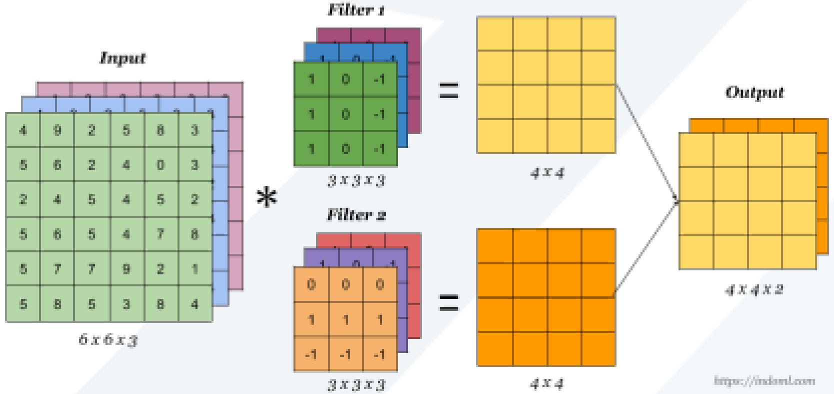
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 6: Vector Spaces: Part B



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4.5 Basis and Dimension

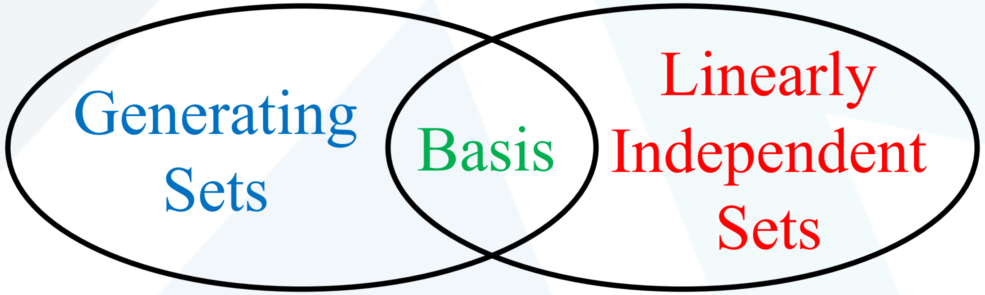
- Basis:

V: a vector space $\qquad S=\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}\subseteq V$

(*a*) S spans V (i.e., $\text{span}(S) = V$)

(*b*) S is linearly independent

$\Rightarrow$ S is called a basis for V



- Notes:

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \qquad \boldsymbol{i}=(1, 0, 0),\ \boldsymbol{j}=(0, 1, 0),\ \boldsymbol{k}=(0, 0, 1)$

(3) the standard basis for R^n:

$$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\} \qquad \mathbf{e}_1 = (1,0,\ldots,0),\ \mathbf{e}_2 = (0,1,\ldots,0),\ \mathbf{e}_n = (0,0,\ldots,1)$$

Ex: R^4 $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

- Finite dimensional:

A vector space V is called finite dimensional, if it has a basis consisting of a finite number of elements.

- Dimension:

The dimension of a finite dimensional vector space V is defined to be the number of vectors in a basis for V.

V: a vector space, S: a basis for V $\Rightarrow \dim(V) = \#(S)$ (the number of vectors in S)

- Notes:

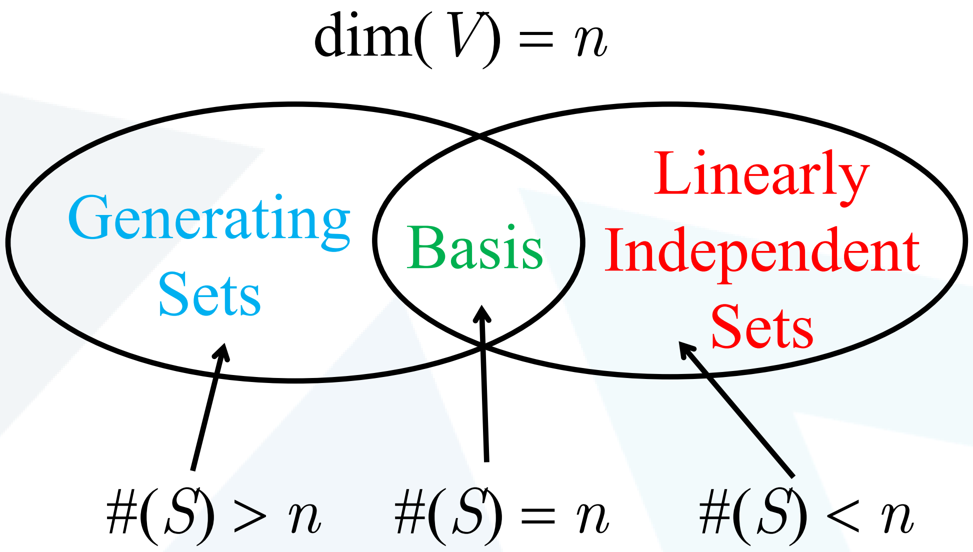
(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\emptyset)$

(2) $\dim(V) = n,\ S \subseteq V$

S: a L.I. set $\Rightarrow \#(S) \leq n$

S: a generating set $\Rightarrow \#(S) \geq n$

S: a basis $\Rightarrow \#(S) = n$



4.6 Rank and Nullity of a Matrix

- **Rank of a Matrix:**

  The **rank** of an mxn matrix A, denoted by rank(A), is the maximum number of linearly independent row vectors in A or the maximum number of linearly independent column vectors in A

- **Nullity of a Matrix:**

  The **nullity** of an mxn matrix A, denoted by nullity(A), is the dimension of the solution space of the linear system $A\boldsymbol{x} = \mathbf{0}$

- **Theorem 4.6:**

  If A is any matrix, then rank(A) = rank(A^T)

- Notes:
  - (1) The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix
  - (2) The number of leading 1's in the reduced row-echelon form of A is equal to the rank of A
  - (3) The number of free variables in the reduced row-echelon form of A is equal to the nullity of A

- Theorem 4.7: (Consistency of $A\boldsymbol{x} = \boldsymbol{b}$)

  If $\text{rank}([A|\boldsymbol{b}]) = \text{rank}(A)$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

- Note:

  A linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$ is consistent iff the rank of A is the same as the rank of the augmented matrix of the system $[A|\boldsymbol{b}]$

- Notes:

  (1) If $\text{rank}(A) = \text{rank}(A|\boldsymbol{b}) = n$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique sol.

  (2) If $\text{rank}(A) = \text{rank}(A|\boldsymbol{b}) < n$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has ∞-many sols.

  (3) If $\text{rank}(A) < \text{rank}(A|\boldsymbol{b})$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent.

- Ex 1: (Rank by Row Reduction)

$$A=\begin{bmatrix}1 & 1 & -1 & 3\\ 2 & -2 & 6 & 8\\ 3 & 5 & -7 & 8\end{bmatrix} \xrightarrow{\text{Gauss Elimination}} \begin{bmatrix}1 & 1 & -1 & 3\\ 0 & 1 & -2 & -\frac{1}{2}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\operatorname{rank}(A)=2$ (2 non-zero rows = 2 non-zero rows)

$\operatorname{nullity}(A)=2$ (2 free variables)

- Ex 2: (Finding the solution set of a nonhomogeneous system)

$$\begin{aligned} x_1 + x_2 - x_3 &= -1\\ x_1 \qquad + x_3 &= 3\\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

Sol:

$$A=\begin{bmatrix}1&1&-1\\1&0&1\\3&2&-1\end{bmatrix}\xrightarrow{\text{Gauss-Jordan Elimination}}\begin{bmatrix}1&0&1\\0&1&-2\\0&0&0\end{bmatrix}$$

$$[A\,\vdots\,\boldsymbol{b}]=\left[\begin{array}{ccc:c}1&1&-1&-1\\1&0&1&3\\3&2&-1&1\end{array}\right]\xrightarrow{\text{Gauss-Jordan Elimination}}\left[\begin{array}{ccc:c}1&0&1&3\\0&1&-2&-4\\0&0&0&0\end{array}\right]$$

$$\begin{aligned}x_1 \quad\quad + x_3 &= 3\\ x_2 - 2x_3 &= -4\end{aligned}\quad\Rightarrow\quad\begin{aligned}x_1 &= 3 - x_3\\ x_2 &= -4 + 2x_3\end{aligned}$$

letting $x_3 = t$, then the solutions are: $\{(3-t,\ -4+2t,\ t) | t \in R\}$

So the system has infinitely many solutions (consistent)

- Check: $\operatorname{rank}(A) = \operatorname{rank}([\,A\,\vdots\,\boldsymbol{b}\,]) = 2$

- Theorem 4.8 (Dimension Theorem for Matrices)

If A is a matrix with n columns, then $\text{rank}(A) + \text{nullity}(A) = n$

- Ex 3: (Rank and nullity of a matrix)

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 3$ (the number of nonzero rows in B)

$\text{nullity}(A) = n - \text{rank}(A) = 5 - 3 = 2$

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- Summary of equivalent conditions for square matrices:

If A is an nxn matrix, then the following conditions are equivalent:

(1) A is invertible

(2) $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for any $n\times 1$ matrix $\boldsymbol{b}$.

(3) $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution

(4) A is row-equivalent to I_n

(5) $|A| \neq 0$

(6) $\text{rank}(A) = n$

(7) The n row vectors of A are linearly independent.

(8) The n column vectors of A are linearly independent.

4.7 Coordinates and Change of Basis

- Coordinate representation relative to a basis

Let $B = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ be an ordered basis for a vector space V and let $\boldsymbol{x}$ be a vector in V such that $\boldsymbol{x} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n}$

The scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\boldsymbol{x}$ relative to the basis B. The coordinate matrix (or coordinate vector) of x relative to B is the column matrix in R^n whose components are the coordinates of $\boldsymbol{x}$.

$$[\boldsymbol{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- Ex 1: (Coordinates and components in R^n)

Find the coordinate matrix of $\boldsymbol{x} = (-2, 1, 3)$ in R^3 relative to the standard basis $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Sol:

$$\boldsymbol{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$$

$$[\boldsymbol{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

- Ex 2: (Finding a coordinate matrix relative to a nonstandard basis)

Find the coordinate matrix of $\boldsymbol{x} = (1, 2, -1)$ in R^3 relative to the (nonstandard) basis

$B' = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$

Sol:

$$\boldsymbol{x} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} \Rightarrow (1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$$

$$\Rightarrow \begin{array}{rcrcrcr} c_1 & & & + & 2c_3 & = & 1 \\ & & -c_2 & + & 3c_3 & = & 2 \\ c_1 & + & 2c_2 & - & 5c_3 & = & -1 \end{array} \quad \text{i.e.} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{rrrr} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{array}\right] \xrightarrow{\text{G. J. Elimination}} \left[\begin{array}{rrrr} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right] \Rightarrow [\boldsymbol{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$

Change of Basis In R^n

Change of basis: Given the coordinates of a vector relative to a basis B, find the coordinates relative to another basis B'.

In Ex 2, let B be the standard basis. Finding the coordinate matrix of $\boldsymbol{x} = (1, 2, -1)$ relative to the basis B' becomes solving for c_1, c_2, and c_3 in the matrix equation

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{[\boldsymbol{x}]_{B'}} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\boldsymbol{x}]_B}$$

P is the transition matrix from B' to B,

$P[\boldsymbol{x}]_{B'} = [\boldsymbol{x}]_B$ Change of basis from B' to B

$[\boldsymbol{x}]_{B'} = P^{-1}[\boldsymbol{x}]_B$ Change of basis from B to B'

$$\underbrace{\begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}}_{[\boldsymbol{x}]_B} = \underbrace{\begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}}_{[\boldsymbol{x}]_{B'}}$$

$$[\boldsymbol{x}]_{B'} = P^{-1}[\boldsymbol{x}]_B$$

Coordinate matrix of $\boldsymbol{x}$ relative to B'	Transition matrix from B to B'	Coordinate matrix of $\boldsymbol{x}$ relative to B

- Theorem 4.25: (The inverse of a transition matrix)

If P is the transition matrix from a basis B' to a basis B in R^n, then

(1) P is invertible

(2) The transition matrix from B to B' is P^{-1}

- Notes:

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}, \qquad B' = \{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$$

$$[\mathbf{v}]_B = [[\mathbf{u}'_1]_B, [\mathbf{u}'_2]_B, \ldots, [\mathbf{u}'_n]_B] \; [\mathbf{v}]_{B'} = P\,[\mathbf{v}]_{B'}$$

$$[\mathbf{v}]_{B'} = [[\mathbf{u}_1]_{B'}, [\mathbf{u}_2]_{B'}, \ldots, [\mathbf{u}_n]_{B'}] \; [\mathbf{v}]_B = P^{-1}\,[\mathbf{v}]_B$$

- **Theorem 4.26: (Transition matrix from B to B')**

Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ and $B' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be two bases for R^n. Then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the nx$2n$ matrix $[B' \vdots B]$ as follows: $[B' \vdots B] \longrightarrow [I_n \vdots P^{-1}]$

- **Ex 3: (Finding a transition matrix)**

$B = \{(-3, 2), (4, -2)\}$ and $B' = \{(-1, 2), (2, -2)\}$ are two bases for R^2

(a) Find the transition matrix from B' to B.

(b) Let $[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find $[\boldsymbol{v}]_B$

(c) Find the transition matrix from B to B'.

Sol: (a) $\left[\begin{array}{cc:cc} -3 & 4 & -1 & 2 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G. J. Elimination}} \left[\begin{array}{cc:cc} 1 & 0 & 3 & -2 \\ 0 & 1 & 2 & -1 \end{array}\right]$

$B \quad B' \qquad\qquad I \quad P$

$$\Rightarrow P = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \text{ (the transition matrix from } B' \text{ to } B\text{)}$$

(b) $[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow [\boldsymbol{v}]_B = P[\boldsymbol{v}]_{B'} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

- Check:

$$[\boldsymbol{v}]_{B'} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \boldsymbol{v} = (1)(-1,2) + (2)(2,-2) = (3,-2)$$

$$[\boldsymbol{v}]_{B} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{v} = (-1)(3,-2) + (0)(4,-2) = (3,-2)$$

(c)

$$\left[\begin{array}{cc:cc} -1 & 2 & -3 & 4 \\ 2 & -2 & 2 & -2 \end{array}\right] \xrightarrow{\text{G. J. Elimination}} \left[\begin{array}{cc:cc} 1 & 0 & -1 & 2 \\ 0 & 1 & -2 & 3 \end{array}\right]$$

B' B I P^{-1}

$$\Rightarrow P^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \quad \text{(the transition matrix from } B \text{ to } B'\text{)}$$

- Check:

$$PP^{-1} = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

- Ex 4: (Finding a transition matrix)

Find the transition matrix from B to B' for The bases for R^3 below.

$B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and $B' = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$

Sol: $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix}$

$$\underset{B' \qquad\qquad B}{\left[\begin{array}{ccc:ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -5 & 0 & 0 & 1 \end{array}\right]} \xrightarrow{\text{G. J. Elimination}} \underset{I \qquad\qquad P^{-1}}{\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 3 & -7 & -3 \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right]}$$

$$\begin{bmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix} \qquad \text{(Ex 2)}$$

4.8 Applications of Vector Spaces

Conic Sections And Rotation

$ax^2 + bxy + cy^2 + dx + ey + f = 0$ General equation of a conic section

performing a rotation of axes that eliminates the xy-term

$a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$

- **Rotation of Axes:**

  The general equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can be written in the form $a'(x')^2 + c'(y')^2 + d'x' + e'y' + f' = 0$ by rotating the coordinate axes counterclockwise through the angle θ, where θ is found using the equation $\cot 2\theta = (a - c)/b$. The coefficients of the new equation are obtained from the substitutions $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.

- Ex 1: (Rotation of a Conic Section)

Perform a rotation of axes to eliminate the xy-term in

$$5x^2 - 6xy + 5y^2 + 14\sqrt{2}x - 2\sqrt{2}y + 18 = 0$$

Sol:

$$\cot 2\theta = \frac{a-c}{b} = \frac{5-5}{-6} = 0 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow \sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$$

$$x = x'\cos\theta - y'\sin\theta = \frac{1}{\sqrt{2}}(x' - y')$$

$$y = x'\sin\theta + y'\cos\theta = \frac{1}{\sqrt{2}}(x' + y')$$

$$\Rightarrow (x')^2 + 4(y')^2 + 6x' - 8y' + 9 = 0$$

- Ex 2: (Rotation of a Conic Section)

$$\frac{(x'+3)^2}{4}+\frac{(y'-1)^2}{1^2}=1$$