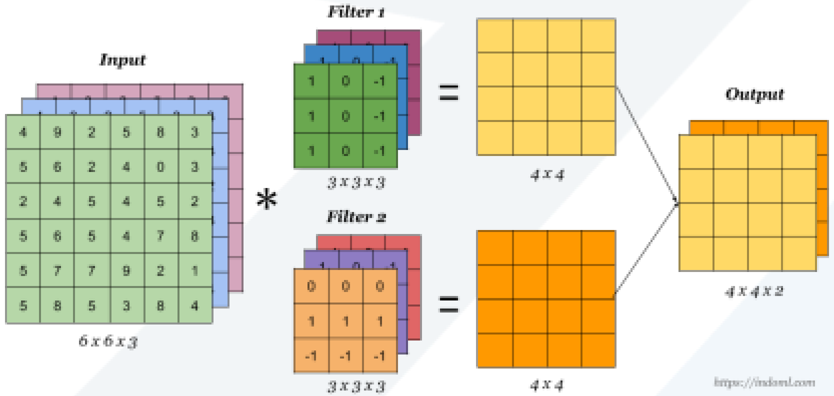
CECC122: Linear Algebra and Matrix Theory

Lecture Notes 9: Linear Transformations: Part A



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Informatics

Manara University

6.1 Introduction to Linear Transformations

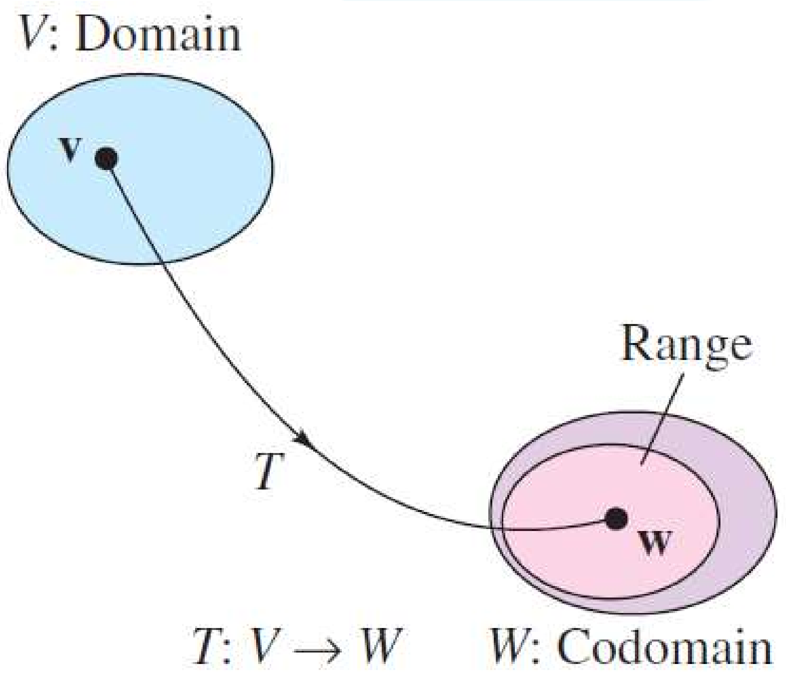
- **Images And Preimages of Functions:**

Function T that maps a vector space V into a vector space W

$T: V \xrightarrow{\text{Mapping}} W,$ V, W: vector spaces

V: the domain of T

W: the codomain of T



- **Image of $\boldsymbol{v}$ under T:**

If $\boldsymbol{v}$ is in V and $\boldsymbol{w}$ is in W such that: $T(\boldsymbol{v}) = \boldsymbol{w}$

Then $\boldsymbol{w}$ is called the image of $\boldsymbol{v}$ under T

- Images And Preimages of Functions:

- The range of T: The set of all images of vectors in V.

- The preimage of $\boldsymbol{w}$: The set of all $\boldsymbol{v}$ in V such that $T(\boldsymbol{v}) = \boldsymbol{w}$.

- Ex 1: (A function from R^2 into R^2)

$$T: R^2 \to R^2 \qquad \boldsymbol{v} = (v_1, v_2) \in R^2$$

$$T(v_1, v_2) = (v_1 - v_2, v_1 + 2v_2)$$

(a) Find the image of $\boldsymbol{v} = (-1, 2)$. (b) Find the preimage of $\boldsymbol{w} = (-1, 11)$

Sol:

$$(a)\ \boldsymbol{v} = (-1, 2) \Rightarrow T(\boldsymbol{v}) = T(-1, 2) = (-1-2, -1+2(2)) = (-3,\ 3)$$

$$(b)\ T(\boldsymbol{v}) = \boldsymbol{w} = (-1, 11) \Rightarrow T(v_1, v_2) = (v_1 - v_2,\ v_1 + 2v_2) = (-1,\ 11)$$

$$\Rightarrow v_1 - v_2 = -1$$

$$v_1 + 2v_2 = 11$$

$$\Rightarrow v_1 = 3,\ \ v_2 = 4$$

Thus $\{(3, 4)\}$ is the preimage of $\boldsymbol{w} = (-1, 11)$.

- Linear Transformation (L.T.):

V, W: vector spaces

$T: V \to W$: Linear Transformation

$$(1)\ T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v}), \quad \forall \boldsymbol{u},\boldsymbol{v} \in V$$

$$(2)\ T(c\boldsymbol{u}) = cT(\boldsymbol{u}), \quad \forall c \in R$$

- Ex 2: (Functions that are not linear transformations)

$(a)\ f(x) = \sin x$

$$\sin(x_1 + x_2) \neq \sin(x_1) + \sin(x_2)$$

$$\sin(\frac{\pi}{2} + \frac{\pi}{3}) \neq \sin(\frac{\pi}{2}) + \sin(\frac{\pi}{3})$$

not a linear transformations

$(b)\ f(x) = x^2$

$$(x_1 + x_2)^2 \neq x_1^2 + x_2^2 \qquad (1+2)^2 \neq 1^2 + 2^2$$

not a linear transformations

$(c)\ f(x) = x + 1$

$$f(x_1 + x_2) = x_1 + x_2 + 1$$

$$f(x_1) + f(x_2) = (x_1 + 1) + (x_2 + 1) = x_1 + x_2 + 2$$

$$f(x_1 + x_2) \neq f(x_1) + f(x_2)$$

not a linear transformations

- Notes: Two uses of the term "linear"

  (1) $f(x) = x + 1$ is called a linear function because its graph is a line.

  (2) $f(x) = x + 1$ is not a linear transformation from a vector space R into R because it preserves neither vector addition nor scalar multiplication

- Zero transformation:

$$T: V \to W \qquad T(\mathbf{v}) = \mathbf{0}, \ \ \forall \mathbf{v} \in V$$

- Identity transformation:

$$T: V \to V \qquad T(\mathbf{v}) = \mathbf{v}, \ \ \forall \mathbf{v} \in V$$

- Theorem 6.1: (Properties of linear transformations)

$$T: V \to W, \quad \boldsymbol{u}, \boldsymbol{v} \in V$$

(1) $T(\mathbf{0}) = \mathbf{0}$

(2) $T(-\boldsymbol{v}) = -T(\boldsymbol{v})$

(3) $T(\boldsymbol{u} - \boldsymbol{v}) = T(\boldsymbol{u}) - T(\boldsymbol{v})$

(4) If $\boldsymbol{v} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$ then

$$\begin{aligned} T(\boldsymbol{v}) &= T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n) \\ &= c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + \cdots + c_nT(\boldsymbol{v}_n) \end{aligned}$$

- Ex 3: (Linear transformations and bases)

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that

$$T(1,0,0)=(2,-1,4), \quad T(0,1,0)=(1,5,-2), \quad T(0,0,1)=(0,3,1)$$

Find $T(2, 3, -2)$

Sol:

$$(2,3,-2)=2(1,0,0)+3(0,1,0)-2(0,0,1)$$

$$\begin{aligned} T(2,3,-2) &= 2T(1,0,0)+3T(0,1,0)-2T(0,0,1) \\ &= 2(2,-1,4)+3(1,5,-2)-2T(0,3,1) \\ &= (7,7,0) \end{aligned}$$

- Ex 4: (A linear transformation defined by a matrix)

The function $T: R^2 \rightarrow R^3$ is defined as $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

(a) Find $T(\mathbf{v})$, where $\mathbf{v} = (2, -1)$

(b) Show that T is a linear transformation from R^2 into R^3

Sol:

$$(a)\ \mathbf{v} = (2, -1) \qquad T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{R^2 \text{ vector}} = \underbrace{\begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}}_{R^3 \text{ vector}} \Rightarrow T(2,-1) = (6, 3, 0)$$

$$(b)\ T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = T(\mathbf{u}) + T(\mathbf{v})$$ (vector addition)

$$T(c\mathbf{u}) = A(c\mathbf{u}) = c(A\mathbf{u}) = cT(\mathbf{u})$$ (scalar multiplication)

- Theorem 6.2: (The linear transformation given by a matrix)

  Let A be an $m\times n$ matrix. The function T defined by $T(\boldsymbol{v}) = A\boldsymbol{v}$ is a linear transformation from R^n into R^m.

- Note:

$$A\boldsymbol{v}=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{R^n \text{ vector}}=\underbrace{\begin{bmatrix} a_{11}v_1+a_{12}v_2+\cdots+a_{1n}v_n \\ a_{21}v_1+a_{22}v_2+\cdots+a_{2n}v_n \\ \vdots \\ a_{m1}v_1+a_{m2}v_2+\cdots+a_{mn}v_n \end{bmatrix}}_{R^m \text{ vector}}$$

$T(\boldsymbol{v}) = A\boldsymbol{v}$ R^n into R^m

$T: R^n \rightarrow R^m$

- Ex 5: (Rotation in the plane)

Show that the L.T. $T: R^2 \to R^2$ given by the matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

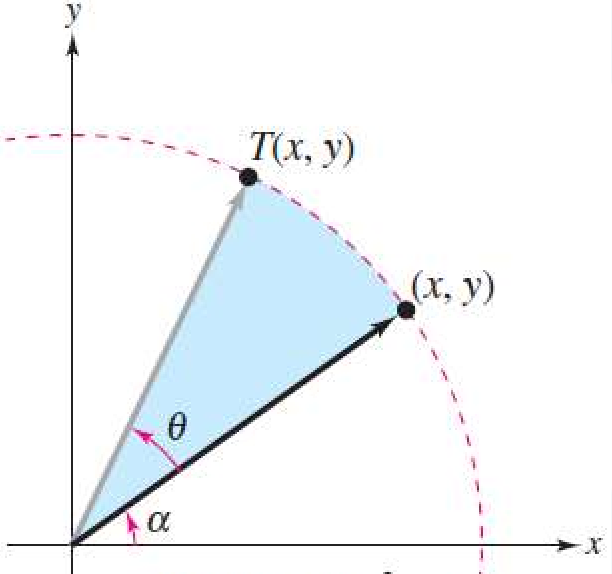
has the property that it rotates every vector in R^2 counterclockwise about the origin through the angle θ.

Sol:

$\boldsymbol{v} = (x, y) = (r\cos\alpha, r\sin\alpha)$ (polar coordinates)

r: the length of $\boldsymbol{v}$

α: the angle from the positive x-axis counterclockwise to the vector $\boldsymbol{v}$



Rotation in R^2

$$T(\boldsymbol{v}) = A\boldsymbol{v} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ r\sin\theta\cos\alpha + r\cos\theta\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix}$$

r: the length of $T(\boldsymbol{v})$

$\theta + \alpha$: the angle from the positive x-axis counterclockwise to the vector $T(\boldsymbol{v})$

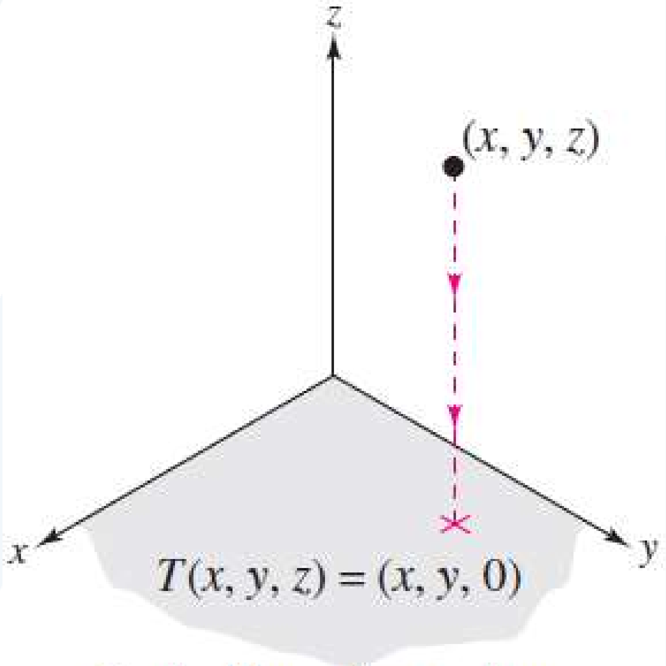
Thus, $T(\boldsymbol{v})$ is the vector that results from rotating the vector $\boldsymbol{v}$ counterclockwise through the angle θ.

- Ex 6: (A projection in R^3)

  The linear transformation $T: R^3 \rightarrow R^3$ is given by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

  is called a projection in R^3.

  If $\boldsymbol{v} = (x, y, z)$ is a vector in R^3, then
  $T(\boldsymbol{v}) = (x, y, 0)$.



Projection onto xy-plane

6.2 The Kernel and Range of a Linear Transformation

- **Kernel of a linear transformation T:**

  Let $T: V \to W$ be a linear transformation. Then the set of all vectors $\boldsymbol{v}$ in V that satisfy $T(\boldsymbol{v}) = \mathbf{0}$ is called the kernel of T and is denoted by ker(T).

$$\ker(T) = \{\boldsymbol{v} \mid T(\boldsymbol{v}) = \mathbf{0},\ \forall \boldsymbol{v} \in V\}$$

- **Ex 1: (The kernel of the zero and identity transformations)**

  $(a)\ T(\boldsymbol{v}) = \mathbf{0}$ (the zero transformation $T: V \to W$)

$$\ker(T) = V$$

  $(b)\ T(\boldsymbol{v}) = \boldsymbol{v}$ (the identity transformation $T: V \to V$)

$$\ker(T) = \{\mathbf{0}\}$$

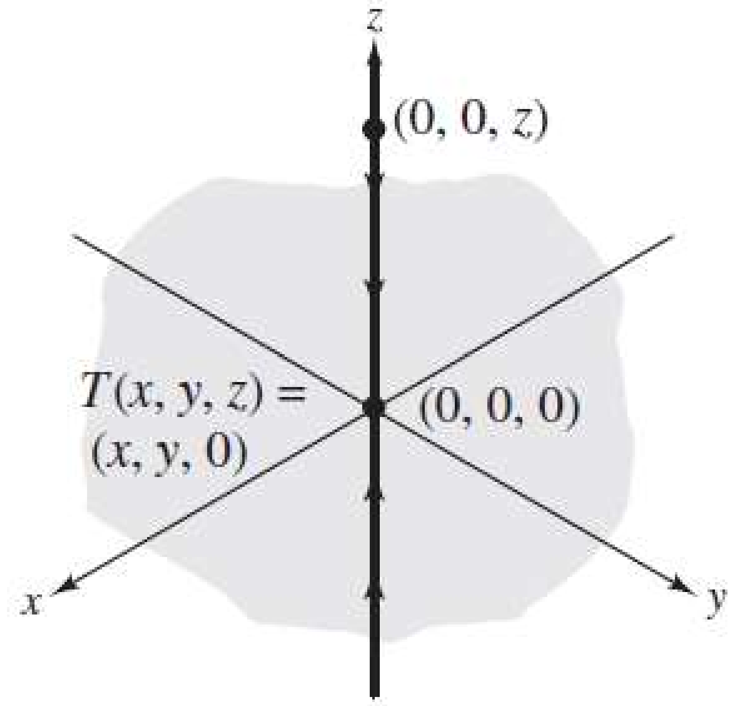
- Ex 2: (Finding the kernel of a L.T.)

$$T(\boldsymbol{v}) = (x, y, 0) \qquad T: R^3 \rightarrow R^3$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(0, 0, z) | \ z \text{ is a real number}\}$$



- Ex 3: (Finding the kernel of a linear transformation)

$$T(\boldsymbol{x}) = A\boldsymbol{x} = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (T: R^3 \rightarrow R^2)$$

$$\ker(T) = ?$$

Sol:

$$\ker(T) = \{(x_1,x_2,x_3) | T(x_1,x_2,x_3) = (0,0),\ \ \boldsymbol{x} = (x_1, x_2, x_3) \in R^3\}$$

$$T(x_1,x_2,x_3) = (0,0)$$

$$\begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \ker(T) \ = \{t(1, -1, 1) | t \text{ is a real number}\} = \text{span}\{(1, -1, 1)$$

- **Theorem 6.3: (The kernel is a subspace of V)**

  The kernel of a linear transformation $T: V \to W$ is a subspace of the domain V.

- **Range of a linear transformation T:**

  Let $T: V \to W$ be a L.T.

  Then the set of all vectors $\boldsymbol{w}$ in W that are images of vectors in V is called the range of T and is denoted by range(T)

$$\text{range}(T) = \{T(\boldsymbol{v}) | \forall \boldsymbol{v} \in V\}$$

- **Theorem 6.4: (The range of T is a subspace of W)**

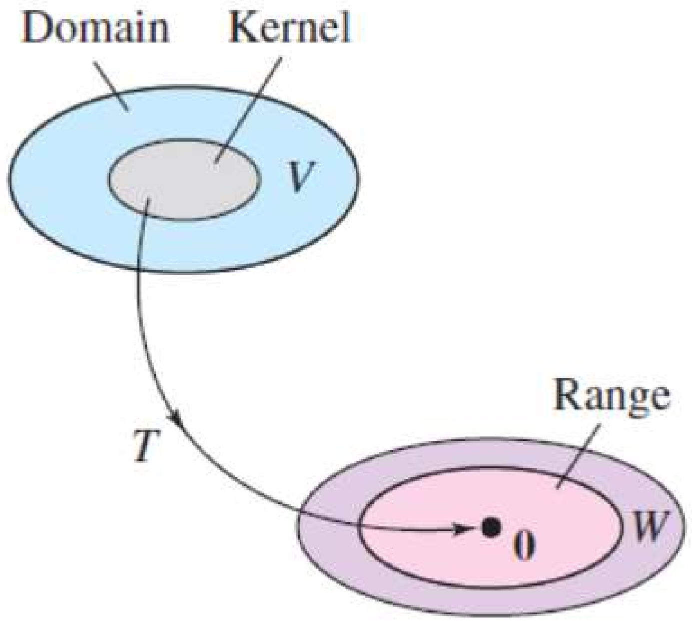
  The range of a linear transformation $T: V \to W$ is a subspace of the W

- Notes:

  $T: V \to W$: is Linear Transformation

  (1) $\ker(T)$ is a subspace of V

  (2) $\text{Range}(T)$ is a subspace of W



- Rank of a linear transformation $T: V \to W$:

  $\text{rank}(T)$ = the dimension of the range of T

- Nullity of a linear transformation $T: V \to W$:

  $\text{nullity}(T)$ = the dimension of the kernel of T

- Note:

  Let $T: R^n \to R^m$ be the L.T. given by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Then

  $\Rightarrow \text{rank}(T) = \text{rank}(A)$, $\quad \text{nullity}(T) = \text{nullity}(A)$

- Theorem 6.5: (Sum of rank and nullity)

Let $T: V \to W$ be a L.T. from an n-dimensional vector space V into a vector space W. Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

$$\dim(\text{range of } T) + \dim(\text{kernel of } T) = \dim(\text{domain of } T)$$

- Ex 4: (Finding rank and nullity of a linear transformation)

Find the rank and nullity of the L.T. $T: R^3 \to R^3$ defined by

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$$\text{rank}(T) = \text{rank}(A) = 2$$

$$\text{nullity}(T) = \dim(\text{domain of } T) - \text{rank}(T) = 3 - 2 = 1$$

- Ex 5: (Finding rank and nullity of a linear transformation)

Let $T: R^5 \rightarrow R^7$ be a linear transformation

(a) Find the dimension of the kernel of T if the dimension of the range is 2

(b) Find the rank of T if the nullity of T is 4

(c) Find the rank of T if $\ker(T) = \{\mathbf{0}\}$

Sol:

(a) $\dim(\text{domain of } T) = 5$

$$\dim(\ker \text{ of } T) = n - \dim(\text{range of } T) = 5 - 2 = 3$$

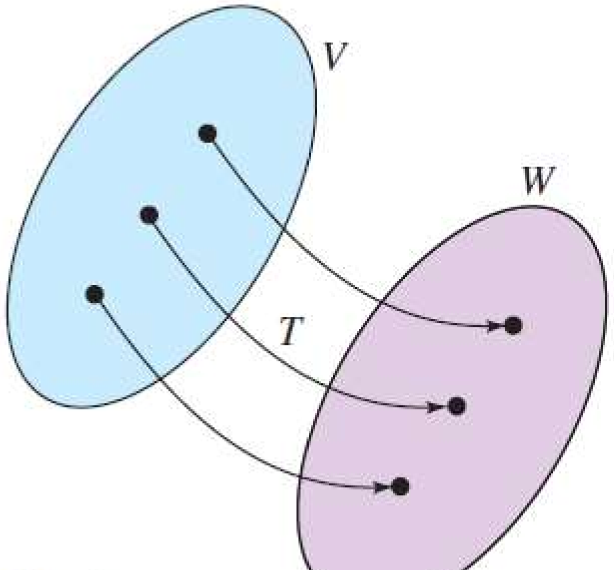
(b) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 4 = 1$

(c) $\text{rank}(T) = n - \text{nullity}(T) = 5 - 0 = 5$

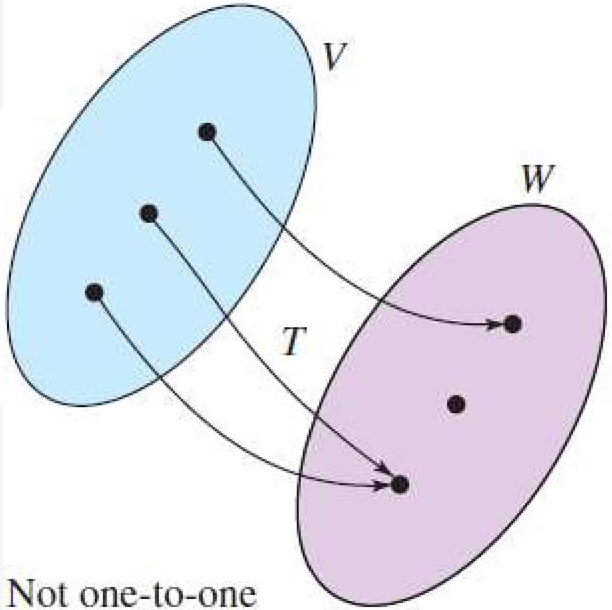
- One-to-one:

  A function $T: V \to W$ is one-to-one when the preimage of every $\boldsymbol{w}$ in the range consists of a single vector

  T is one-to-one if and only if, for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in V, $T(\boldsymbol{u}) = T(\boldsymbol{v})$ implies $\boldsymbol{u} = \boldsymbol{v}$.



One-to-one



Not one-to-one

- **Onto:**

A function $T: V \rightarrow W$ is onto when every element in W has a preimage in V. (T is onto W when W is equal to the range of T)

- **Theorem 6.6: (One-to-one linear transformation)**

Let $T: V \rightarrow W$ be a linear transformation. Then T is one-to-one iff $\ker(T) = \{\mathbf{0}\}$

- **Ex 6: (One-to-one and not one-to-one linear transformation)**

(a) The linear transformation $T: M_{3x2} \rightarrow M_{2x3}$ given by $T(A) = A^T$ is one-to-one because its kernel consists of only the mxn zero matrix

(b) The zero transformation $T: R^3 \to R^3$ is not one-to-one because its kernel is all of R^3

- **Theorem 6.7: (Onto linear transformation)**

  Let $T: V \to W$ be a linear transformation, where W is finite dimensional Then T is onto iff the rank of T is equal to the dimension of W.

- **Theorem 6.8: (One-to-one and onto linear transformation)**

  Let $T: V \to W$ be a linear transformation, with vector space V and W both of dimension n. Then T is one-to-one iff it is onto.

- Ex 7:

Let $T: R^n \to R^m$ be a L.T. given by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither

$$(a)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},\ (b)\ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},\ (c)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix},\ (d)\ A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Sol:

$T: R^n \to R^m$	dim(domain of T)	rank(T)	nullity(T)	one-to-one	onto
$(a)\ T: R^3 \to R^3$	3	3	0	Yes	Yes
$(b)\ T: R^2 \to R^3$	2	2	0	Yes	No
$(c)\ T: R^3 \to R^2$	3	2	1	No	Yes
$(d)\ T: R^3 \to R^3$	3	2	1	No	No