

Structural Mechanics (2)

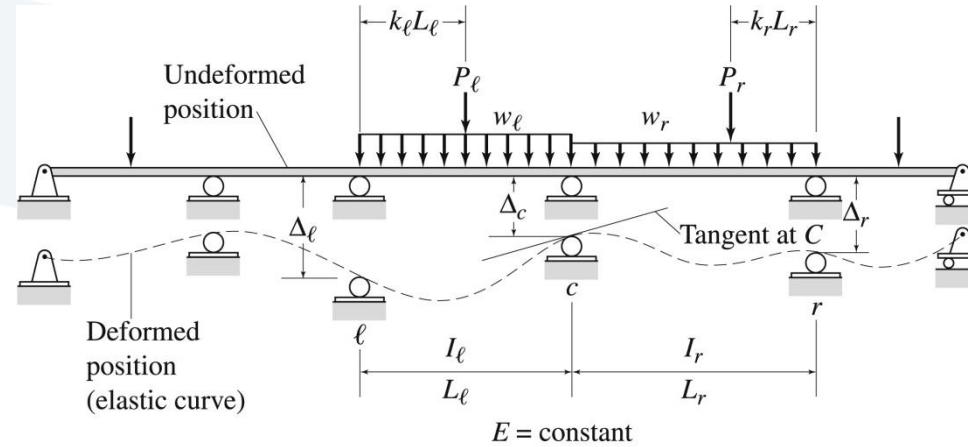
Week No-02

Analysis of Indeterminate Structures - Force Method

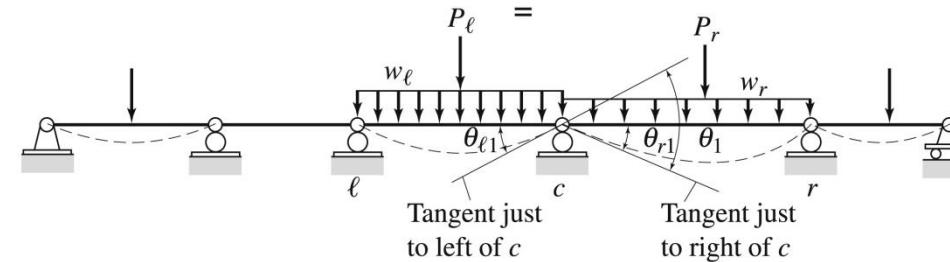
- Indeterminate Structures vs. Determinate Structures
- Analysis of Indeterminate Structures.
- Structures with single Degree of Indeterminacy (Beams & Frames)
- Structures with single Degree of Indeterminacy (Trusses: Int. & Ext.)
- Structures with multiple Degrees of Indeterminacy
- Support Settlements
- Three-Moment Equation for Continuous Beams

Derivation of three-Moment Equation

A continuous beam under: ex. loads & sup. settlements.



(a) Continuous Beam



(b) Primary Structure Subjected to External Loading

From ex. Loads: $\theta_1 = \theta_h + \theta_{r1}$

From sup. settlements: $\theta_2 = \theta_{l2} + \theta_{r2}$

From sup. bending moments: $\theta_3 = \theta_B + \theta_{r3}$

Comp. Equation at support c is: $\theta_1 + \theta_2 + \theta_3 = 0$

Or in detail: $\theta_h + \theta_{r1} + \theta_{l2} + \theta_{r2} + \theta_B + \theta_{r3} = 0$

Supports: I, c, r . Spans: L_l, L_r . Mo. of In.: I_l, I_r

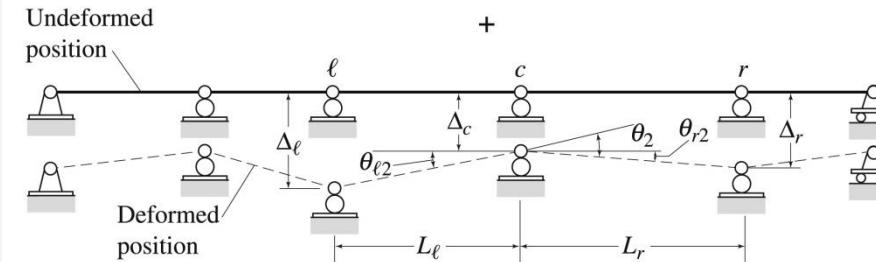
Con. Loads: P_l at $k_l L_l$ from I & P_r at $k_r L_r$ from r .

Dis. Loads: w_l over L_l and w_r over L_r

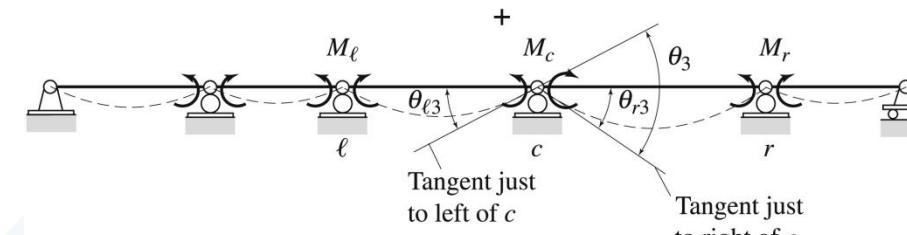
Sup. Settlements: Δ_l at I , Δ_c at c , Δ_r at r .

Redundants are internal moments at supports:

..., M_l, M_c, M_r, \dots Compatibility Equations are slope continuities at the supports.



(c) Primary Structure Subjected to Support Settlements



(d) Primary Structure Loaded with Redundant Bending Moments

Derivation of three-Moment Equation

$$\theta_{l1} = \sum \frac{P_l L_l^2 k_l (1 - k_l^2)}{6EI_l} + \frac{w_l L_l^3}{24EI_l} \quad \theta_{l2} = \frac{\Delta_l - \Delta_c}{L_l}$$

$$\theta_{r1} = \sum \frac{P_r L_r^2 k_r (1 - k_r^2)}{6EI_r} + \frac{w_r L_r^3}{24EI_r} \quad \theta_{r2} = \frac{\Delta_r - \Delta_c}{L_r}$$

$$\theta_{l3} = \frac{M_l L_l}{6EI_l} + \frac{M_c L_l}{3EI_l}; \quad \theta_{r3} = \frac{M_c L_r}{3EI_r} + \frac{M_r L_r}{6EI_r}$$

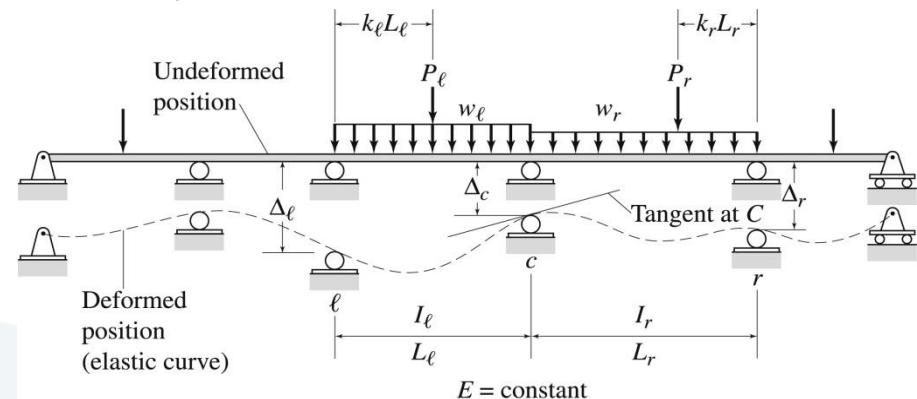
Substituting in to the comp. equation at support c:

$$\theta_h + \theta_{l1} + \theta_{l2} + \theta_{r1} + \theta_{r2} + \theta_{r3} = 0 \text{ we get}$$

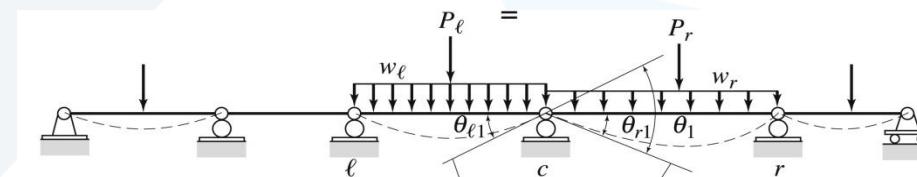
$$\sum \frac{P_l L_l^2 k_l (1 - k_l^2)}{6EI_l} + \frac{w_l L_l^3}{24EI_l} + \sum \frac{P_r L_r^2 k_r (1 - k_r^2)}{6EI_r} + \frac{w_r L_r^3}{24EI_r} + \frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r} + \frac{M_l L_l}{6EI_l} + \frac{M_c L_l}{3EI_l} + \frac{M_c L_r}{3EI_r} + \frac{M_r L_r}{6EI_r} = 0$$

Keeping the unknowns in LS and simplifying we get the three-moment equation

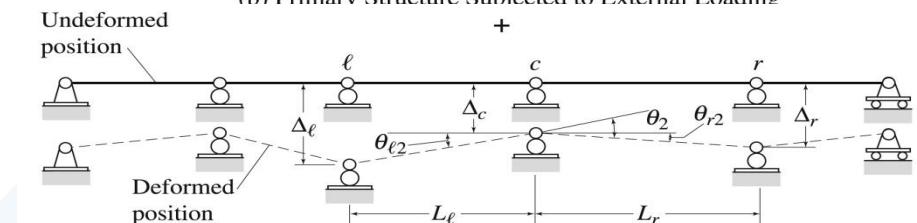
$$\frac{M_l L_l}{I_l} + 2M_c \left(\frac{L_l}{I_l} + \frac{L_r}{I_r} \right) + \frac{M_r L_r}{I_r} = -\sum \frac{P_l L_l^2 k_l (1 - k_l^2)}{I_l} - \sum \frac{P_r L_r^2 k_r (1 - k_r^2)}{I_r} - \frac{w_l L_l^3}{4I_l} - \frac{w_r L_r^3}{4I_r} - 6E \left(\frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r} \right)$$



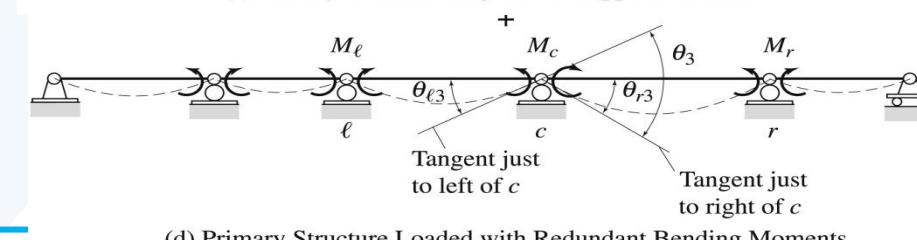
(a) Continuous Beam



(b) Primary Structure Subjected to External Loading



(c) Primary Structure Subjected to Support Settlements



(d) Primary Structure Loaded with Redundant Bending Moments

Derivation of three-Moment Equation

$$\frac{M_l L_l}{I_l} + 2M_c \left(\frac{L_l}{I_l} + \frac{L_r}{I_r} \right) + \frac{M_r L_r}{I_r} = -\sum \frac{P_l L_l^2 k_l (1-k_l^2)}{I_l} - \sum \frac{P_r L_r^2 k_r (1-k_r^2)}{I_r} - \frac{w_l L_l^3}{4I_l} - \frac{w_r L_r^3}{4I_r} - 6E \left(\frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r} \right)$$

If the continuous beam has a constant section ($I_l = I_r = I$) the three-moment equation simplifies to

$$M_l L_l + 2M_c (L_l + L_r) + M_r L_r = -\sum P_l L_l^2 k_l (1-k_l^2) - \sum P_r L_r^2 k_r (1-k_r^2) - \frac{1}{4} (w_l L_l^3 + w_r L_r^3) - 6EI \left(\frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r} \right)$$

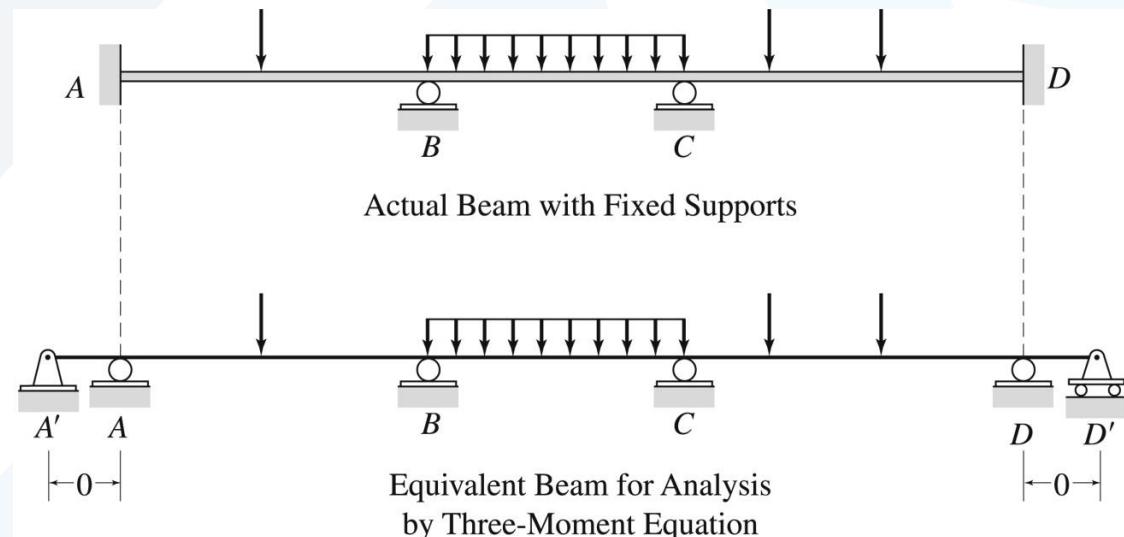
If in addition the two spans are equal ($L_l = L_r = L$) the three-moment equation becomes

$$M_l + 4M_c + M_r = -\sum P_l L k_l (1-k_l^2) - \sum P_r L k_r (1-k_r^2) - \frac{L^2}{4} (w_l + w_r) - \frac{6EI}{L^2} (\Delta_l - 2\Delta_c + \Delta_r)$$

The foregoing forms of the three-moment equation are applicable to any three consecutive supports l , c & r of a continuous beam, provided that there are no discontinuities, such as internal hinges, in the beam between the left support l and the right support r .

Fixed Supports

The above three-moment equations were derived to satisfy comp. equ. at any interior support. For a comp. equ. of zero slope at a fixed end support, the beam can be imaginary modified by adjoining end spans of zero length simply supported as shown



Three-Moment Equation - Examples

Ex-01. Determine the reactions and draw the shear force & bending moment diagrams for the beam shown in the figure by using the three-moment equation

Solution:

The continuous beam ABC has one degree of indeterminacy. It has two different spans of different rigidity. There is no support settlement. So we use the following three-moment equation.

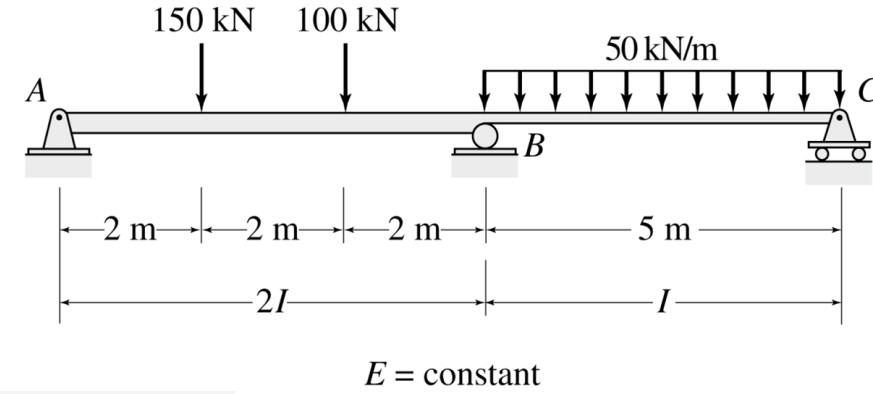
$$\frac{M_l L_l}{I_l} + 2M_c \left(\frac{L_l}{I_l} + \frac{L_r}{I_r} \right) + \frac{M_r L_r}{I_r} = -\sum \frac{P_l L_l^2 k_l (1 - k_l^2)}{I_l} - \sum \frac{P_r L_r^2 k_r (1 - k_r^2)}{I_r} - \frac{w_l L_l^3}{4I_l} - \frac{w_r L_r^3}{4I_r}$$

Considering the support B as the central one with two null moments at A & B , we write

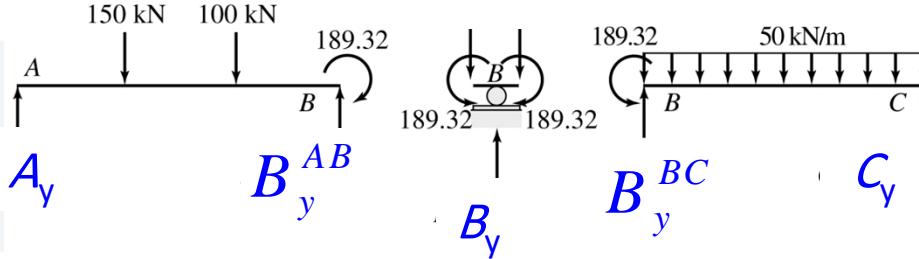
$$0 + 2M_B \left(\frac{6}{2I} + \frac{5}{I} \right) + 0 = -\frac{\{(150)(6)^2(2/6)[1-(2/6)^2] + (100)(6)^2(4/6)[1-(4/6)^2]\}}{2I} - \frac{(50)(5)^3}{4I}$$

Canceling I in both sides and simplifying we get

$$16M_B = -3029.17 \quad \Rightarrow \quad M_B = -189.32 \text{ kN}\cdot\text{m}$$

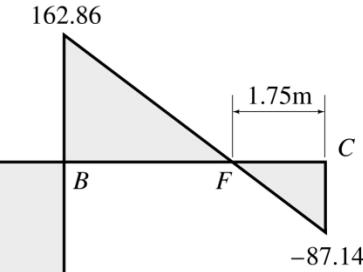
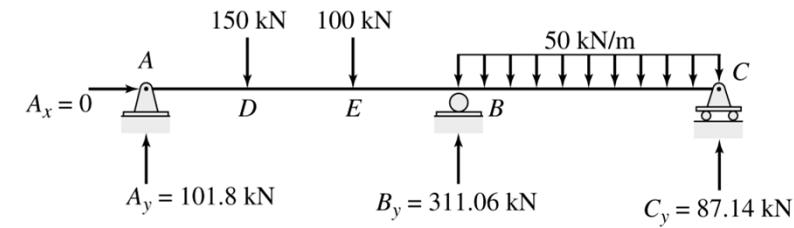
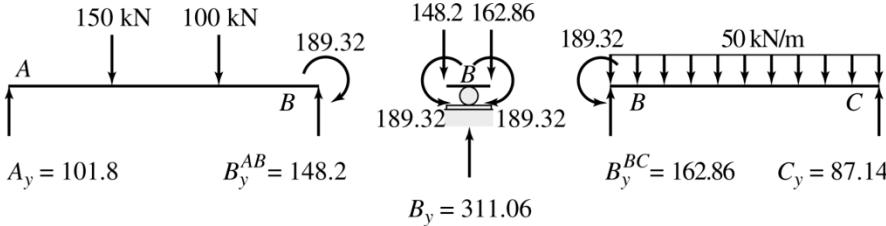


Three-Moment Equation - Examples

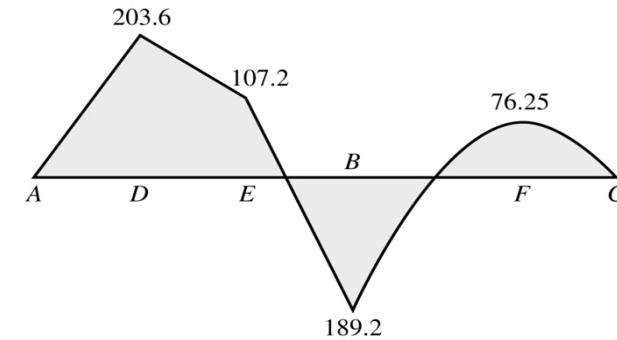


(b) Span End Moments and Shears

$$+\downarrow \sum M_B = 0 \Rightarrow -A_y(6) + 150(4) + 100(2) - 189.32 = 0 \\ \Rightarrow A_y = 101.8 \text{ kN}$$



Shear diagram (k)



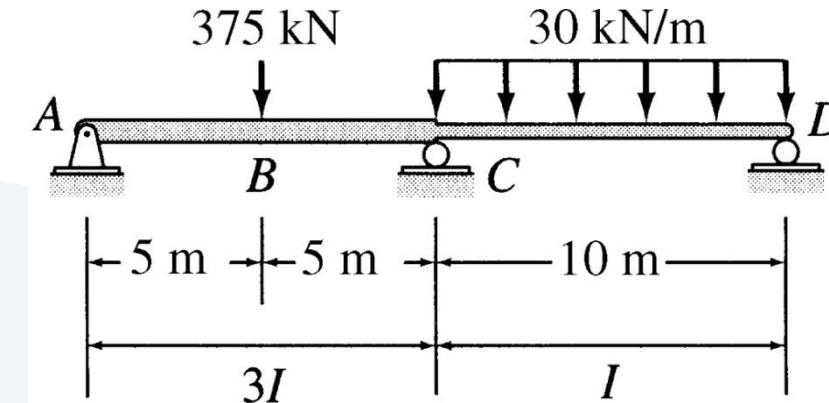
Bending moment diagram (kN-m)

Three-Moment Equation - Examples

Ex. 02. Determine the reaction for the continuous beam ABCD shown in the figure.

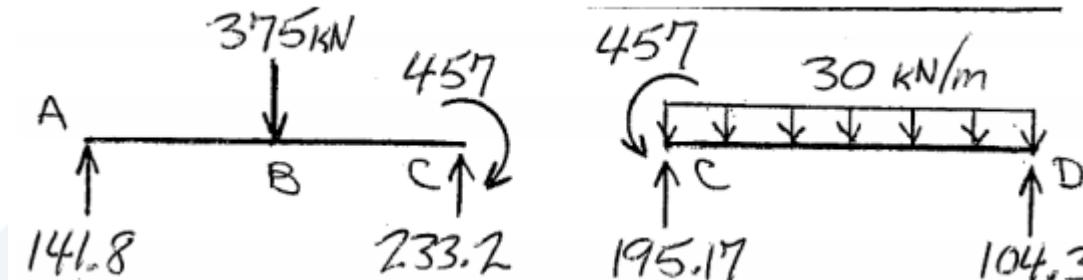
Solution:

The continuous beam *ABCD* has one degree of indeterminacy. It has two spans of different rigidity. There is no support settlement. So we use the following three-moment equation.



$$\frac{M_l L_l}{I_l} + 2M_c \left(\frac{L_l}{I_l} + \frac{L_r}{I_r} \right) + \frac{M_r L_r}{I_r} = - \sum \frac{P_l L_l^2 k_l (1 - k_l^2)}{I_l} - \sum \frac{P_r L_r^2 k_r (1 - k_r^2)}{I_r} - \frac{w_l L_l^3}{4I_l} - \frac{w_r L_r^3}{4I_r}$$

$$M_c = -457 \text{ kN}\cdot\text{m}$$



Three-Moment Equation - Examples

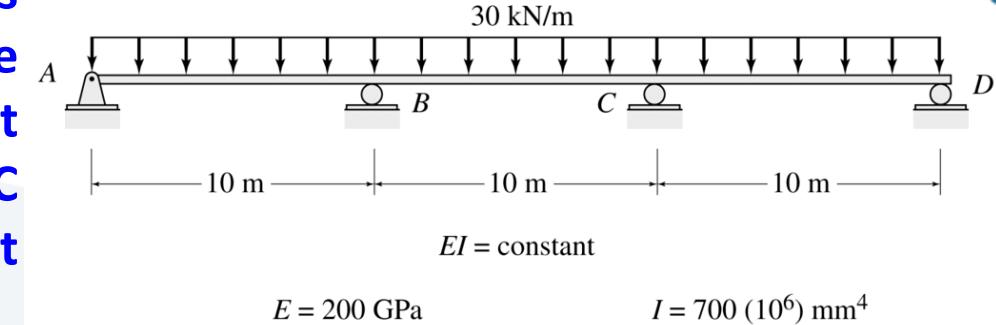
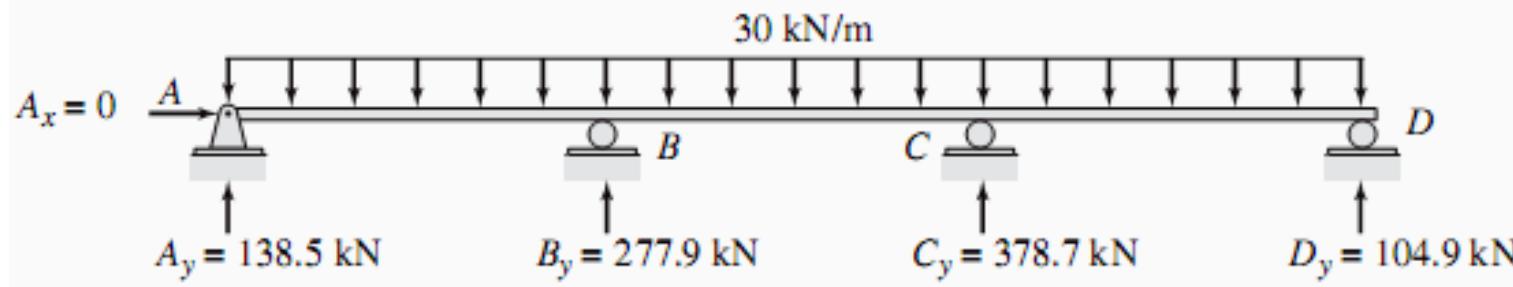
Ex. 03. Determine the reactions for the continuous beam ABCD shown in the figure, due to the uniformly distributed load and due to support settlement of 10 mm at A, 50 mm at B, 20 mm at C and 40 mm at D. Using the three-moment equation

Solution:

$$M_l + 4M_c + M_r = -\sum P_l L k_l (1 - k_l^2) - \sum P_r L k_r (1 - k_r^2) - \frac{L^2}{4} (w_l + w_r) - \frac{6EI}{L^2} (\Delta_l - 2\Delta_c + \Delta_r)$$

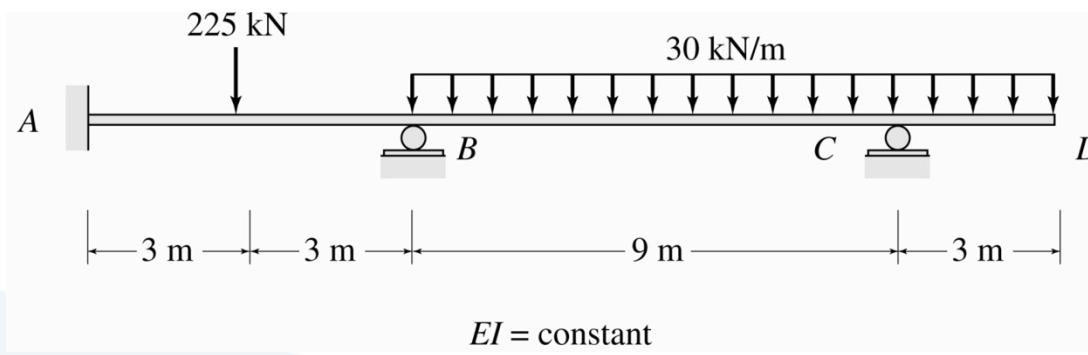
$$M_B = -115.2 \text{ kN-m}$$

$$M_C = -451.2 \text{ kN-m}$$



Ex. 04. Determine the reaction for the continuous beam ABCD shown in the figure,

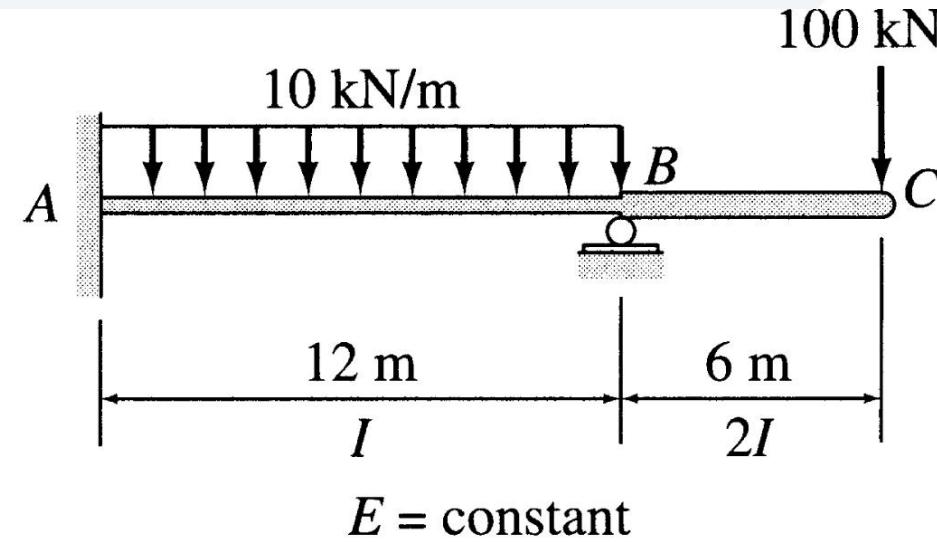
Fixed Support & Extended Cantilever



$$M_l L_l + 2M_c (L_l + L_r) + M_r L_r = -\sum P_l L_l^2 k_l (1 - k_l^2) - \sum P_r L_r^2 k_r (1 - k_r^2) - \frac{1}{4} (w_l L_l^3 + w_r L_r^3) - 6EI \left(\frac{\Delta_l - \Delta_c}{L_l} + \frac{\Delta_r - \Delta_c}{L_r} \right)$$

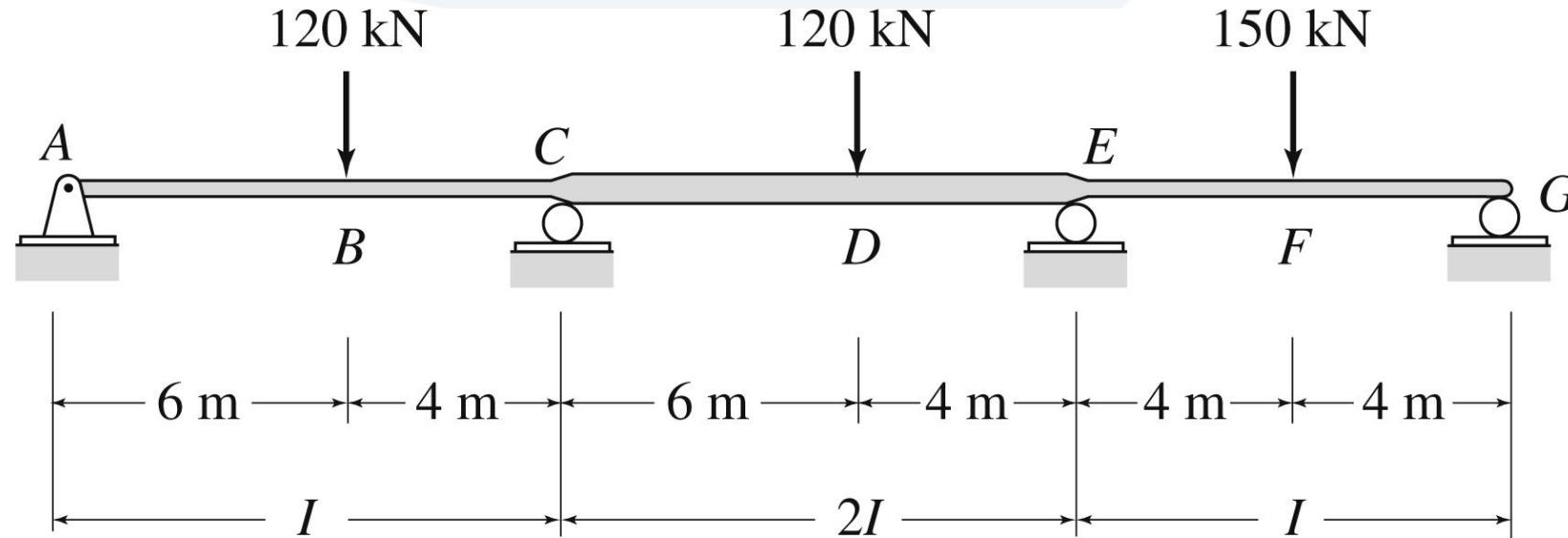
Homework

Pr-02: Determine the reactions and draw shear force & bending moment diagrams for the continuous beam shown in the figure, due to the concentrated loads Using three moment equation method.



Homework

Pr-03: Determine the reactions for the continuous beam shown in the figure, due to the concentrated loads and due to support settlement of 10 mm at A, 65 mm at C, 40 mm at E and 25 mm at G. Using the three-moment equation method.



$$E = 200 \text{ GPa}$$

$$I = 500 (10^6) \text{ mm}^4$$