}{z^3} - \frac{1}{3!z}}^{\text{principal part}} + \overbrace{\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \cdots}^{\text{analytic part}}$$

The analytic part of the series converges for $|z| < \infty$. The principal part is valid for $|z| > 0 \Rightarrow$ the series converges for all z except at $z = 0$ $(0 < |z| < \infty)$.

- Theorem 10 (Laurent's Theorem): Let f be analytic within the annular domain D defined by $r < |z - z_0| < R$. and let z_0 be a point in D. Then f has the series representation:

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$



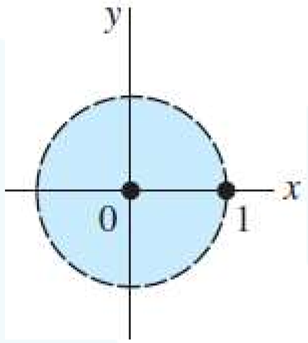
valid for $r < |z - z_0| < R$. The coefficients a_k are given by:

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}}\, ds, \quad k = 0, \pm 1, \pm 2, \ldots$$

where C is a simple closed curve that lies entirely within D and has z_0 in its interior.

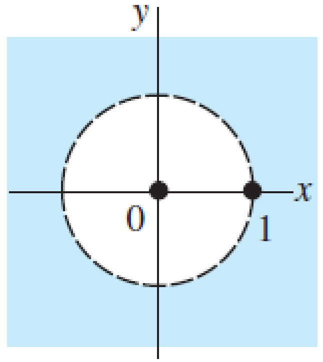
- Example 10: Laurent Expansions

Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for (a) $0 < |z| < 1$, (b) $1 < |z|$, (c) $0 < |z-1| < 1$, and (d) $1 < |z-1|$.



(a) $$f(z) = -\frac{1}{z}\frac{1}{1-z} = -\frac{1}{z}\underbrace{[1 + z + z^2 + z^3 + \cdots]}_{|z|<1}$$

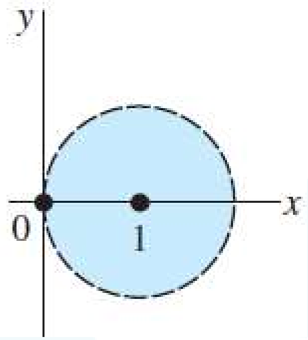
$$= -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$ converges for $0 < |z| < 1$



(b) $$f(z) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}} = \frac{1}{z^2}\underbrace{\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]}_{\left|\frac{1}{z}\right|<1}$$

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots$$ converges for $1 < |z|$

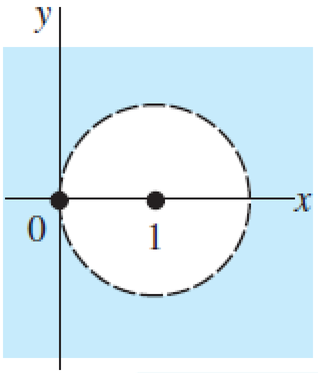
(c) $$f(z) = \frac{1}{1-1+z}\,\frac{1}{z-1} = \frac{1}{z-1}\,\frac{1}{1+(z-1)}$$



$$= \frac{1}{z-1}\underbrace{[1-(z-1)+(z-1)^2-(z-1)^3+\cdots]}_{|z-1|<1}$$

$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots$$ converges for $0 < |z-1| < 1$

(d) $$f(z) = \frac{1}{z-1}\,\frac{1}{1+(z-1)} = \frac{1}{(z-1)^2}\,\frac{1}{1+\dfrac{1}{z-1}}$$



$$f(z)=\frac{1}{(z-1)^2}\underbrace{\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right]}_{\left|\frac{1}{z-1}\right|<1}$$

$$f(z)=\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\frac{1}{(z-1)^5}+\cdots$$ converges for $1<|z-1|$

- Example 11: Laurent Expansions

Expand $f(z)=\dfrac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for (a) $0<|z-1|<2$, (b) $0<|z-3|<2$.

(a) $f(z) = \dfrac{1}{(z-1)^2(z-3)} = \dfrac{1}{(z-1)^2}\dfrac{1}{-2+(z-1)} = \dfrac{-1}{2(z-1)^2}\dfrac{1}{1-\dfrac{z-1}{2}}$

$$f(z) = \frac{-1}{2(z-1)^2}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right]$$

$$= -\frac{1}{2(z-1)^2}-\frac{1}{4(z-1)}-\frac{1}{8}-\frac{1}{16}(z-1)-\cdots$$ valid for $0<|z-1|<2$

(b) $f(z) = \dfrac{1}{(z-1)^2(z-3)} = \dfrac{1}{z-3}\dfrac{1}{[2+(z-3)]^2} = \dfrac{1}{4(z-3)}\left[1+\dfrac{z-3}{2}\right]^{-2}$

using the general binomial theorem:

$$(1+z)^m = 1+mz+\frac{m(m-1)}{2!}z^2+\frac{m(m-1)(m-2)}{3!}z^3+\cdots, \quad |z|<1,\ m\in Q$$

$$f(z)=\frac{1}{4(z-3)}\left[1+\frac{(-2)}{1!}\left(\frac{z-3}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2+\cdots\right]$$

$$f(z)=\frac{1}{4(z-3)}-\frac{1}{4}+\frac{3}{16}(z-3)-\frac{1}{8}(z-3)^2+\cdots$$ valid for $0<|z-3|<2$

- Example 12: Laurent Expansions

Expand $f(z)=\dfrac{8z+1}{z(1-z)}$ in a Laurent series valid for $0<|z|<1$.

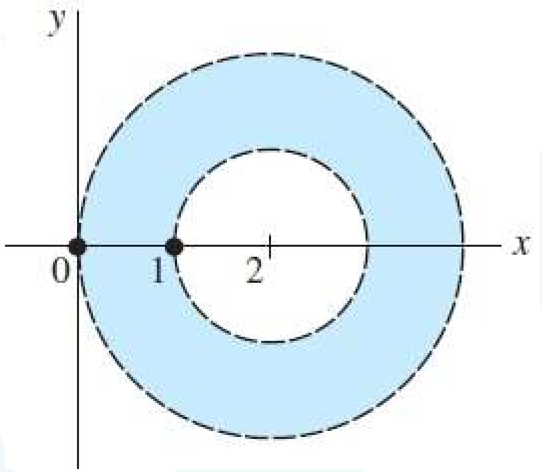
$$f(z)=\frac{8z+1}{z(1-z)}=\frac{8z+1}{z}\frac{1}{1-z}=\left(8+\frac{1}{z}\right)(1+z+z^2+z^3+\cdots)$$

$$f(z)=\frac{1}{z}+9+9z+9z^2+\cdots$$ valid for $0<|z|<1$

- Example 13: Laurent Expansions

Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$.

Find two series involving integer powers of $z - 2$: one converging for $1 < |z - 2|$ and the other converging for $|z - 2| > 2$.



$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z)$$

$$f_1(z) = -\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2}\frac{1}{1+\dfrac{z-2}{2}} = -\frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \cdots\right]$$

$$f_1(z) = -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$$ converges for $|z-2| < 2$

$$f_2(z) = \frac{1}{1+z-2} = \frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}} = \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right]$$

$$f_2(z) = \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots$$ converges for $1 < |z-2|$

$$f(z) = \cdots + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2} - \frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots$$

converges for $1 < |z-2| < 2$

- Example 14: Laurent Expansions

Expand $f(z) = e^{3/z}$ in a Laurent series valid for $|z| > 0$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots$$ valid for $|z| > 0$