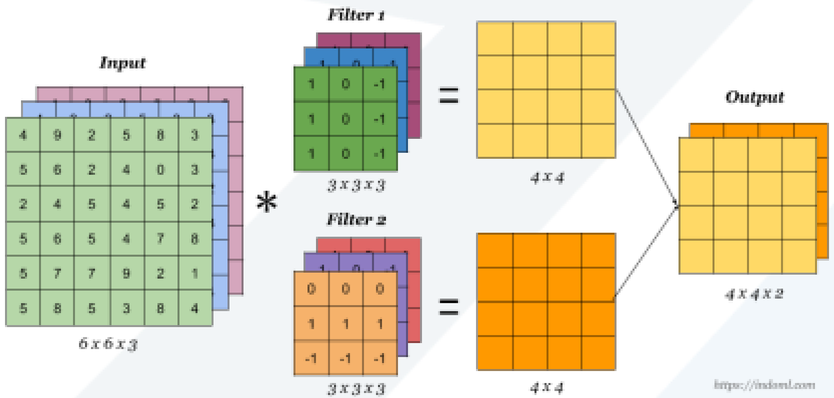
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 1: Systems of Linear Equations



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Chapter 1

Systems of Linear Equations

1.1 Introduction to Systems of Linear Equations

- A linear equation in n variables: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

  $a_1, a_2, \ldots, a_n, b$: real numbers

  a_1: leading coefficient

  x_1: leading variable

- Notes:

(1) Linear equations have no products or roots of variables and no variables involved in trigonometric, exponential, or logarithmic functions.

(2) Variables appear only to the first power.

- Ex 1: (Linear or Nonlinear)

Linear $(a)\ 3x + 2y = 7$

$(b)\ \frac{1}{2}x + y - \pi z = \sqrt{2}$ Linear

Linear $(c)\ x_1 - 2x_2 + 10x_3 + x_4 = 0$

$(d)\ (\sin\frac{\pi}{2})x_1 - 4x_2 = e^2$ Linear

NonLinear $(e)\ xy + z = 2$

Products

Exponential

$(f)\ e^x - 2y = 4$ NonLinear

NonLinear $(g)\ \sin x_1 + 2x_2 - 3x_3 = 0$

Trigonometric functions

$(h)\ \frac{1}{x} + \frac{1}{y} = 4$ NonLinear

Not the first power

- A solution of a linear equation in n variables:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$x_1 = s_1,\ x_2 = s_2, \ldots, x_n = s_n$ such that: $a_1s_1 + a_2s_2 + \cdots + a_ns_n = b$

- Solution set: the set of all solutions of a linear equation

- Ex 2: (Parametric representation of a solution set)

$x_1 + 2x_2 = 4$ (2, 1) is a solution, i.e. $x_1 = 2,\ x_2 = 1$

If you solve for x_1 in terms of x_2, you obtain $x_1 = 4 - 2x_2$

By letting $x_2 = t$ you can represent the solution set as $x_1 = 4 - 2t$

And the solutions are $\{(4 - 2t,\ t) | t \in R\}$ or $\{(s,\ 2 - ½s) | s \in R\}$

In vector form: $(x_1,\ x_2) = (4, 0) + t\,(-2, 1) = (0, 2) + s\,(1, -½)$

- A system of m linear equations in n variables:

$$\begin{array}{cccccccc} a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2n}x_n & = & b_2 \\ & \vdots & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + \cdots + & a_{mn}x_n & = & b_m \end{array}$$

- Consistent: A system of linear equations has <u>at least one solution</u>.
- Inconsistent: A system of linear equations has <u>no solution</u>.
- Notes: Every system of linear equations has either

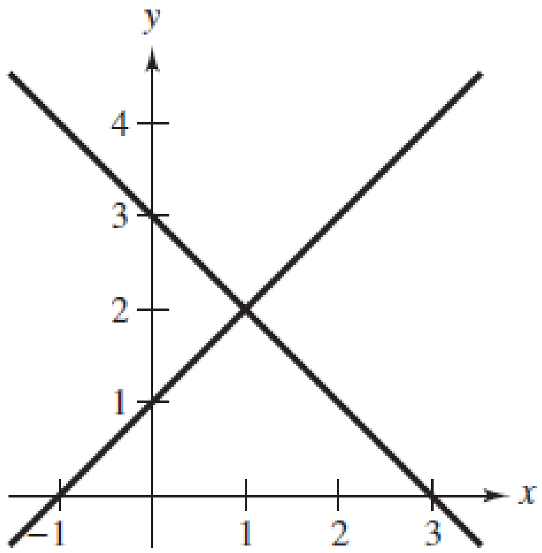
  (1) exactly one solution,

  (2) infinitely many solutions, or

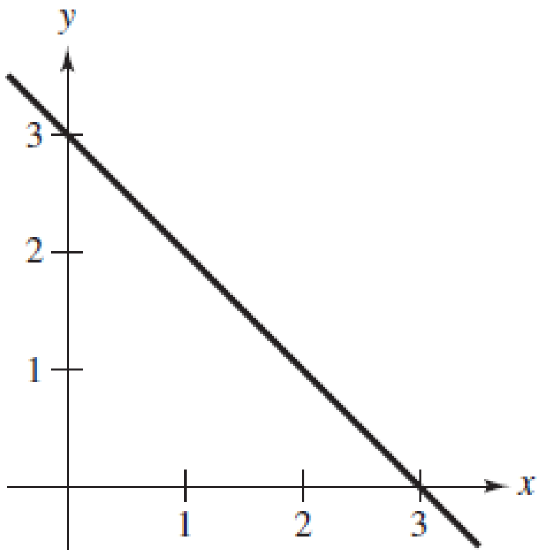
  (3) no solution.

- Ex 3: (Solution of a system of linear equations)

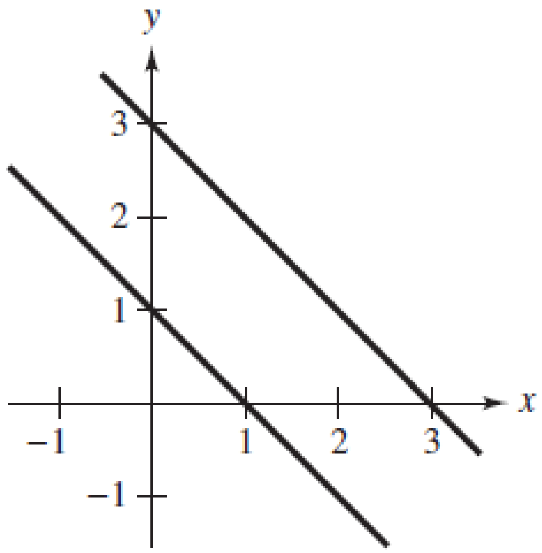
two intersecting lines	two coincident lines	two parallel lines
$\begin{array}{rcrcr} x & + & y & = & 3 \\ x & - & y & = & -1 \end{array}$	$\begin{array}{rcrcr} x & + & y & = & 3 \\ 2x & + & 2y & = & 6 \end{array}$	$\begin{array}{rcrcr} x & + & y & = & 3 \\ x & + & y & = & 1 \end{array}$



exactly one solution



infinite number



no solution

- Ex 4: (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y &= 5 \quad (1) \\ y &= -2 \quad (2) \end{aligned}$$

Sol: By substituting $y = -2$ into (1), you obtain

$$\begin{aligned} x - 2(-2) &= 5 \\ x &= 1 \end{aligned}$$

The system has exactly one solution: $x = 1,\ y = -2$

- Ex 5: (Using back substitution to solve a system in row echelon form)

$$\begin{aligned} x - 2y + 3z &= 9 \quad (1) \\ y + 3z &= 5 \quad (2) \\ z &= 2 \quad (3) \end{aligned}$$

Sol: Substitute $z = 2$ into (2)

$$\begin{aligned} y + 3(2) &= 5 \\ y &= -1 \end{aligned}$$

and substitute $y = -1$ and $z = 2$ into (1)

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ x &= 1 \end{aligned}$$

The system has exactly one solution: $x = 1,\ y = -1,\ z = 2$

- Equivalent:

Two systems of linear equations are called equivalent if they have precisely the same solution set

- Notes: Each of the following operations on a system of linear equations produces an equivalent system.

  (1) Interchange two equations.

  (2) Multiply an equation by a nonzero constant.

  (3) Add a multiple of an equation to another equation.

- Ex 6: Solve a system of linear equations (consistent system)

$$\begin{array}{rcrcrcr}
x & - & 2y & + & 3z & = & 9 \\
-x & + & 3y & & & = & -4 \\
2x & - & 5y & + & 5z & = & 17
\end{array}
\qquad
\begin{array}{c}(1)\\(2)\\(3)\end{array}$$

Sol:

$(1)+(2)\rightarrow(2)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ 2x & - & 5y & + & 5z & = & 17 \end{array} \qquad (4)$$

$(1)\times(-2)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & -y & - & z & = & -1 \end{array} \qquad (5)$$

$(4)+(5)\rightarrow(5)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \end{array} \qquad (6)$$

$(6)\times\frac{1}{2}\rightarrow(6)$

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$$

So the solution is: $x=1,\ y=-1,\ z=2$

- Ex 7: Solve a system of linear equations (inconsistent system)

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ 2x_1 & - & x_2 & - & 2x_3 & = & 2 \\ x_1 & + & 2x_2 & - & 3x_3 & = & -1 \end{array} \qquad \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Sol: $(1)\times(-2)+(2)\rightarrow(2)$ $\quad$ $(1)\times(-1)+(3)\rightarrow(3)$

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & x_3 & = & 1 \\ & & 5x_2 & - & 4x_3 & = & 0 \\ & & 5x_2 & - & 4x_3 & = & -2 \end{array} \qquad \begin{array}{c} \\ (4) \\ (5) \end{array}$$

$$(4)\times(-1)+(5)\rightarrow(5)$$

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 1 \\ 5x_2 - 4x_3 &= 0 \\ \boxed{0 = -2} \end{aligned}$$

(a false statement)

- Ex 8: Solve a system of linear equations (infinitely many solutions)

$$\begin{aligned} x_2 - x_3 &= 0 && (1) \\ x_1 - 3x_3 &= -1 && (2) \\ -x_1 + 3x_2 &= 1 && (3) \end{aligned}$$

Sol: $(1)\leftrightarrow(2)$

$$\begin{aligned} x_1 - 3x_3 &= -1 && (1) \\ x_2 - x_3 &= 0 && (2) \\ -x_1 + 3x_2 &= 1 && (3) \end{aligned}$$

$(1)+(3) \to (3)$

$$\begin{array}{rcrcr} x_1 & & -\ 3x_3 & = & -1 \\ & x_2 & -\ x_3 & = & 0 \\ & 3x_2 & -\ 3x_3 & = & 0 \end{array} \qquad (4)$$

$(2)\times(-3)+(4) \to (4)$

$$\begin{array}{rcrcr} x_1 & & -\ 3x_3 & = & -1 \\ & x_2 & -\ x_3 & = & 0 \\ & & \boxed{0 = 0} & & \end{array}$$

(a True statement)

$$\Rightarrow x_2 = x_3, \quad x_1 = -1 + 3x_3$$

letting $x_3 = t$, then the solutions are: $\{(3t-1,\ t,\ t) \mid t \in R\}$

1.2 Gaussian Elimination and Gauss-Jordan Elimination

- $m \times n$ matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad m \text{ rows}$$

$$n \text{ columns}$$

- Notes:

(1) Every entry a_{ij} in a matrix is a number.

(2) A matrix with $\underline{m \text{ rows}}$ and $\underline{n \text{ columns}}$ is said to be of size $m \times n$.

(3) If $m = n$, then the matrix is called square of order n.

(4) For a square matrix, $a_{11}, a_{22}, \ldots, a_{nn}$ are called the main diagonal entries.

- Ex 1: Matrix Size

Matrix	Size
$\begin{bmatrix} 2 \end{bmatrix}$	1×1
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	2×2
$\begin{bmatrix} 1 & -3 & 0 & \frac{1}{2} \end{bmatrix}$	1×4
$\begin{bmatrix} e & \pi \\ 2 & \sqrt{2} \\ -7 & 4 \end{bmatrix}$	3×2

- Note: One very common use of matrices is to represent a system of linear equations.

- A system of m equations in n variables:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Matrix form: $A\boldsymbol{x} = \boldsymbol{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Augmented matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vline & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vline & b_2 \\ & \vdots & & & \vline & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vline & b_m \end{bmatrix} = [\,A \,|\, \boldsymbol{b}\,]$$

- Coefficient matrix:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = A$$

- Elementary row operation:
  - (1) Interchange two rows. $r_{ij}: R_i \leftrightarrow R_j$
  - (2) Multiply a row by a nonzero constant. $r_i^{(k)}: (k)R_i \to R_i$
  - (3) Add a multiple of a row to another row. $r_{ij}^{(k)}: (k)R_i + R_j \to R_j$

- Row equivalent: Two matrices are said to be row equivalent if one can be obtained from the other by a finite sequence of elementary row operation.

- Ex 2: (Elementary row operation)

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix} \xrightarrow{r_{12}} \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})}} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{r_{13}^{(-2)}} \begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

- Ex 3: Using elementary row operations to solve a system

Linear System	Augmented Matrix	Elementary Row Operation
$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ 2x - 5y + 5z &= 17 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$	$r_{12}^{(1)}: (1)R_1 + R_2 \rightarrow R_2$
$\begin{aligned} x - 2y + 3z &= 9 \\ y + 3z &= 5 \\ -y - z &= -1 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$	$r_{13}^{(-2)}: (-2)R_1 + R_3 \rightarrow R_3$

Linear System	Associated Augmented Matrix	Elementary Row Operation
$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & 2z & = & 4 \end{array}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$	$r_{23}^{(1)}: (1)R_2 + R_3 \to R_3$
$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 9 \\ & & y & + & 3z & = & 5 \\ & & & & z & = & 2 \end{array}$	$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$	$r_3^{(\frac{1}{2})}: (\frac{1}{2})R_3 \to R_3$

$$\longrightarrow \quad \begin{array}{rcr} x & = & 1 \\ y & = & -1 \\ z & = & 2 \end{array}$$

- Row-echelon form: (1, 2, 3)
- Reduced row-echelon form: (1, 2, 3, 4)

(1) All row consisting entirely of zeros occur at the bottom of the matrix.

(2) For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a leading 1).

(3) For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

(4) Every column that has a leading 1 has zeros in every position above and below its leading 1.

- Notes:

(1) Each column of the coefficient matrix corresponds to a variable in the system of equations, we call each variable associated to a leading 1 in the RREF a leading variable.

(2) A variable (if any) that is not a leading variable is called a free variable.

- Ex 4:

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ & & 2x_2 & - & 4x_3 & + & x_4 & = & 5 \\ x_1 & - & 2x_2 & + & 2x_3 & - & 3x_4 & = & 4 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_3 are leading variables and x_2, x_4 are free variables

- Ex 5: (Row-echelon form or reduced row-echelon form)

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$ row-echelon form

$$\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row-echelon form

$$\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ row-echelon form

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ reduced row-echelon form

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$$

- Gaussian elimination:

  The procedure for reducing a matrix to a row-echelon form.

- Gauss-Jordan elimination:

  The procedure for reducing a matrix to a reduced row-echelon form.

- Notes:

  (1) Every matrix has an unique reduced row echelon form.

  (2) A row-echelon form of a given matrix is not unique.

  (Different sequences of row operations can produce different row-echelon forms.)

- Ex 6: Solve a system by Gauss-Jordan elimination method (one solution)

$$\begin{aligned} x &- 2y + 3z = 9 \\ -x &+ 3y = -4 \\ 2x &- 5y + 5z = 17 \end{aligned}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{r_{12}^{(1)},\ r_{13}^{(-2)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow{r_{23}^{(1)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow{r_3^{\left(\frac{1}{2}\right)}} \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{31}^{(-3)},\ r_{32}^{(-3)},\ r_{21}^{(2)}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{aligned} x &= 1 \\ y &= -1 \\ z &= 2 \end{aligned}$$

row-echelon form

reduced row-echelon form

- Ex 7: Solve a system by G.J. elimination method (infinitely many solutions)

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 &= 0 \\ 3x_1 + 5x_2 \qquad &= 1 \end{aligned}$$

Sol:

augmented matrix

$$\begin{bmatrix} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{r_1^{(\frac{1}{2})},\ r_{12}^{(-3)},\ r_2^{(-1)},\ r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

reduced row-echelon form

$$\longrightarrow \begin{aligned} x_1 \qquad + 5x_3 &= 2 \\ x_2 \quad -3x_3 &= -1 \end{aligned}$$

leading variables: x_1, x_2

free variable: x_3

$$\begin{aligned} x_1 &= 2 - 5x_3 \\ x_2 &= -1 + 3x_3 \end{aligned}$$

$x_3 = t$, then the solutions are: $\{(2 - 5t, -1 + 3t, t) | t \in R\}$

So the system has infinitely many solutions.

- Ex 8: Solve a system by Gauss-Jordan elimination method (no solution)

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 2x_3 & = & 4 \\ x_1 & & & + & x_3 & = & 6 \\ 2x_1 & - & 3x_2 & + & 5x_3 & = & 4 \\ 3x_1 & + & 2x_2 & - & x_3 & = & 1 \end{array}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \xrightarrow{r_{12}^{(-1)},\ r_{13}^{(-2)},\ r_{14}^{(-3)},\ r_{23}^{(1)}} \begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & -2 \\ 0 & 5 & -7 & -11 \end{bmatrix}$$

$$
\begin{aligned}
x_1 - x_2 + 2x_3 &= 4 \\
x_2 - x_3 &= 2 \\
0 &= -2 \\
5x_2 - 7x_3 &= -11
\end{aligned}
$$

Because the third equation is not possible, the system has no solution.

- **Homogeneous systems of linear equations:**

A system of linear equations is said to be homogeneous if all the constant terms are zero.

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\
 & \vdots & & & & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0
\end{array}
$$

- Trivial solution: $x_1 = x_2 = \cdots = x_n = 0$
- Nontrivial solution: other solutions

- Notes:

(1) Every homogeneous system of linear equations is consistent.

(2) If the homogenous system has fewer equations than variables, then it must have an infinite number of solutions.

(3) For a homogeneous system, exactly one of the following is true.

(a) The system has only the trivial solution.

(b) The system has infinitely many nontrivial solutions in addition to the trivial solution.

- Ex 9: Solve the following homogeneous system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 3x_3 & = & 0 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 0 \end{array}$$

Sol:

augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 2 & -1 & 3 & 0 \end{bmatrix} \xrightarrow{r_{12}^{(-2)},\ r_2^{(-\frac{1}{3})},\ r_{21}^{(-1)}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{array}{rcrcr} x_1 & & + \ 2x_3 & = & 0 \\ & x_2 & + \ \ x_3 & = & 0 \end{array}$$

reduced row-echelon form

leading variables: $x_1,\ x_2$

free variable: x_3

letting $x_3 = t$, then the solutions are:

$$\{(-2t,\ -t,\ t) | t \in R\}$$

When $t = 0$, $x_1 = x_2 = x_3 = 0$ (trivial solution)

- The Number of Solutions of a Homogeneous System

Every homogeneous system of linear equations is consistent. Moreover, if the system has fewer equations than variables, then it must have infinitely many solutions.

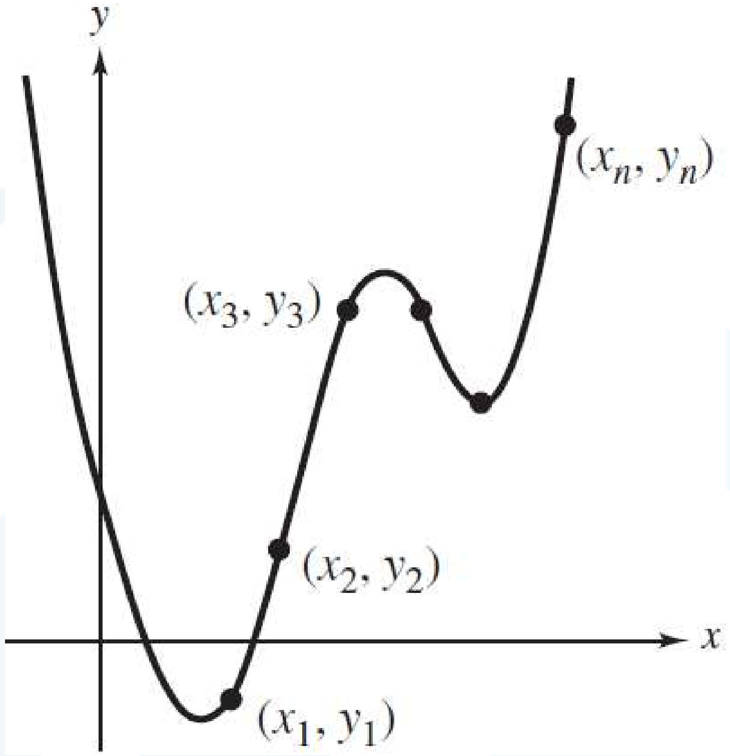
1.3 Applications of Systems of Linear Equations

- Polynomial Curve Fitting

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$$

$$\begin{aligned} a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_{n-1} x_1^{n-1} &= y_1 \\ a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_{n-1} x_2^{n-1} &= y_2 \\ &\vdots \\ a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_{n-1} x_n^{n-1} &= y_n \end{aligned}$$



- Ex 1: (Polynomial Curve Fitting)

Find a polynomial that fits the points: (–2, 3), (–1, 5), (0, 1), (1, 4), and (2, 10)

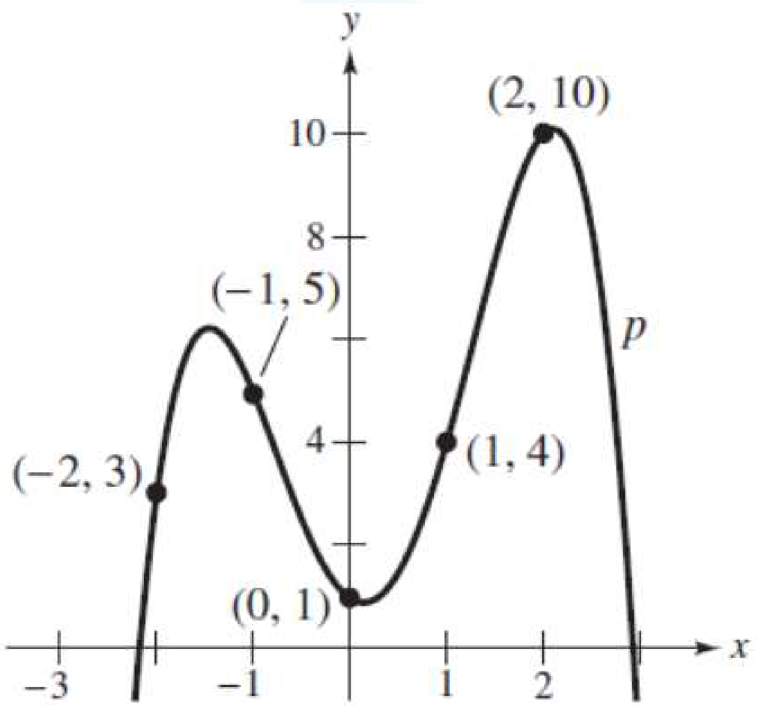
Sol:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\begin{aligned}
a_0 - 2a_1 + 4a_2 - 8a_3 + 16a_4 &= 3 \\
a_0 - a_1 + a_2 - a_3 + a_4 &= 5 \\
a_0 &= 1 \\
a_0 + a_1 + a_2 + a_3 + a_4 &= 4 \\
a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 &= 10
\end{aligned}$$

$$a_0 = 1,\ a_1 = -\tfrac{5}{4},\ a_2 = \tfrac{101}{24},\ a_3 = \tfrac{3}{4},\ a_4 = -\tfrac{17}{24}$$

$$\Rightarrow p(x) = 1 - \tfrac{5}{4}x + \tfrac{101}{24}x^2 + \tfrac{3}{4}x^3 - \tfrac{17}{24}x^4$$



- Analysis of an Electrical Network

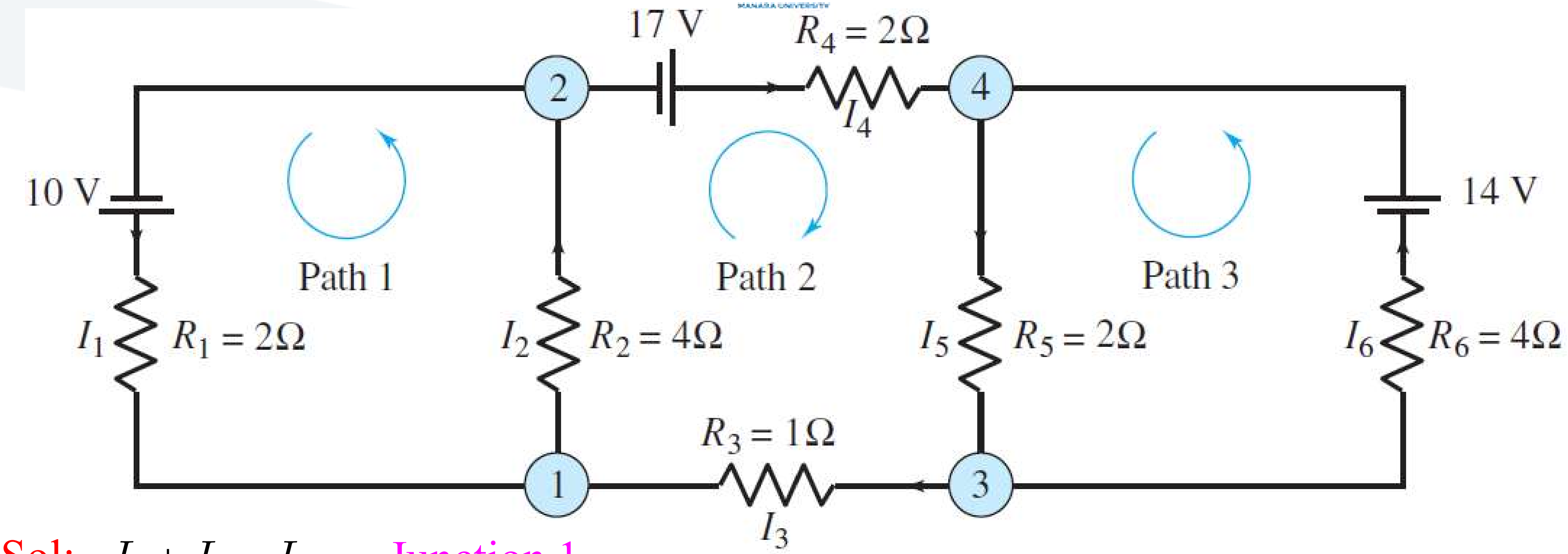
Ohm's Law: The current I and the voltage drop V across a resistance R are related by the equation $V = RI$

Kirchhoff's Laws:

1. (Junction Rule) The current flow into a junction equals the current flow out of that junction.
2. (Circuit Rule) The algebraic sum of the voltage drops (due to resistances) around any closed circuit of the network must equal the sum of the voltage increases around the circuit

- Ex 2: (Analysis of an Electrical Network)

Determine the currents I_1, I_2, I_3, I_4, I_5, and I_6 for the electrical network:



Sol:

$I_1 + I_3 = I_2$ Junction 1

$I_1 + I_4 = I_2$ Junction 2

$I_3 + I_6 = I_5$ Junction 3

$I_4 + I_6 = I_5$ Junction 4

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\begin{aligned}
2I_1 + 4I_2 \qquad\qquad\qquad\qquad &= 10 \quad \text{Path 1}\\
4I_2 + I_3 + 2I_4 + 2I_5 \qquad &= 17 \quad \text{Path 2}\\
2I_5 + 4I_6 &= 14 \quad \text{Path 3}
\end{aligned}
$$

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\begin{aligned}
I_1 - I_2 + I_3 &= 0 \\
I_1 - I_2 + I_4 &= 0 \\
I_3 - I_5 + I_6 &= 0 \\
I_4 - I_5 + I_6 &= 0 \\
2I_1 + 4I_2 &= 10 \\
4I_2 + I_3 + 2I_4 + 2I_5 &= 17 \\
2I_5 + 4I_6 &= 14
\end{aligned}
\quad \Rightarrow \quad
\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}
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