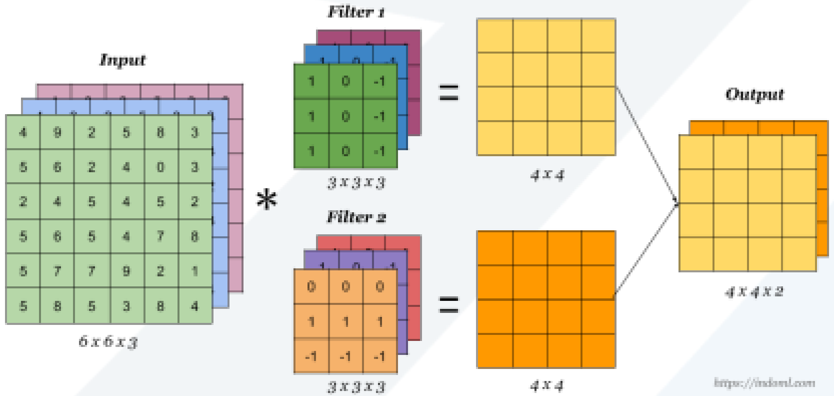
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 8: Inner Product Spaces: Part B



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5.3 Orthonormal Bases: Gram-Schmidt Process

- Orthogonal:

A set S of vectors in an inner product space V is called an orthogonal set if every pair of vectors in the set is orthogonal.

$$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\} \subseteq V$$

$$< \boldsymbol{v}_i , \boldsymbol{v}_j > = 0, \quad i \neq j$$

- Orthonormal:

An orthogonal set in which each vector is a unit vector is called orthonormal

$$S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_n\} \subseteq V$$

$$< \boldsymbol{v}_i , \boldsymbol{v}_j > = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Note:

  If S is a basis, then it is called an orthogonal basis or an orthonormal basis.

- Ex 1: (A nonstandard orthonormal basis for R^3)

  Show that the following set is an orthonormal basis.

$$S = \left\{ \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}_{v_1}, \underbrace{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}_{v_2}, \underbrace{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)}_{v_3} \right\}$$

Sol:

$$v_1 . v_2 = -\frac{1}{6} + \frac{1}{6} + 0 = 0$$

$$v_1 . v_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$v_2 . v_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Show that the three vectors are mutually orthogonal.

$$\|v_1\| = \sqrt{v_1 . v_1} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

$$\|v_2\| = \sqrt{v_2 . v_2} = \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{9}} = 1$$

$$\|v_3\| = \sqrt{v_3 . v_3} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$$

Show that each vector is of length 1

Thus S is an orthonormal set

- Theorem 5.9: (Orthogonal sets are linearly independent)

  If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

- Corollary to Theorem 5.9:

  If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

- Ex 2: (Using orthogonality to test for a basis)

Show that the following set is a basis for R^4

$$S = \{\overset{v_1}{(2, 3, 2, -2)}, \overset{v_2}{(1, 0, 0, 1)}, \overset{v_3}{(-1, 0, 2, 1)}, \overset{v_4}{(-1, 2, -1, 1)}\}$$

Sol:

v_1, v_2, v_3, v_4: nonzero vectors

$$v_1 \cdot v_2 = 2 + 0 + 0 - 2 = 0 \qquad v_2 \cdot v_3 = -1 + 0 + 0 + 1 = 0$$

$$v_1 \cdot v_3 = -2 + 0 + 4 - 2 = 0 \qquad v_2 \cdot v_4 = -1 + 0 + 0 + 1 = 0$$

$$v_1 \cdot v_4 = -2 + 6 - 2 - 2 = 0 \qquad v_3 \cdot v_4 = 1 + 0 - 2 + 1 = 0$$

$\Rightarrow S$ is orthogonal $\Rightarrow S$ is a basis for R^4

- Theorem 5.10: (Coordinates relative to an orthonormal basis)

  If $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\boldsymbol{w}$ with respect to B is

$$\boldsymbol{w} = <\boldsymbol{w}, \boldsymbol{v}_1> \boldsymbol{v}_1 + <\boldsymbol{w}, \boldsymbol{v}_2> \boldsymbol{v}_2 + \cdots + <\boldsymbol{w}, \boldsymbol{v}_n> \boldsymbol{v}_n$$

- Note:

  If $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ is an orthonormal basis for V and $\boldsymbol{w} \in V$, then the corresponding coordinate matrix of $\boldsymbol{w}$ relative to B is

$$[\boldsymbol{w}]_B = \begin{bmatrix} <\boldsymbol{w}, \boldsymbol{v}_1> \\ <\boldsymbol{w}, \boldsymbol{v}_2> \\ \vdots \\ <\boldsymbol{w}, \boldsymbol{v}_n> \end{bmatrix}$$

- Ex 3: (Representing vectors relative to an orthonormal basis)

Find the coordinates of vector $w = (5, -5, 2)$ relative to the following orthonormal basis for R^3 .

$$B = \{(\frac{3}{5}, \frac{4}{5}, 0), (-\frac{4}{5}, \frac{3}{5}, 0), (0, 0, 1)\}$$

Sol:

$$< w, v_1 > = w.v_1 = (5, -5, 2) . (\frac{3}{5}, \frac{4}{5}, 0) = -1$$

$$< w, v_2 > = w.v_2 = (5, -5, 2) . (-\frac{4}{5}, \frac{3}{5}, 0) = -7 \quad \Rightarrow [w]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

$$< w, v_3 > = w.v_3 = (5, -5, 2) . (0, 0, 1) = 2$$

- Theorem 5.11: (Gram-Schmidt orthonormalization process)

(1) Let $B = \{v_1, v_2, \ldots, v_n\}$ is a basis for an inner product space V

(2) Let $B' = \{w_1, w_2, \ldots, w_n\}$, where

$$w_1 = v_1$$

$$w_2 = v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >} w_1$$

$$w_3 = v_3 - \frac{< v_3, w_1 >}{< w_1, w_1 >} w_1 - \frac{< v_3, w_2 >}{< w_2, w_2 >} w_2$$

$$\vdots$$

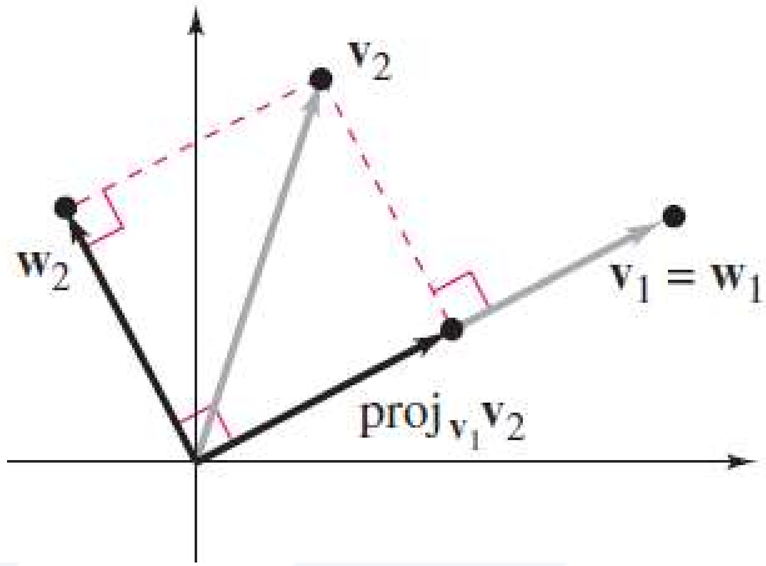
$$w_n = v_n - \sum_{i=1}^{n-1} \frac{< v_n, w_i >}{< w_i, w_i >} w_i$$

Then B' is an orthogonal basis for V

(3) Let $\boldsymbol{u}_i = \dfrac{\boldsymbol{w}_i}{\|\boldsymbol{w}_i\|}$

Then $B'' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is an orthonormal basis for V

Also, $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ for $k = 1, 2, \ldots, n$



- Ex 4: (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt orthonormalization process to the basis B for R^2

$$B = \{\overset{v_1}{(1,1)}, \overset{v_2}{(0,1)}\}$$

Sol:

$$w_1 = v_1 = (1,1)$$

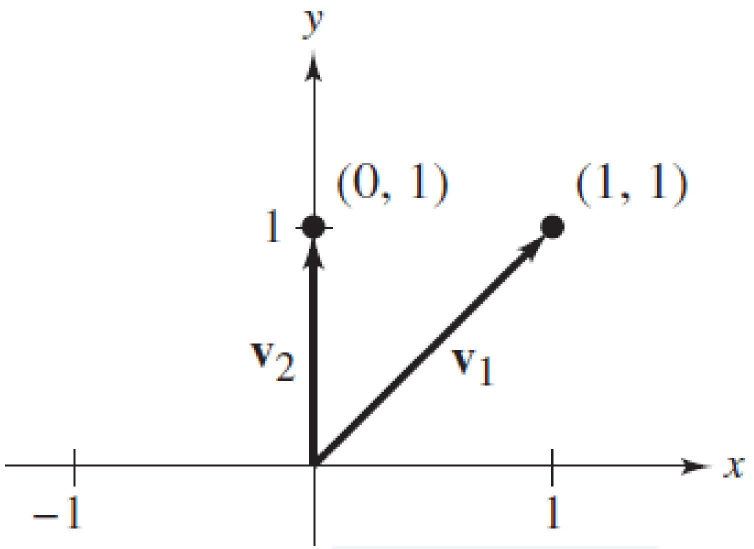
$$w_2 = v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >} w_1 = (0,1) - \frac{1}{2}(1,1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$

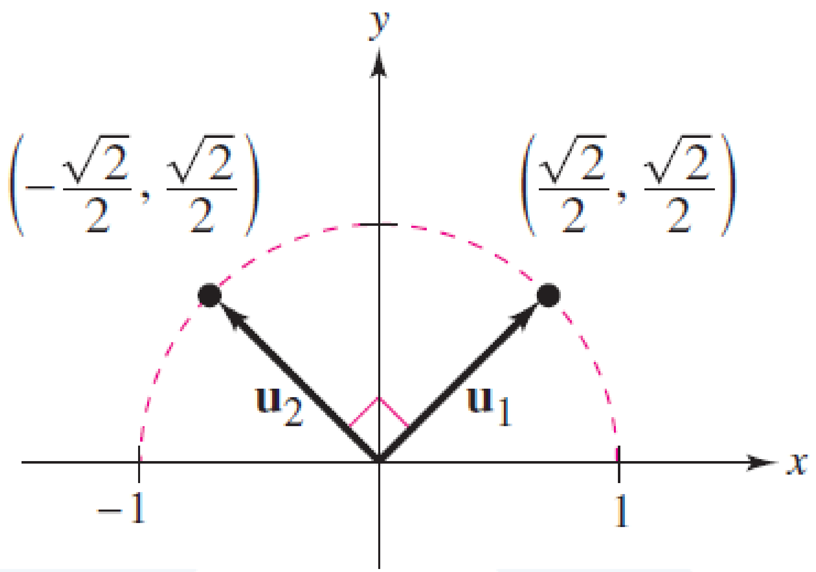
The set $B' = \{w_1, w_2\}$ is an orthogonal basis for R^2

$$u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

The set $B'' = \{u_1, u_2\}$ is an orthonormal basis for R^2





- Ex 5: (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt orthonormalization process to the basis B for R^3

$$B = \{\overset{v_1}{(1,1,0)}, \overset{v_2}{(1,2,0)}, \overset{v_3}{(0,1,2)}\}$$

Sol:

$$w_1 = v_1 = (1,1,0)$$

$$w_2 = v_2 - \frac{<v_2, w_1>}{<w_1, w_1>} w_1 = (1,2,0) - \frac{3}{2}(1,1,0) = (-\frac{1}{2}, \frac{1}{2}, 0)$$

$$w_3 = v_3 - \frac{<v_3, w_1>}{<w_1, w_1>} w_1 - \frac{<v_3, w_2>}{<w_2, w_2>} w_2$$

$$= (1,2,0) - \frac{1}{2}(1,1,0) - \frac{1/2}{1/2}(-\frac{1}{2}, \frac{1}{2}, 0) = (0,0,2)$$

The set $B' = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3\}$ is an orthogonal basis for R^3

$$\boldsymbol{u}_1 = \frac{\boldsymbol{w}_1}{\|\boldsymbol{w}_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$$

$$\boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{\|\boldsymbol{w}_2\|} = \frac{1}{1/\sqrt{2}}(-\frac{1}{2}, \frac{1}{2}, 0) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$$

$$\boldsymbol{u}_3 = \frac{\boldsymbol{w}_3}{\|\boldsymbol{w}_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1)$$

The set $B'' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthonormal basis for R^3

5.4 Mathematical Models and Least Square Analysis

- Best Approximation; Least Squares:

Least Squares Problem: Given $A\boldsymbol{x} = \boldsymbol{b}$ of m equations in n unknowns, find $\boldsymbol{x}$ in R^n that minimizes $\|\boldsymbol{b} - A\boldsymbol{x}\|$ with respect to the Euclidean inner product on R^m. We call $\boldsymbol{x}$, if it exists, a least squares solution of $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{b} - A\boldsymbol{x}$ the least squares error vector, and $\|\boldsymbol{b} - A\boldsymbol{x}\|$ the least squares error

$$\boldsymbol{b} - A\boldsymbol{x} = \begin{bmatrix} e_1 \\ e_1 \\ \vdots \\ e_m \end{bmatrix} \Rightarrow \|\boldsymbol{b} - A\boldsymbol{x}\|^2 = e_1^2 + e_2^2 + \cdots + e_m^2$$

- Finding Least Squares Solutions: $A^T A\boldsymbol{x} = A^T\boldsymbol{b}$

- Ex 1: Finding Least Squares Solutions

Find the Least Squares Solution, the least squares error vector, and the least squares error of the linear system

$$\begin{aligned} x &- y &= 4 \\ 3x &+ 2y &= 1 \\ -2x &+ 4y &= 3 \end{aligned}$$

Sol:

$$A^T A = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix}$$

$$A^T \boldsymbol{b} = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$$A^T A \boldsymbol{x} = A^T \boldsymbol{b} \Rightarrow \begin{bmatrix} 14 & -3 \\ -3 & 21 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17/95 \\ 143/285 \end{bmatrix}$$

$$\boldsymbol{b} - A\boldsymbol{x} = \begin{bmatrix} 1232/285 \\ -154/285 \\ 77/57 \end{bmatrix}, \quad \text{and} \quad \|\boldsymbol{b} - A\boldsymbol{x}\| \approx 4.556$$

- Theorem 5.12:

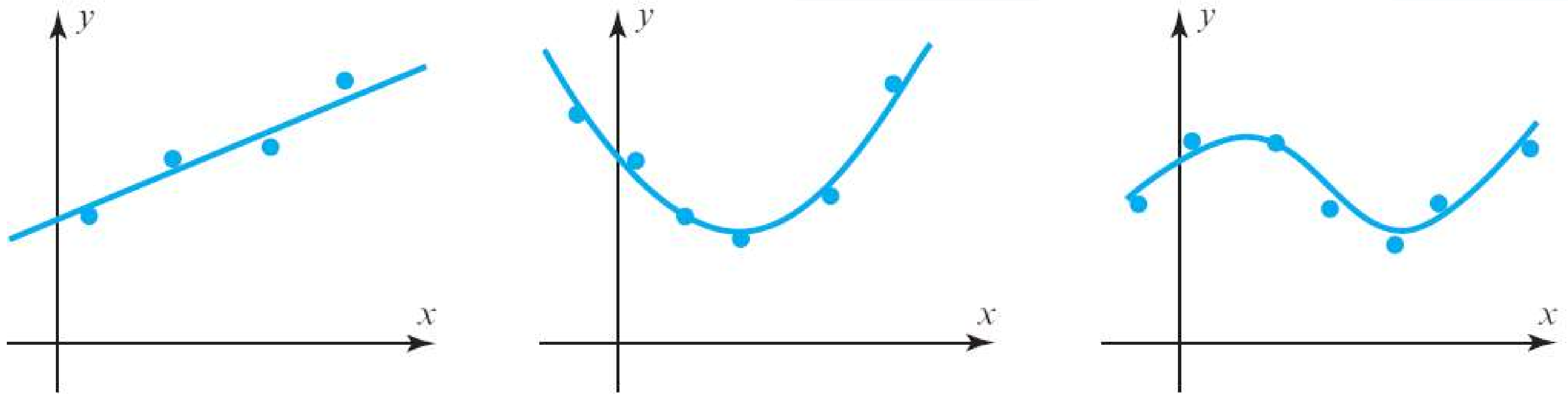
  If A is an $m \times n$ matrix with linearly independent column vectors, then for every $m \times 1$ matrix $\boldsymbol{b}$, the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique least squares solution. This solution is given by

$$\boldsymbol{x} = (A^T A)^{-1} A^T \boldsymbol{b}$$

- Mathematical Modeling Using Least Squares

Fitting a Curve to Data

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$



(a) $y = a + bx$ (b) $y = a + bx + cx^2$ (c) $y = a + bx + cx^2 + dx^3$

mathematical model

Least Squares Fit of a Straight Line $y = a + bx$

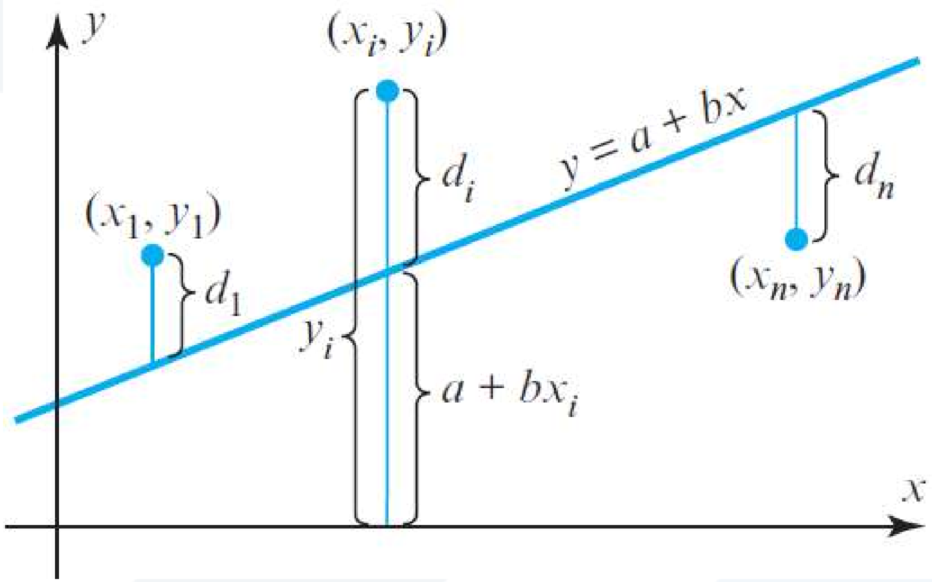
$$\begin{aligned} y_1 &= a + bx_1 \\ y_2 &= a + bx_2 \\ &\vdots \\ y_n &= a + bx_n \end{aligned} \quad \Rightarrow \quad M\boldsymbol{v} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y}$$

$$M\boldsymbol{v} = \boldsymbol{y} \quad \Rightarrow \quad M^T M\boldsymbol{v} = M^T\boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{v}^* = \begin{bmatrix} a^* \\ b^* \end{bmatrix} = (M^T M)^{-1} M^T \boldsymbol{y}$$

$y = a^* + b^* x$ Least squares line of best fit or the regression line

It minimizes $\left\|\boldsymbol{y} - M\boldsymbol{v}\right\|^2 = [y_1 - (a + bx_1)]^2 + [y_2 - (a + bx_2)]^2 + \cdots + [y_n - (a + bx_n)]^2$

$$d_1 = |y_1 - (a + bx_1)|,\ d_2 = |y_2 - (a + bx_2)|, \cdots, d_n = |y_n - (a + bx_n)| \quad \text{residuals.}$$

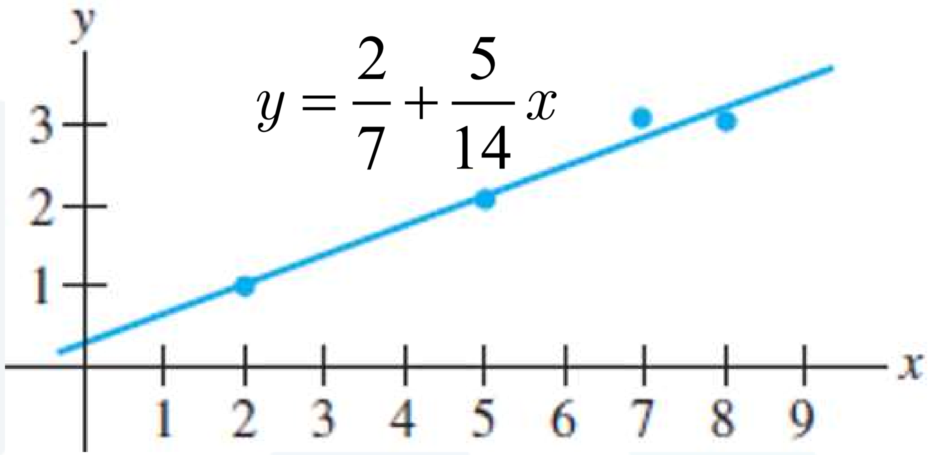


- Ex 2: Least Squares Straight Line Fit

Find the least squares straight line fit to the 4 points (2, 1), (5, 2), (7, 3), and (8, 3)

Sol:

$$M^T M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}$$



$$y = \frac{2}{7} + \frac{5}{14}x$$

$$M^T \boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

$$\boldsymbol{v}^* = (M^T M)^{-1} M^T \boldsymbol{y} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix} \begin{bmatrix} 9 \\ 57 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{5}{14} \end{bmatrix}$$

Least Squares Fit of a Polynomial $y = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

$$\begin{aligned}
a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_m x_1^m &= y_1 \\
a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_m x_2^m &= y_2 \\
&\vdots \\
a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_m x_n^m &= y_n
\end{aligned}$$

$$M\boldsymbol{v} = \begin{bmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^m \\ 1 & x_2 & x_2^2 & \ldots & x_2^m \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \ldots & x_n^m \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \boldsymbol{y}$$

$$M\boldsymbol{v} = \boldsymbol{y} \quad \Rightarrow \quad M^T M \boldsymbol{v} = M^T \boldsymbol{y} \quad \Rightarrow \quad \boldsymbol{v}^* = (M^T M)^{-1} M^T \boldsymbol{y}$$

Ex 3: Fitting a Quadratic Curve to Data

Newton's second law of motion $s = s_0 + v_0 t + \frac{1}{2} g t^2$

Laboratory experiment

Time t (sec)	.1	.2	.3	.4	.5
Displacement s (ft)	-0.18	0.31	1.03	2.48	3.73

Approximate g

Sol:

Let $s = a_0 + a_1 t + a_2 t^2$

(0.1,–0.18), (0.2, 0.31), (0.3, 1.03), (0.4, 2.48), (0.5, 3.73)

$$M = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \\ 1 & t_5 & t_5^2 \end{bmatrix} = \begin{bmatrix} 1 & 0.1 & 0.01 \\ 1 & 0.2 & 0.04 \\ 1 & 0.3 & 0.09 \\ 1 & 0.4 & 0.16 \\ 1 & 0.5 & 0.25 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} = \begin{bmatrix} -0.18 \\ 0.31 \\ 1.03 \\ 2.48 \\ 3.73 \end{bmatrix}$$

$$\boldsymbol{v}^* = \begin{pmatrix} a_0^* \\ a_1^* \\ a_2^* \end{pmatrix} = (M^T M)^{-1} M^T \boldsymbol{y} = \begin{pmatrix} -0.4 \\ 0.35 \\ 16.1 \end{pmatrix}$$

$$g = 2a_2^* = 2(16.1) = 32.2 \text{ feet}/s^2$$

$$s_0 = a_0^* = -0.4 \text{ feet} \qquad v_0 = a_1^* = 0.35 \text{ feet/s}$$