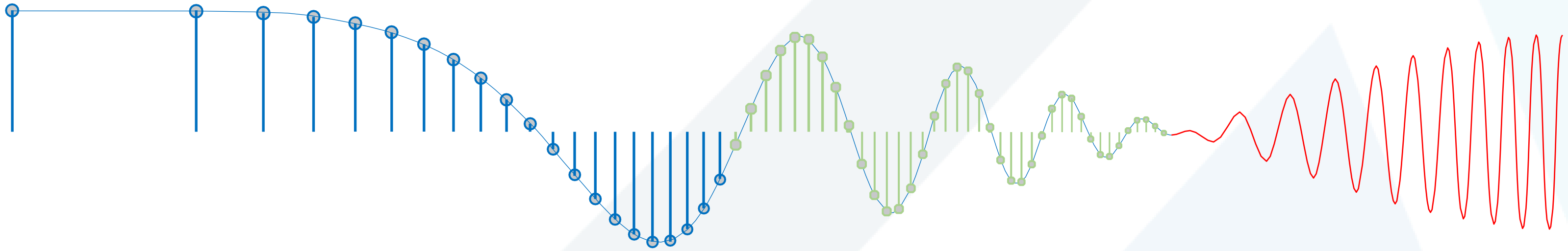
CEDC606: Digital Signal Processing

Lecture Notes 5: Transform analysis of LTI systems



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Chapter 5

Transform analysis of LTI systems

1. Sinusoidal response of LTI systems
2. Fourier representation of continuous-time signals
3. Distortion of signals passing through LTI systems
4. Ideal and practical filters
5. Frequency response for rational transfer functions
6. Allpass systems
7. Invertibility and minimum-phase systems

1. Sinusoidal response of LTI systems

$$x[n] = z^n \xrightarrow{\mathcal{T}} y[n] = H(z)z^n, \text{ all } n \qquad \text{where } H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- **Eigenfunctions of LTI systems** If the system is stable, the ROC of $H(z)$ contains the unit circle. In this case, and for $z = e^{j\omega}$, the result is:

$$x[n] = e^{j\omega n} \xrightarrow{\mathcal{T}} y[n] = H(e^{j\omega})e^{j\omega n}, \text{ all } n$$

where $H(e^{j\omega}) = H(z)\big|_{z=j\omega} = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$

The complex exponentials are the only eigenfunctions of LTI systems. Thus:

$$y[n] = H(e^{j\omega})x[n], \text{ all } n$$

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

$$x[n] = Ae^{j(\omega n+\phi)} \xrightarrow{\mathcal{T}} y[n] = A|H(e^{j\omega})|e^{j[\omega n+\phi+\angle H(e^{j\omega})]}$$

- Therefore, the response of a stable LTI system to a complex exponential sequence is a complex exponential sequence with the same frequency; only the amplitude and phase are changed by the system.

Sinusoidal response of real LTI systems Suppose that the input is a real sinusoidal sequence

$$x[n] = A_x \cos(\omega n + \phi_x) = \frac{A_x}{2} e^{j\phi_x} e^{j\omega n} + \frac{A_x}{2} e^{-j\phi_x} e^{-j\omega n} = x_1[n] + x_2[n]$$

$$y_1[n] = \frac{A_x}{2} |H(e^{j\omega})| e^{j\phi_x} e^{j[\omega n + \angle H(e^{j\omega})]}, \qquad y_2[n] = \frac{A_x}{2} |H(e^{-j\omega})| e^{-j\phi_x} e^{j[-\omega n + \angle H(e^{-j\omega})]}$$

$$y[n] = \frac{A_x}{2} |H(e^{j\omega})| e^{j\phi_x} e^{j[\omega n + \angle H(e^{j\omega})]} + \frac{A_x}{2} |H(e^{-j\omega})| e^{-j\phi_x} e^{j[-\omega n + \angle H(e^{-j\omega})]}$$

If we assume that the impulse response $h[n]$ is real-valued, we have $|H(e^{-j\omega})| = |H(e^{j\omega})|$ and $\angle H(e^{-j\omega}) = -\angle H(e^{j\omega})$. Hence:

$$y[n] = A_x|H(e^{j\omega})|\cos[\omega n + \phi_x + \angle H(e^{j\omega})] = A_y\cos(\omega n + \phi_y)$$

where $A_y = A_x|H(e^{j\omega})|, \quad \phi_y = \phi_x + \angle H(e^{j\omega})$

- Note: The quantity $|H(e^{j\omega})|$ is known as the magnitude response or gain of the system, and ϕ_y is called the phase response of the system.
- Example 1: A stable system described by the first-order difference equation

$$y[n] = ay[n-1] + bx[n], \quad -1 < a < 1$$

The frequency response function is: $H(e^{j\omega}) = \dfrac{b}{1 - ae^{-j\omega}}$

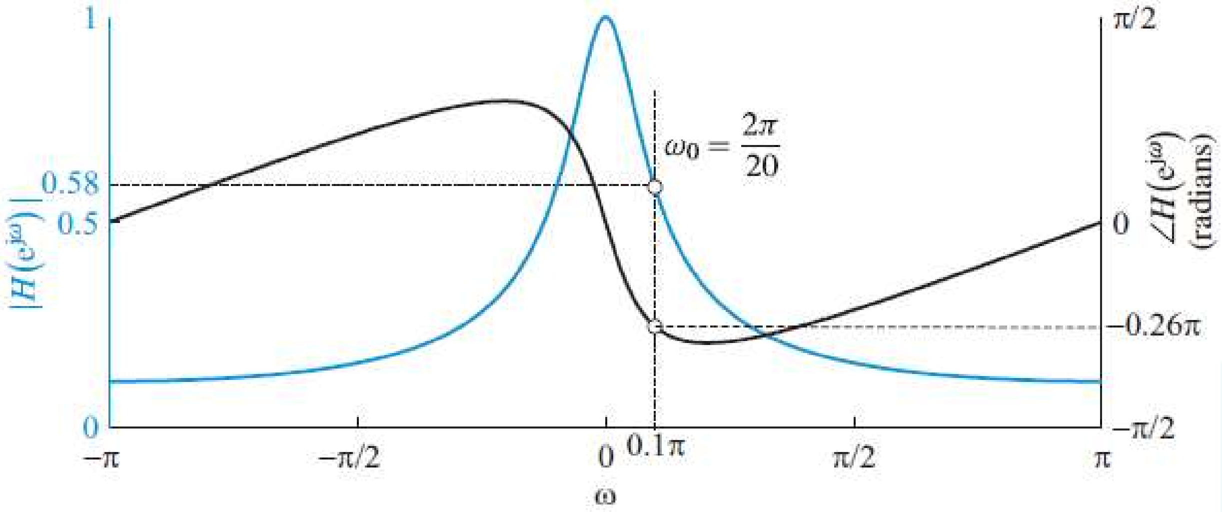
$$|H(e^{j\omega})| = \frac{|b|}{\sqrt{1 - 2a\cos(\omega) + a^2}}, \quad \angle H(e^{j\omega}) = \angle b - \tan^{-1}\frac{a\sin(\omega)}{1 - a\cos(\omega)}$$

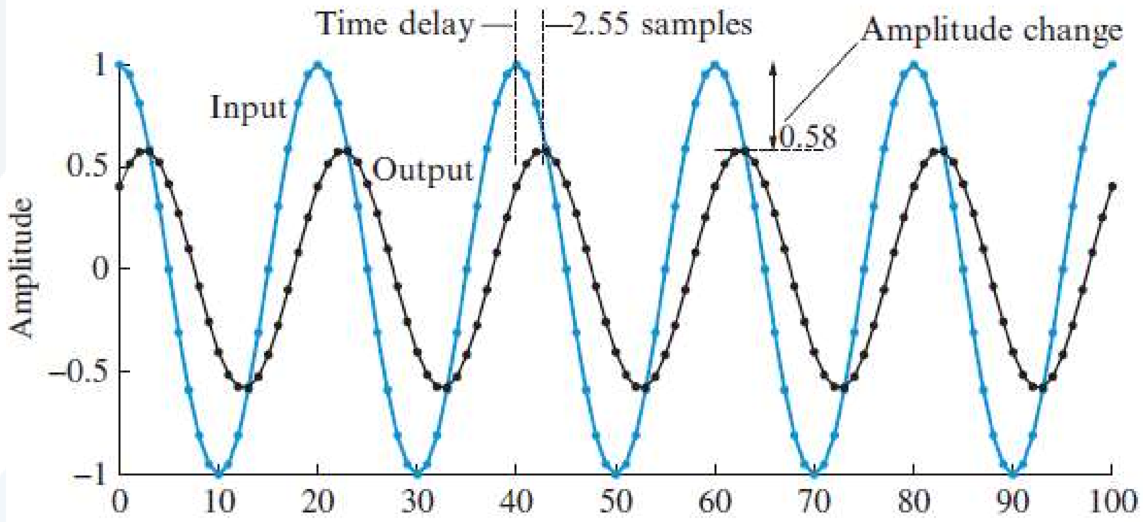
Figure below shows plots of magnitude and phase response functions for $a = 0.8$ and an input-output pair for the frequency $\omega_0 = 2\pi/20$. ($x[n] = \cos(0.1\pi n)$)

Continuous and principal phase functions The phase angle of any complex number is not uniquely defined, since any integer multiple of 2π can be added.

When the phase is numerically computed with the use of an arctangent subroutine, the principal value is typically obtained.

We will denote the principal value of the phase of $H(e^{j\omega})$ as $\text{Arg}[H(e^{j\omega})]$, where $-\pi < \text{Arg}[H(e^{j\omega})] \leq \pi$.





- If the phase response exceeds the limits of $(-\pi .. \pi]$, the function Arg $[H(e^{j\omega})]$ is discontinuous. We refer to Arg$[H(e^{j\omega})]$ as the "wrapped" phase.
- The continuous (unwrapped) phase function is denoted as $\Psi(\omega)$ or arg $[H(e^{j\omega})]$. $\angle H(e^{j\omega})$ is used to denote the phase response function of a system, in general.

Steady-state and transient response The eigenfunction property holds if the input sequence $x[n]$ is a complex exponential and sinusoidal that exists over the entire interval $-\infty < n < \infty$. In such a case, the response that we observe at the output of the LTI system is the steady-state response. There is no transient response in this case.

- However, in practice every input starts at a finite time. Consider a complex exponential starting at time $n = 0$, that is, $x[n] = e^{j\omega n}u[n]$, the response of a causal system to the causal input $x[n]$ is:

$$y[n]=\sum_{k=0}^{n}h[k]x[n-k]=\sum_{k=0}^{n}h[k]e^{j\omega(n-k)}=\left(\sum_{k=0}^{\infty}h[k]e^{-j\omega k}\right)e^{j\omega n}-\left(\sum_{k=n+1}^{\infty}h[k]e^{-j\omega k}\right)e^{j\omega n}$$

$$y[n]=\underbrace{H(e^{j\omega})e^{j\omega n}}_{y_{ss}[n]}-\underbrace{\left(\sum_{k=n+1}^{\infty}h[k]e^{-j\omega k}\right)e^{j\omega n}}_{y_{tr}[n]}$$

If the system is stable, we have: $|y_{tr}[n]|\le\sum_{k=n+1}^{\infty}|h[k]|\le\sum_{k=0}^{\infty}|h[k]|<\infty$

For large values of n the transient response of a stable system decays towards zero, that is, $\lim_{n\to\infty}y[n]=H(e^{j\omega})e^{j\omega n}=y_{ss}[n]$

- Note: In practice, the eigenfunction property holds after the transient response has diminished.

- Example 2: A stable system described by the first-order difference equation

$$y[n] = ay[n-1] + x[n]$$

This system's response to any input $x[n]$ applied at $n = 0$ is given as:

$$y[n] = y[-1]a^{n+1} + \sum_{k=0}^{n} a^k x[n-k], \quad \text{for } n \geq 0$$

Now, let the input to the system is the complex exponential $x[n] = Ae^{j\omega n}$, $n \geq 0$

$$y[n] = y[-1]a^{n+1} + A\sum_{k=0}^{n} a^k e^{j\omega(n-k)} = y[-1]a^{n+1} + A\left[\sum_{k=0}^{n} (ae^{-j\omega})^k\right]e^{j\omega n}, \quad \text{for } n \geq 0$$

$$y[n] = y[-1]a^{n+1} - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1 - ae^{-j\omega}}e^{j\omega n} + \frac{A}{1 - ae^{-j\omega}}e^{j\omega n}, \quad \text{for } n \geq 0$$

The system is BIBO stable if $|a| < 1$. The two terms involving $a^{n+1} \to 0$ as $n \to \infty$.

$$y_{tr}[n] = y[-1]a^{n+1} - \frac{Aa^{n+1}e^{-j\omega(n+1)}}{1 - ae^{-j\omega}}e^{j\omega n}, \quad \text{for } n \geq 0$$

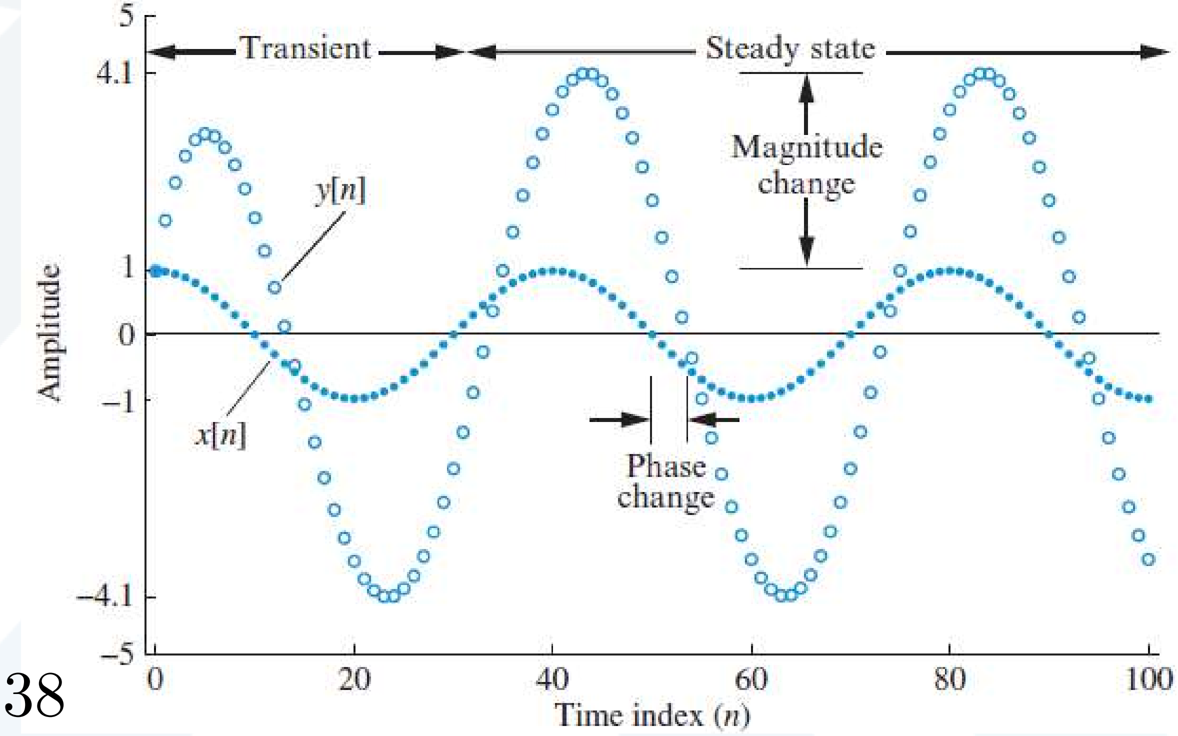
$$y_{ss}[n] = \lim_{n\to\infty} y[n] = \frac{A}{1-ae^{-j\omega}} e^{j\omega n} = AH(e^{j\omega})e^{j\omega n}$$

- Example 3: Acausal and stable system described by the first-order difference equation: $y[n] = 0.8y[n-1] + x[n]$.

  The response $y[n]$ of the system to the input: $x[n] = \cos(0.05\pi n)u[n]$:

$$y[n] = \underbrace{-2.515(0.8)^n u[n]}_{y_{tr}[n]} + \underbrace{4.093\cos(0.05\pi n - 0.538)u[n]}_{y_{ss}[n]}$$

$$|H(e^{j0.05\pi})| = 4.093, \quad \angle H(e^{j0.05\pi}) = -0.538$$



2. Response of LTI systems in the frequency domain

Response to periodic inputs

- Consider a periodic input $x[n] = x[n+N]$ with fundamental period N. Then $x[n]$ can be expressed as a sum of complex exponentials using the IDTFS:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} c_k^{(x)} e^{j\frac{2\pi}{N}kn}$$

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k^{(x)} e^{j\frac{2\pi}{N}kn} \xrightarrow{\mathcal{T}} \tilde{y}[n] = \sum_{k=0}^{N-1} \tilde{c}_k^{(x)} H(j\frac{2\pi}{N}kn) e^{j\frac{2\pi}{N}kn} = \sum_{k=0}^{N-1} \tilde{c}_k^{(y)} e^{j\frac{2\pi}{N}kn}$$

$$\tilde{c}_k^{(y)} = \tilde{c}_k^{(x)} H(j\tfrac{2\pi}{N}kn), \qquad -\infty < k < \infty$$

Therefore, the response of an LTI system to a periodic input sequence is a periodic sequence with the same fundamental period.

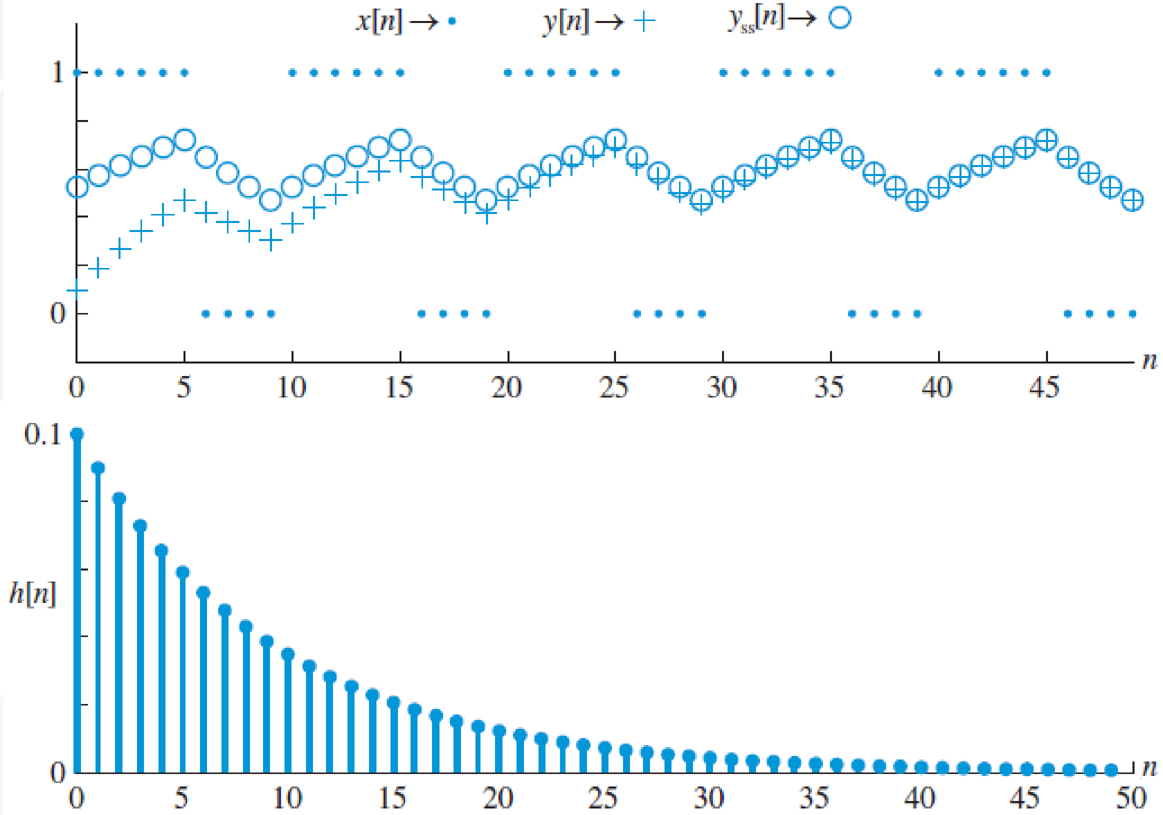
$$\mathcal{P}_y = \frac{1}{N}\sum_{n=0}^{N-1}|\tilde{y}[n]|^2 = \sum_{k=0}^{N-1}\left|\tilde{c}_k^{(y)}\right|^2 = \sum_{k=0}^{N-1}\left|H(j\tfrac{2\pi}{N}kn)\right|^2\left|\tilde{c}_k^{(x)}\right|^2$$

- Example 4: Zero-state and steady-state responses

$$y[n] = 0.9y[n-1] + 0.1x[n], \quad y[-1] = 0$$

The system is excited by a periodic sequence, with $N = 10$, given by:

$$x[n] = \begin{cases} 1, & 0 \le 0 < 6 \\ 0, & 6 \le n < 10 \end{cases}$$



Response to aperiodic inputs

- The convolution theorem provides the desired frequency-domain relationship for determining the output of an LTI system to an aperiodic finite-energy signal. $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$
- A finite-energy aperiodic signal contains a continuum of frequency components.
- The LTI system, through its frequency response function, attenuates some frequency components of the input signal and amplifies others.
- If the input signal spectrum is changed by the system in an undesirable way, we say that the system has caused magnitude and phase distortion.
- The output of a linear time-invariant system cannot contain frequency components that are not contained in the input signal.

- It takes either a linear time-variant system or a nonlinear system to create frequency components that are not necessarily contained in the input signal.
- Figure below illustrates the time-domain and frequency-domain relationships that can be used in the analysis of BIBO-stable LTI systems.

$$y[n] = \mathcal{F}^{-1}\{Y(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})\, e^{j\omega n} d\omega$$

$$\mathcal{E}_y = \sum_{n=-\infty}^{\infty} |y[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 \left|H(e^{j\omega})\right|^2 d\omega$$

$$\xrightarrow[X(e^{j\omega})]{x[n]} \boxed{\begin{array}{c} LTI\ system \\ h[n],\ H(e^{j\omega}) \end{array}} \xrightarrow[Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})]{y[n] = x[n] * h[n]}$$

3. Distortion of signals passing through LTI systems

Distortionless response systems A system has distortionless response if the input signal $x[n]$ and the output signal $y[n]$ have the same "shape." This is possible if $x[n]$ and $y[n]$ satisfy the condition: $y[n] = Gx[n - n_d],\ G > 0$

where G and n_d are constants

$$Y(e^{j\omega}) = Ge^{-j\omega n_d} X(e^{j\omega}) \Rightarrow H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = Ge^{-j\omega n_d}$$

$$|H(e^{j\omega})| = G, \quad \angle H(e^{j\omega}) = -\omega n_d$$

- For a LTI system to have a distortionless response, the magnitude response $|H(e^{j\omega})|$ must be a constant and the phase response $\angle H(e^{j\omega})$ must be a linear function of ω (pass through the origin $\omega = 0$) with slope $-n_d$, where n_d is the delay of the output with respect to the input.

Magnitude distortion A system introduces magnitude distortion if $|H(e^{j\omega})| \neq G$.

- Note: Systems without magnitude distortion are known as allpass systems.

Phase or delay distortion If the phase response is not a linear function of frequency, that is, $\angle H(e^{j\omega}) \neq -\omega n_d$.

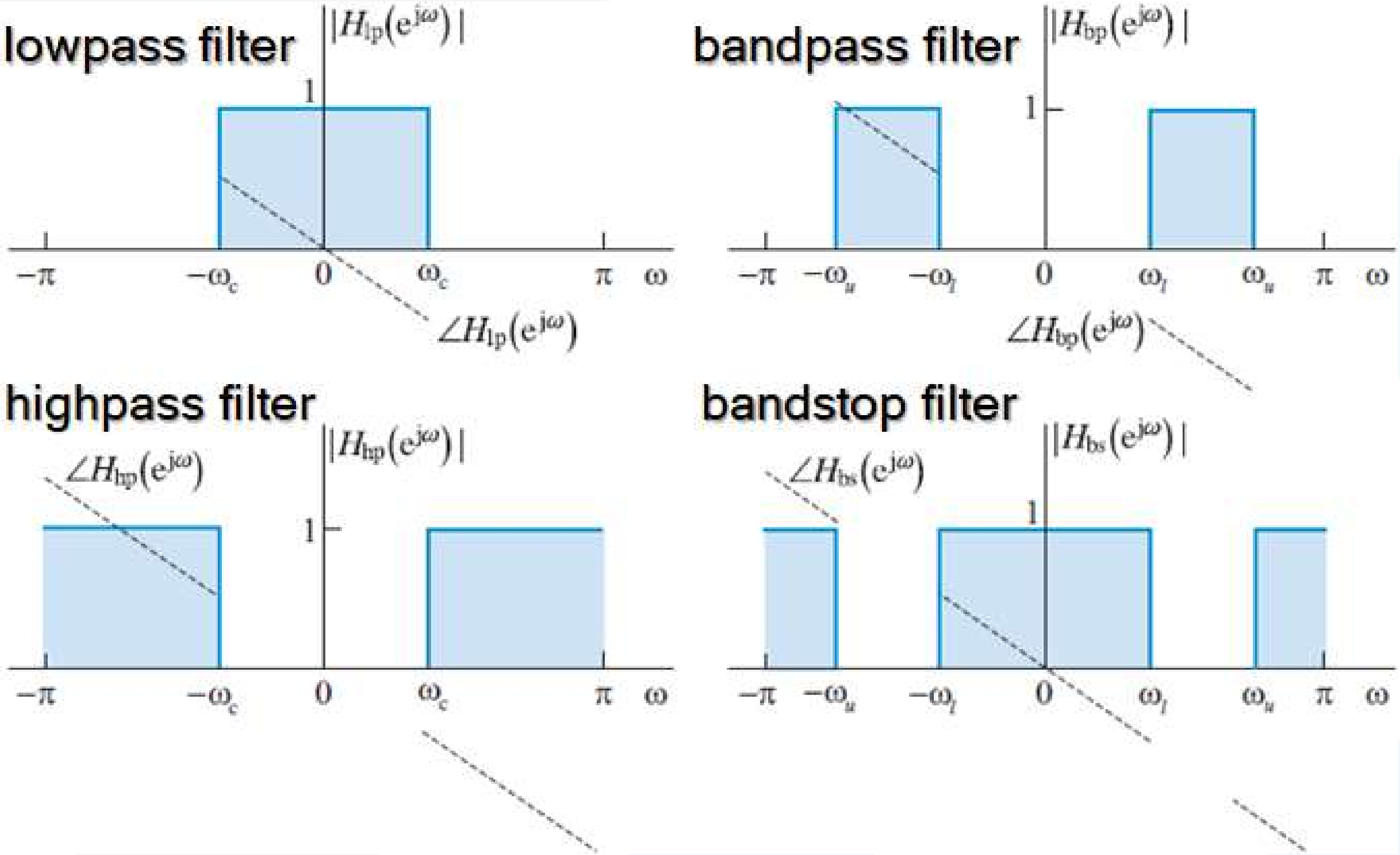
- Note: The derivative of the phase with respect to ω has the units of delay.

Group delay A convenient way to check the linearity of phase response is to use the group delay, defined as the negative of the slope of the phase as follows:

$$\tau_{gd}(\omega) = -\frac{d\Psi(\omega)}{d\omega}$$

- The derivative in this definition requires that the phase response is a continuous function of frequency. Therefore, to compute the group delay, we should use the unwrapped phase response $\Psi(\omega)$.
- We interpret $\tau_{gd}(\omega)$ as the time delay that a signal component of frequency ω undergoes as it passes from the input to the output of the system.
- Note that when $\Psi(\omega) = -\omega n_d$ is linear, $\tau_{gd}(\omega) = n_d =$ constant. In this case all frequency components of the input signal undergo the same time delay.

4. Ideal and practical filters

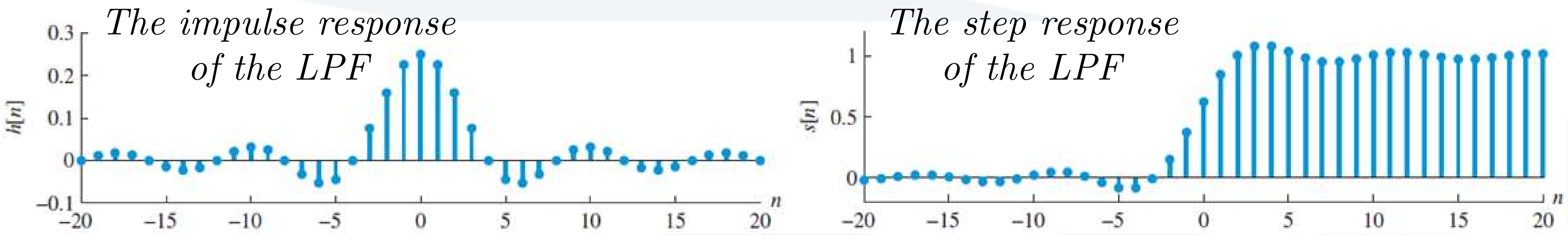


- Filters are usually classified according to their frequency-domain characteristics as lowpass, highpass, bandpass, and bandstop filters.
- These ideal filters have a constant-gain passband characteristic and zero gain in their stopband, and a linear phase response.
- The parameters ω_l and ω_u, which specify the end points of the passband, are called the lower and upper cutoff frequencies.
- The bandwidth of the filter, defined as the width of the passband: $\Delta\omega = \omega_u - \omega_l$.
- An ideal lowpass filter with $\omega_l = 0$, whereas an ideal highpass filter has $\omega_u = \pi$.

For example, an ideal lowpass filter (LPF) with linear phase is defined by:

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c = \omega_u \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \qquad h_{lp}[n] = \frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}$$

- The impulse response and the step response of the ideal lowpass filter are illustrated in the figure below for $n_d = 0$. ($h_{lp}[n]$ extends from $-\infty$ to ∞)



- The impulse response $h_{lp}[n]$ has a DTFT $H_{lp}(e^{j\omega})$ because it has finite energy. However, $h_{lp}[n]$ is not absolutely summable, that is, $\sum_{n=-\infty}^{\infty} \left|h_{lp}[n]\right| = \infty$.
- Therefore, the LPF is unstable. Furthermore, since $r^{-n}h_{lp}[n]$ is not absolutely summable for any value of r, the sequence $h_{lp}[n]$ does not have a z-transform. In conclusion, the ideal lowpass filter is unstable and practically unrealizable.

- Note: The existence of the DTFT does not always imply the existence of the z-transform.
- The impulse response of the ideal bandpass filter can be obtained by modulating the impulse response of an ideal lowpass filter with $\omega_c = (\omega_u - \omega_l)/2 = \Delta\omega/2$ using a carrier with frequency $\omega_0 = (\omega_u + \omega_l)/2$. The result is:

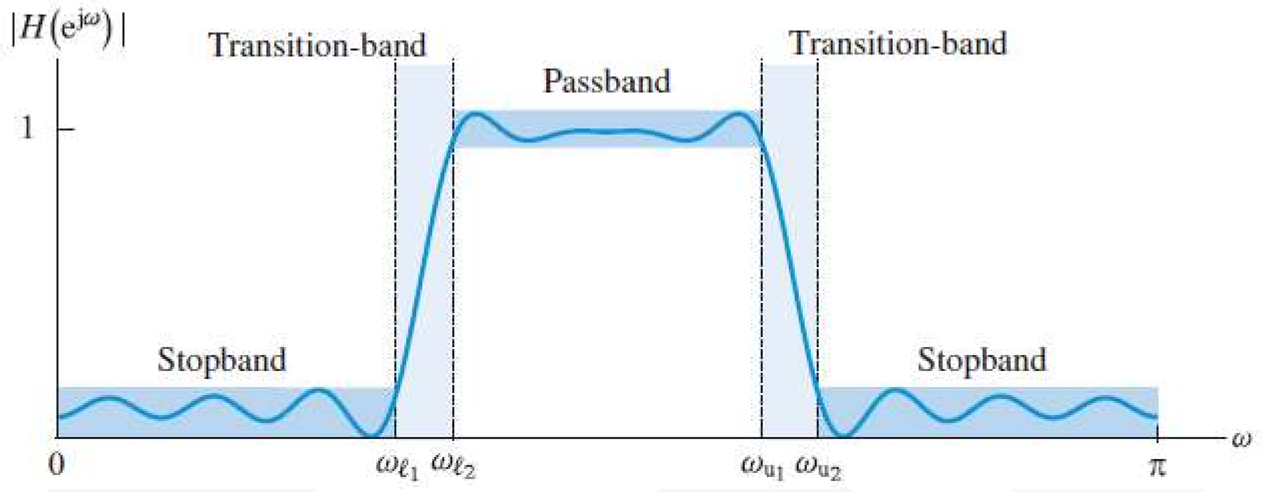
$$h_{lp}[n] = 2\frac{\sin\omega_c(n-n_d)}{\pi(n-n_d)}\cos\omega_0 n$$

- The impulse responses of the ideal highpass and bandstop filters are given by:

$$h_{hp}[n] = \delta[n] - h_{lp}[n], \qquad h_{bs}[n] = \delta[n] - h_{bp}[n],$$

- Note: all ideal filters are unstable and unrealizable. Since all ideal filters can be expressed in terms of ideal lowpass filter $H_{lp}(e^{j\omega})$ referred as prototype filter.

- Since ideal filters are not realizable in practice, they must be approximated by practical (nonideal) filters. This is usually done by minimizing some approximation error between the nonideal filter and a prototype ideal filter.
- A natural question: how to evaluate the quality of a practical filter.
- A good filter should have only a small ripple in the passband, high attenuation in the stopband, and very narrow transition bands.
- In some applications, the specifications of phase or time-domain characteristics (for ex., the overshoot of the step response) are also important.



Typical characteristics of a practical bandpass filter

5. Frequency response for rational transfer functions

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \Rightarrow H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

For a stable system, the system function converges on the unit circle.

$$H(e^{j\omega}) = \left.\frac{B(z)}{A(z)}\right|_{z=e^{j\omega}} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}}$$

$$H(e^{j\omega}) = b_0 \left.\frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}\right|_{z=e^{j\omega}} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})}$$

Each of these first-order terms can be expressed as $C(\omega) = (1 - \alpha e^{j\beta} e^{-j\omega})$.

$$|C(\omega)|^2 = 1 + \alpha^2 - 2\alpha\cos(\omega - \beta), \quad \angle C(\omega) = \tan^{-1}\frac{\alpha\sin(\omega - \beta)}{1 - \alpha\cos(\omega - \beta)}$$

$$\tau_{gd}(\omega) = -\frac{d\Psi(\omega)}{d\omega} = \frac{\alpha^2 - \alpha\cos(\omega - \beta)}{1 + \alpha^2 - 2\alpha\cos(\omega - \beta)}$$

Expressing the zeros and poles in polar notation as $z_k = q_k e^{j\theta_k}$ and $p_k = r_k e^{j\phi_k}$

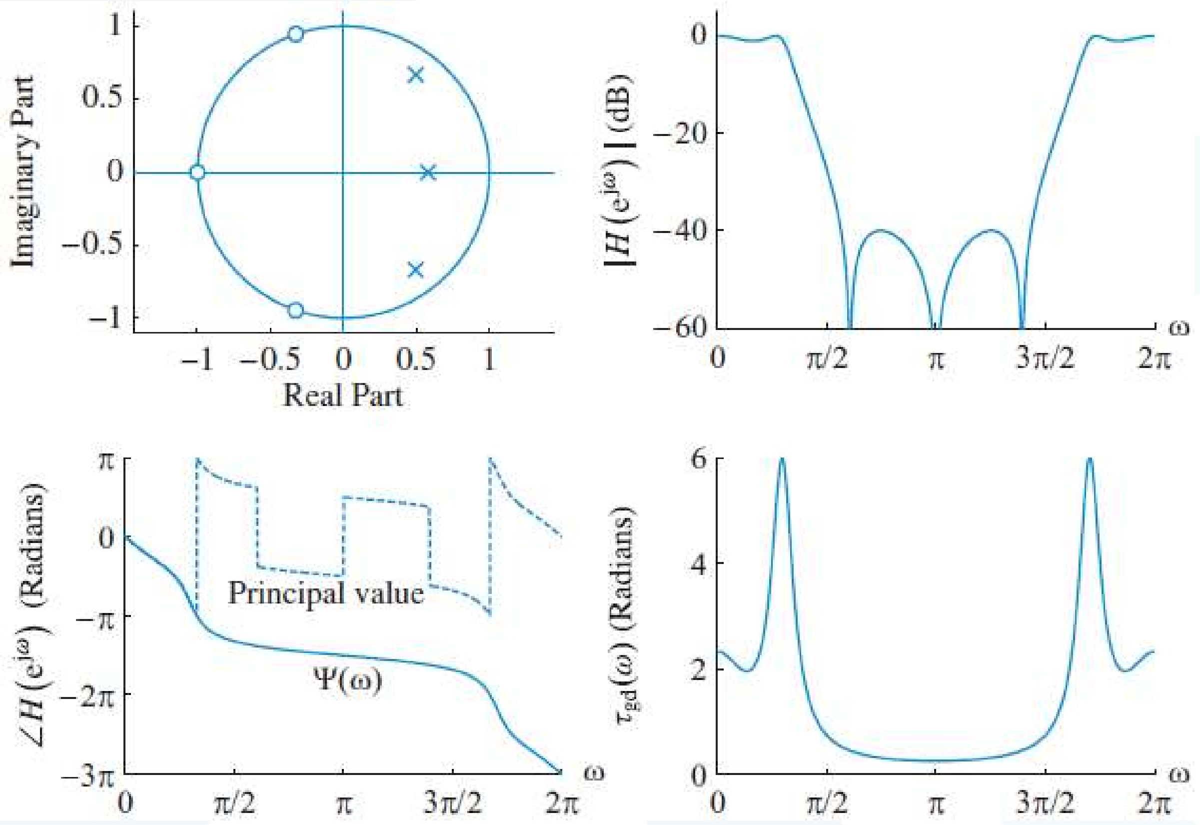
$$|H(e^{j\omega})| = |b_0| \frac{\prod_{k=1}^{M}\sqrt{1 + q_k^2 - 2q_k\cos(\omega - \theta_k)}}{\prod_{k=1}^{N}\sqrt{1 + r_k^2 - 2r_k\cos(\omega - \phi_k)}}$$

$$\angle H(e^{j\omega}) = \angle b_0 + \sum_{k=1}^{M}\tan^{-1}\frac{q_k\sin(\omega - \theta_k)}{1 - q_k\cos(\omega - \theta_k)} - \sum_{k=1}^{N}\tan^{-1}\frac{r_k\sin(\omega - \phi_k)}{1 - r_k\cos(\omega - \phi_k)}$$

$$\tau_{gd}(\omega) = \sum_{k=1}^{N}\frac{r_k^2 - r_k\cos(\omega - \phi_k)}{1 + r_k^2 - 2r_k\cos(\omega - \phi_k)} - \sum_{k=1}^{M}\frac{q_k^2 - q_k\cos(\omega - \theta_k)}{1 + q_k^2 - 2q_k\cos(\omega - \theta_k)}$$

Figure below shows the pole-zero plot, the magnitude response, the phase response (principal value and continuous function), and the group delay of the system:

$$H(z)=\frac{1+1.655z^{-1}+1.665z^{-2}+z^{-3}}{1-1.57z^{-1}+1.264z^{-2}-0.4z^{-3}}$$



The principal phase value jumps by a multiple of 2π when $|\Psi(\omega)|>\pi$. This explains the 2π jumps at the first and last discontinuities. The remaining three discontinuities of size π result from sign reversals due to the real zero at $\omega=\pi$ and the complex conjugate zeros at $\omega=\pm 3\pi/5$.

6. Allpass systems

- The frequency response of an allpass system has constant magnitude ($G > 0$) at all frequencies, that is, $|H(e^{j\omega})| = G$.
- The simplest allpass systems simply scale and delay the input signal: $H_{ap}(z) = Gz^{-k}$.
- A non-trivial family of allpass systems (dispersive allpass systems) can be obtained by noting that $(1 - p_k e^{-j\omega})$ and its complex conjugate $(1 - p_k^* e^{j\omega})$ have the same magnitude.

$$H_k(z) = z^{-1} \frac{1 - p_k^* z}{1 - p_k z^{-1}} = \frac{z^{-1} - p_k^*}{1 - p_k z^{-1}}$$

- Note: The term z^{-1}, has been introduced to make the system causal.
- Higher order allpass systems can be obtained by cascading multiple first-order sections, as:

$$H_{ap}(z) = e^{j\beta}\prod_{k=1}^{N}\frac{z^{-1}-p_k^*}{1-p_k z^{-1}}$$ where β is a constant (usually $\beta = 0$).

- Note: Parallel connection of allpass systems is, in general, not allpass. For ex., the systems $H_1(z) = 1$ and $H_2(z) = z^{-1}$ are allpass but $H(z) = H_1(z) + H_2(z) = 1 + z^{-1}$ is not allpass because $|H(e^{j\omega})| = 2\cos(\omega/2)$.
- We note that each pole p_k of an allpass system should be accompanied by a complex reciprocal zero $1/p_k^*$.
- Since causal and stable allpass systems must have all poles inside the unit circle, all zeros are outside the unit circle.
- The magnitude, phase, and group-delay responses of a first-order allpass system with $p_k = r_k e^{j\phi_k}$, are given by:

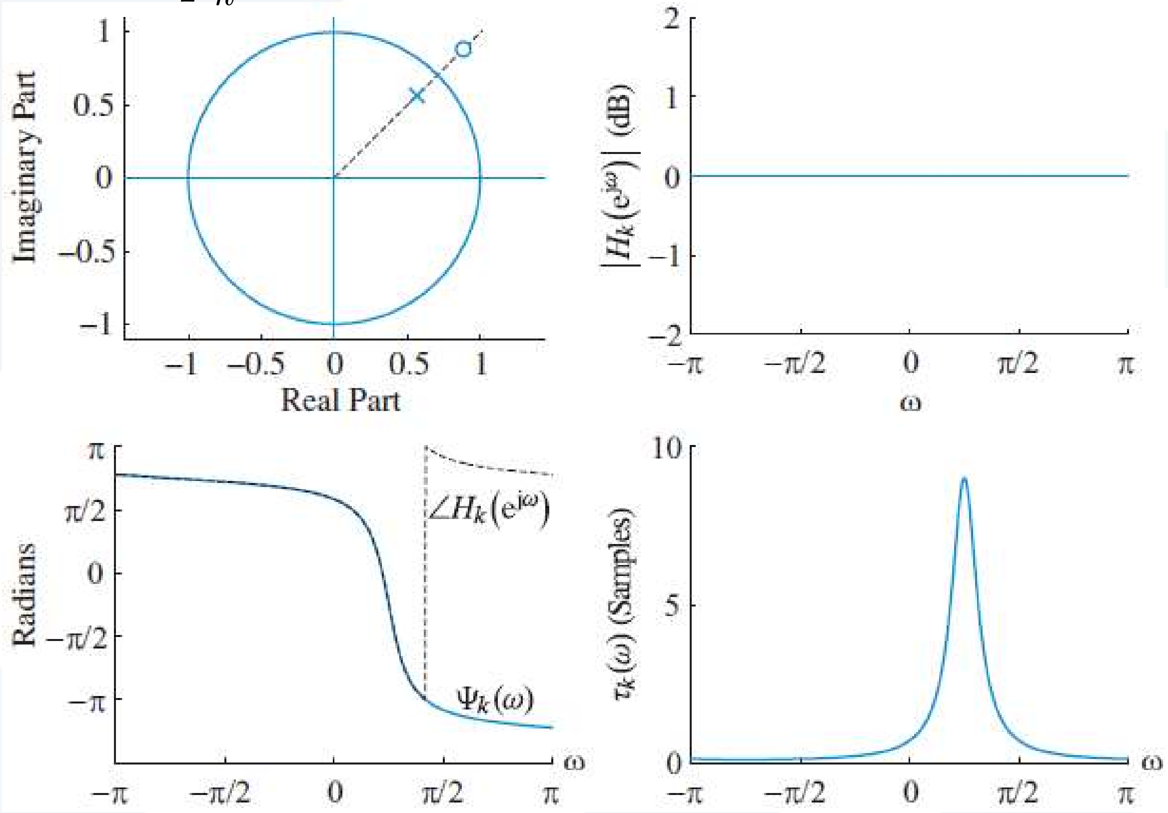
$$\left|H_k(e^{j\omega})\right| = 1, \qquad \angle H_k(e^{j\omega}) = -\omega - 2\tan^{-1}\frac{r_k \sin(\omega - \phi_k)}{1 - r_k\cos(\omega - \phi_k)}$$

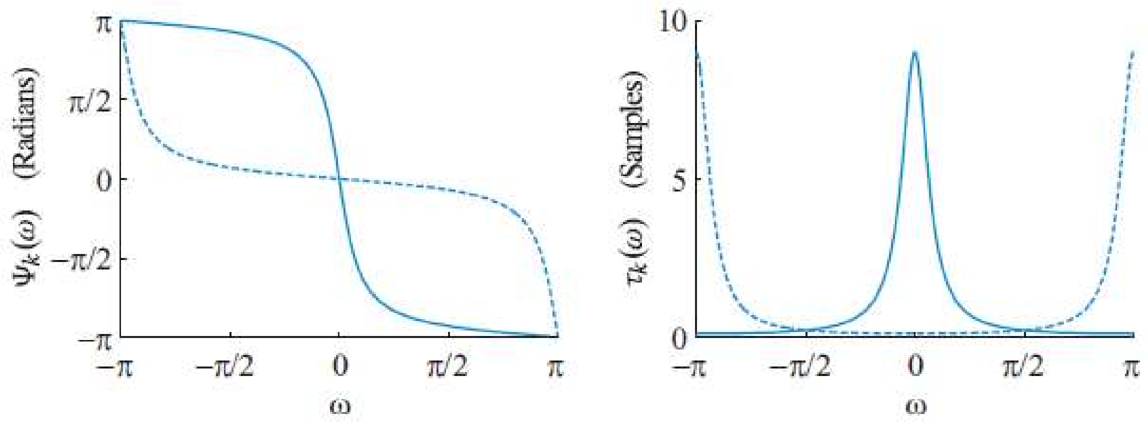
$$\tau_k(\omega) = \frac{1 - r_k^2}{1 + r_k^2 - 2r_k\cos(\omega - \phi_k)}$$

For a causal and stable system,

$r_k < 1 \Rightarrow \tau_k(\omega) \geq 0.$

$p_k = 0.8\, e^{j\pi/4}$





$p = 0.8$ (solid line), $p = -0.8$ (dashed line)

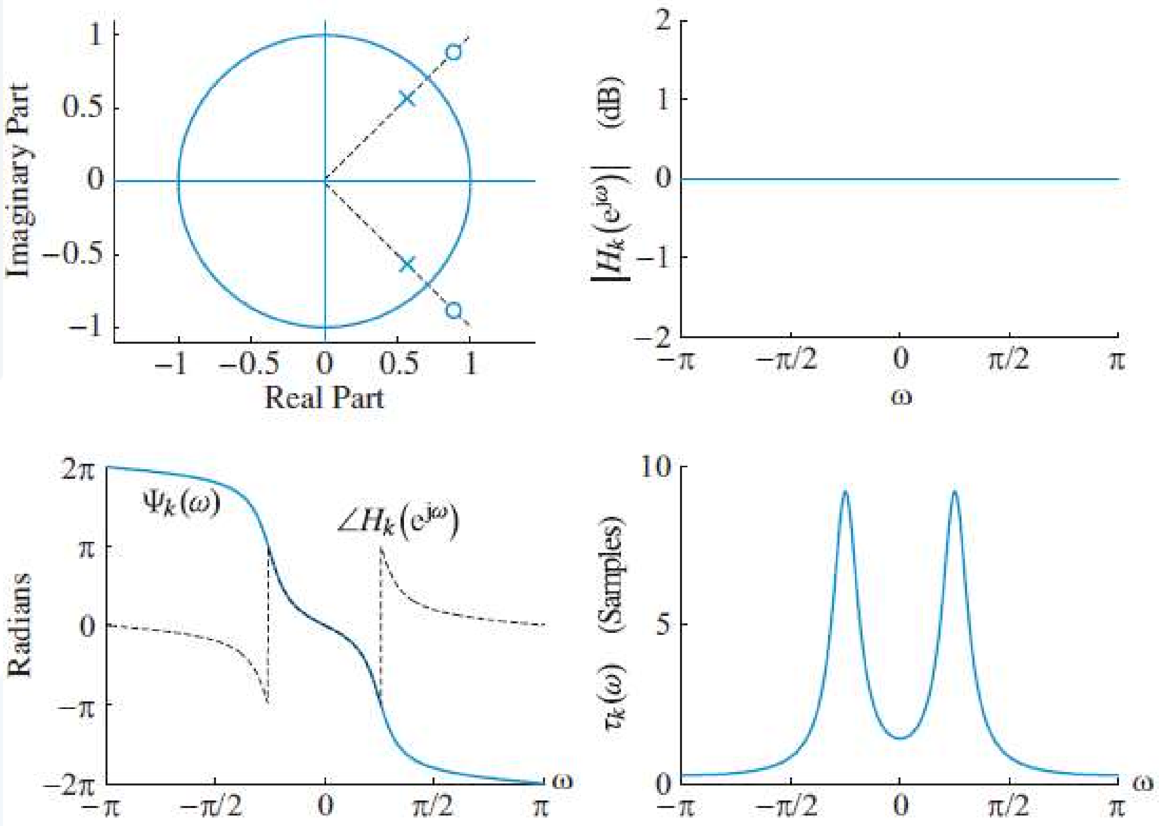
- A system function of a second-order allpass system, with $\beta = 0$ and $N = 2$, can be expressed as:

$$H_{ap}(z) = \frac{a_2^* + a_1^* z^{-1} + z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$= z^{-2} \frac{1 + a_1^* z + a_2^* z^2}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

where $a_1 = -(p_1 + p_2)$ and $a_2 = p_1 p_2$.

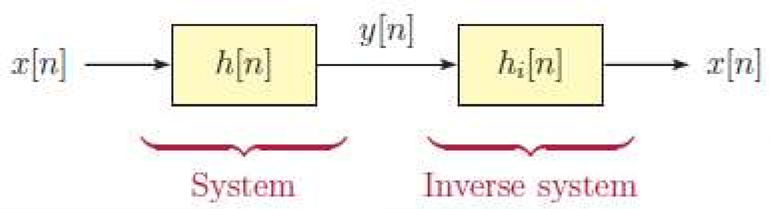
$p_k = 0.8 e^{\pm j\pi/4}$



- Note: For real and complex conjugate poles, the phase response has odd symmetry about $\omega = 0$ and the group delay has even symmetry about $\omega = 0$.

7. Invertibility and minimum-phase systems

- An LTI system $H(z)$ with input $x[n]$ and output $y[n]$ is said to be invertible if we can uniquely determine $x[n]$ from $y[n]$.
- The system $H_i(z)$ that produces $x[n]$ when excited by $y[n]$ is called the inverse system. $h[n] * h_i[n] = \delta[n] \Rightarrow H(z)H_i(z) = 1$



For rational system function: $H_i(z) = \dfrac{1}{H(z)} = \dfrac{A(z)}{B(z)}$

- Example 5: Inverse system

Determine the inverse of the system with impulse response $h[n] = \left(\frac{1}{2}\right)^n u[n]$

$$H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{2}$$ This system is both causal and stable

$$H_i(z) = 1 - \tfrac{1}{2} z^{-1} \Rightarrow h_i[n] = \delta[n] - \tfrac{1}{2}\delta[n-1]$$

- Example 6: Inverse system

Determine the inverse of the system with $h[n] = \delta[n] - \frac{1}{2}\delta[n-1]$

$$H(z) = 1 - \tfrac{1}{2}z^{-1}, \text{ ROC: } |z| > 0 \Rightarrow H_i(z) = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}$$ Two possible ROCs

$|z| > \frac{1}{2} \Rightarrow h_i[n] = \left(\frac{1}{2}\right)^n u[n]$ This system is both causal and stable

$|z| < \frac{1}{2} \Rightarrow h_i[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$ This system is anticausal and unstable

Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

- A causal and stable system $H(z)$ should have its poles inside the unit circle; its zeros can be anywhere.
- A causal and stable LTI system with a causal and stable inverse is known as a minimum-phase system.

- Thus, the impulse response sequences $h[n]$ and $h_i[n]$ should be causal and absolutely summable. Sometimes, to allow for poles or zeros on the unit circle, we only require for the impulse responses to have finite energy.
- A rational system is minimum-phase if both its poles and zeros are inside the unit circle.

Minimum phase and allpass decomposition We shall now show that any system with a rational transfer function can be decomposed into a minimum-phase system and an allpass system.

- We demonstrate the validity of this assertion for the class of causal and stable systems.
- Suppose that $H(z)$ has one zero $z = 1/a^*$, where $|a| < 1$ outside the unit circle, and all other poles and zeros are inside the unit circle. Then:

$$H(z) = H_1(z)\left(z^{-1} - a^*\right)$$ where, by definition, $H_1(z)$ is minimum phase

$$H(z) = H_1(z)\left(1 - az^{-1}\right)\frac{z^{-1} - a^*}{1 - az^{-1}}$$

where $H_1(z)(1 - az^{-1})$ is minimum phase and $(z^{-1} - a^*)/(1 - az^{-1})$ is allpass

- If we repeat this process for every zero outside the unit circle, we obtain:

$$H(z) = H_{min}(z)H_{ap}(z)$$

- Example 7: Minimum-phase/allpass decomposition $H(z) = \dfrac{1 + 5z^{-1}}{1 + \frac{1}{2}z^{-1}}$

$$H(z) = 5\frac{z^{-1} + \frac{1}{5}}{1 + \frac{1}{2}z^{-1}} = 5\frac{z^{-1} + \frac{1}{5}}{1 + \frac{1}{2}z^{-1}}\frac{1 + \frac{1}{5}z^{-1}}{1 + \frac{1}{5}z^{-1}} = \underbrace{5\frac{1 + \frac{1}{5}z^{-1}}{1 + \frac{1}{2}z^{-1}}}_{H_{min}(z)}\underbrace{\frac{z^{-1} + \frac{1}{5}}{1 + \frac{1}{5}z^{-1}}}_{H_{ap}(z)}$$

- Note: a quick way to obtain the minimum-phase system is by replacing each factor $(1 + az^{-1})$, where $|a| > 1$, by a factor of the form $a(1 + \frac{1}{a} z^{-1})$.
- Based on the decomposition of a nonminimum-phase system, we can express the group delay of $H(z)$ as: $\tau_g(\omega) = \tau_g^{min}(\omega) + \tau_g^{ap}(\omega)$

  Since $\tau_g^{ap}(\omega) \geq 0$ for $0 \leq \omega \leq \pi$, it follows that $\tau_g(\omega) \geq \tau_g^{min}(\omega)$, $0 \leq \omega \leq \pi$.

  We conclude that among all pole-zero systems having the same magnitude response, the minimum-phase system has the smallest group delay.

Maximum- and mixed-phase systems A causal and stable system with a rational TF is called maximum phase if all its zeros are outside the unit circle.

- A system with arbitrary $H(z)$ is maximum phase, if it is causal, stable, and $H(z) \neq 0$ for $|z| < 1$.

- A system that is neither minimum phase nor maximum phase is called a mixed-phase system.

- Example 8: Minimum-, maximum-, and mixed-phase systems

  Consider a minimum-phase system with transfer function:

$$H_{min}(z) = (1 - az^{-1})(1 - bz^{-1}) = 1 - (a+b)z^{-1} + abz^{-2} \quad \text{where } -1 < a,\ b < 1$$

  This system has two zeros inside the unit circle at $z = a$, $z = b$. If we only reflect one zero outside the unit circle, we obtain the following mixed-phase systems:

$$H_{mix1}(z) = a(1 - a^{-1}z^{-1})(1 - bz^{-1}), \qquad H_{mix2}(z) = b(1 - az^{-1})(1 - b^{-1}z^{-1})$$

  Reflecting both zeros outside the unit circle yields the maximum-phase system:

$$H_{max}(z) = ab(1 - a^{-1}z^{-1})(1 - b^{-1}z^{-1}) = ab - (a+b)z^{-1} + z^{-2}$$

$$H_{max}(z) = z^{-2}H_{min}(1/z)$$

Which illustrates how the zeros of $H_{min}(z)$, which are inside the unit circle, are reflected outside the unit circle to become the zeros of $H_{max}(z)$.

From figure below we observe that:

- All systems have the same magnitude response,
- The minimum (maximum) phase system has the minimum (maximum) group delay, and
- The group delay of mixed-phase systems are between those of minimum- and maximum-phase systems.