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State Space Model and State Equation

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State Space Representations of Linear Physical Systems

Introduction

As systems become more complex, representing them with differential equations or transfer functions becomes cumbersome. This is even more true if the system has multiple inputs and outputs.The state space representation of a system replaces an n^{th} order differential equation with a single first order *matrix* differential equation. The state space representation of a system is given by two equations :

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{u}(t)$$

The first equation is called the state equation, the second equation is called the output equation. For an n^{th} order system (i.e., it can be represented by an n^{th} order differential equation) with *r* inputs and *m* outputs the size of each of the matrices is as follows:

q is $n{\times}1$ (n rows by 1 column); **q** is called the state vector, it is a function of time

A is $n{\times}n$; **A** is the state matrix, a constant

B is $n{\times}r$; **B** is the input matrix, a constant

u is $r{\times}1$; **u** is the input, a function of time

C is $m{\times}n$; **C** is the output matrix, a constant

D is $m{\times}r$; **D** is the direct transition matrix, a constant

y is $m{\times}1$; **y** is the output, a function of time

Note several features:

- The state equation has a single first order derivative of the state vector on the left, and the state vector, **q**(t), and the input **u**(t) on the right. There are no derivatives on the right hand side.
- The output equation has the output on the left, and the state vector, **q**(t), and the input **u**(t) on the right. There are no derivatives on the right hand side.

Example

Consider an 4^{th} order system represented by a single 4^{th} order differential equation with input x and output y.

$$\ddddot{y} + a_1\dddot{y} + a_2\ddot{y} + a_3\dot{y} + a_4 y = b_0 x$$

We can define 4 new variables, q1 through q4.

$$q_1 = y$$
$$q_2 = \dot{q}_1 = \dot{y}$$
$$q_3 = \dot{q}_2 = \ddot{y}$$
$$q_4 = \dot{q}_3 = \dddot{y}, \quad \text{so}$$
$$\ddddot{y} + a_1 q_4 + a_2 q_3 + a_3 q_2 + a_4 q_1 = b_0 x$$

$$\dot{q}_4 = \ddddot{y}, \quad \text{so}$$
$$\ddddot{y} = \dot{q}_4 = -a_4 q_1 - a_3 q_2 - a_2 q_3 - a_1 q_4 + b_0 x$$

We can now rewrite the 4th order differential equation as 4 first order equations

$$\dot{q}_1 = q_2 = \dot{y}$$

$$\dot{q}_2 = q_3 = \ddot{y}$$

$$\dot{q}_3 = q_4 = \dddot{y}$$

$$\dot{q}_4 = -a_4 q_1 - a_3 q_2 - a_2 q_3 - a_1 q_4 + b_0 x$$

This is compactly written in state space format as

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}u \qquad \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = \mathbf{C}\mathbf{q} + \mathrm{D}u \qquad y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

The State Transition Matrix

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where for two or more simultaneous differential equations, A and C are 2×2 or higher order matrices, and **B** and **D** are column vectors with two or more rows. In this section we will introduce the *state transition matrix* e^{At}, and we will prove that the solution of the matrix differential equation

$$\dot{x} = Ax + Bu \quad \text{with initial conditions} \quad x(t_0) = x_0$$

is obtained from the relation

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

Proof:

Let A be any $n \times n$ matrix whose elements are constants. Then, another $n \times n$ matrix denoted as $\varphi(t)$, is said to be the state transition matrix of $\dot{x} = Ax + Bu$ if it is related to the matrix A as the matrix power series

$$\varphi(t) \equiv e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \ldots + \frac{1}{n!}A^nt^n$$

where I is the $n \times n$ identity matrix.

$$\varphi(0) = e^{A0} = I + A0 + \ldots = I$$

$$\varphi'(t) = \frac{d}{dt}e^{At} = 0 + A \cdot 1 + A^2t + \ldots = A + A^2t + \ldots$$

$$\frac{d}{dt}e^{At} = Ae^{At}$$

The initial condition is satisfied from the relation

$$x(t_0) = e^{A(t_0 - t_0)}x_0 + e^{At_0}\int_{t_0}^{t_0} e^{-A\tau}Bu(\tau)d\tau = e^{A0}x_0 + 0 = Ix_0 = x_0$$

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$\dot{x}(t) = Ae^{A(t-t_0)}x_0 + Ae^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau + e^{At}e^{-At}Bu(t)$$

$$\dot{x}(t) = A\left[e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau\right] + e^{At}e^{-At}Bu(t)$$

$$\dot{x} = Ax + Bu$$

In summary, if A is an $n \times n$ matrix whose elements are constants, $n \geq 2$, and b is a column vector with n elements, the solution of

$$\dot{x} = Ax + Bu$$

with initial condition

$$x_0 = x(t_0)$$

is

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

Computation of the State Transition Matrix

Let A be an $n \times n$ matrix, and I be the $n \times n$ identity matrix. By definition, the *eigenvalues* λ_i, $i = 1, 2, \ldots, n$ of A are the roots of the nth order polynomial

$$\det[A - \lambda I] = 0$$

Evaluation of the state transition matrix e^{At} is based on the *Cayley–Hamilton theorem*. This theorem states that a matrix can be expressed as an $(n-1)$th degree polynomial in terms of the matrix A as

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1}$$

where the coefficients a_i are functions of the eigenvalues λ.

Distinct Eigenvalues

If $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \ldots \neq \lambda_n$, that is, if all eigenvalues of a given matrix A are distinct, the coefficients a_i are found from the simultaneous solution of the following system of equations:

$$
\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \ldots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}
$$

Example

Compute the state transition matrix e^{At} given that $A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

Solution:

We must first find the eigenvalues λ of the given matrix A. These are found from the expansion of

$$\det[A - \lambda I] = 0$$

$$\det[A - \lambda I] = \det\left\{\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = \det\begin{bmatrix} -2-\lambda & 1 \\ 0 & -1-\lambda \end{bmatrix} = 0$$

$$= (-2-\lambda)(-1-\lambda) = 0$$

$$\lambda_1 = -1 \text{ and } \lambda_2 = -2$$

Since A is a 2×2 matrix

$$e^{At} = a_0 I + a_1 A$$

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t} \qquad a_0 + a_1(-1) = e^{-t}$$

$$a_0 + a_1\lambda_2 = e^{\lambda_2 t} \qquad a_0 + a_1(-2) = e^{-2t}$$

Simultaneous solution

$$a_0 = 2e^{-t} - e^{-2t}$$

$$a_1 = e^{-t} - e^{-2t}$$

$$e^{At} = (2e^{-t} - e^{-2t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{-t} - e^{-2t})\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-2t} & e^{-t} - e^{-2t} \\ 0 & e^{-t} \end{bmatrix}$$

In summary, we compute the state transition matrix e^{At} for a given matrix A using the following procedure:

1. We find the eigenvalues λ from $\det[A - \lambda I] = 0$. We can write $[A - \lambda I]$ at once by subtracting λ from each of the main diagonal elements of A. If the dimension of A is a 2×2 matrix, it will yield two eigenvalues; if it is a 3×3 matrix, it will yield three eigenvalues, and so on.

2. If the dimension of A is a 2×2 matrix, we use only the first *2* terms of the right side of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1}$$

If A matrix is a 3×3 matrix, we use the first 3 terms

3. We obtain the a_i coefficients from

$$
\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
a_0 + a_1\lambda_2 + a_2\lambda_2^2 + \ldots + a_{n-1}\lambda_2^{n-1} &= e^{\lambda_2 t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}
$$

We use as many equations as the number of the eigenvalues, and we solve for the coefficients a_i .

4. We substitute the a_i coefficients into the state transition matrix

Example

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} 5 & 7 & -5 \\ 0 & 4 & -1 \\ 2 & 8 & -3 \end{bmatrix}$$

Solution:

1. We first compute the eigenvalues from $\det[A - \lambda I] = 0$. We obtain $[A - \lambda I]$ at once, by subtracting λ from each of the main diagonal elements of A. Then,

$$\det[A - \lambda I] = \det\begin{bmatrix} 5-\lambda & 7 & -5 \\ 0 & 4-\lambda & -1 \\ 2 & 8 & -3-\lambda \end{bmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

We will use MATLAB **roots(p)** function to obtain the roots

```
p=[1 -6 11 -6];
Lambda=roots(p)
```

and thus the eigenvalues are

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3$$

2. Since A is a 3×3 matrix, we use the first 3 terms

$$e^{At} = a_0 I + a_1 A + a_2 A^2$$

3. We obtain the coefficients a_0, a_1, and a_2 from

$$\begin{aligned} a_0 + a_1\lambda_1 + a_2\lambda_1^2 &= e^{\lambda_1 t} \\ a_0 + a_1\lambda_2 + a_2\lambda_2^2 &= e^{\lambda_2 t} \\ a_0 + a_1\lambda_3 + a_2\lambda_3^2 &= e^{\lambda_3 t} \end{aligned} \qquad \begin{aligned} a_0 + a_1 + a_2 &= e^{t} \\ a_0 + 2a_1 + 4a_2 &= e^{2t} \\ a_0 + 3a_1 + 9a_2 &= e^{3t} \end{aligned}$$

```
syms t;
B=[1 1 1; 1 2 4; 1 3 9];
b=[exp(t); exp(2*t); exp(3*t)];
a=inv(B)*b;
disp('a0 = ');
disp(a(1));
disp('a1 = ');
disp(a(2));
disp('a2 = ');
disp(a(3))
```

```
a0 =
exp(3*t) - 3*exp(2*t) + 3*exp(t)

a1 =
4*exp(2*t) - (3*exp(3*t))/2 - (5*exp(t))/2

a2 =
exp(3*t)/2 - exp(2*t) + exp(t)/2
```

4. We also use MATLAB to perform the substitution into the state transition matrix, and to perform the matrix multiplications.

```
syms t;
a0 = 3*exp(t)+exp(3*t)-3*exp(2*t);
a1 = -5/2*exp(t)-3/2*exp(3*t)+4*exp(2*t);
a2 = 1/2*exp(t)+1/2*exp(3*t)-exp(2*t);
A = [5 7 -5; 0 4 -1; 2 8 -3];
eAt=a0*eye(3)+a1*A+a2*A^2
```

```
eAt =
[-2*exp(t)+2*exp(2*t)+exp(3*t),   -6*exp(t)+5*exp(2*t)+exp(3*t),
4*exp(t)-3*exp(2*t)-exp(3*t)]
[-exp(t)+2*exp(2*t)-exp(3*t),   -3*exp(t)+5*exp(2*t)-exp(3*t),
2*exp(t)-3*exp(2*t)+exp(3*t)]
[-3*exp(t)+4*exp(2*t)-exp(3*t),  -9*exp(t)+10*exp(2*t)-exp(3*t),
6*exp(t)-6*exp(2*t)+exp(3*t)]
```

Thus,

$$e^{At} = \begin{bmatrix} -2e^{t}+2e^{2t}+e^{3t} & -6e^{t}+5e^{2t}+e^{3t} & 4e^{t}-3e^{2t}-e^{3t} \\ -e^{t}+2e^{2t}-e^{3t} & -3e^{t}+5e^{2t}-e^{3t} & 2e^{t}-3e^{2t}+e^{3t} \\ -3e^{t}+4e^{2t}-e^{3t} & -9e^{t}+10e^{2t}-e^{3t} & 6e^{t}-6e^{2t}+e^{3t} \end{bmatrix}$$

Complex Eigenvalues

Example

Compute the state transition matrix e^{At} given that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\lambda_1 = +j \text{ and } \lambda_2 = -j$$

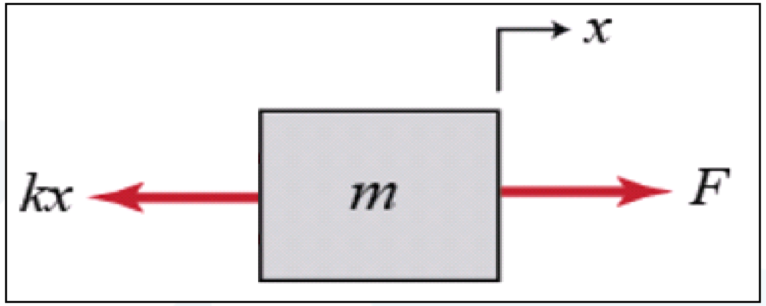
$$\begin{aligned} e^{jt} &= \cos(t) + j\sin(t) = \alpha_0 + \alpha_1 j \\ e^{-jt} &= \cos(t) - j\sin(t) = \alpha_0 - \alpha_1 j, \end{aligned}$$

$$\alpha_0 = \cos(t) \text{ and } \alpha_1 = \sin(t)$$

$$e^{\mathbf{A}t} = \cos(t)\mathbf{I} + \sin(t)\mathbf{A} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t)$$

m=1;
k=1;
F=0.1;



$$F(t) - kx = m\ddot{x}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$x(0) = x_0 = 0.01 \text{ m}$$
$$\dot{x}(0) = v_0 = 0 \text{ m/s}$$

$$\lambda_1 = +j \text{ and } \lambda_2 = -j$$

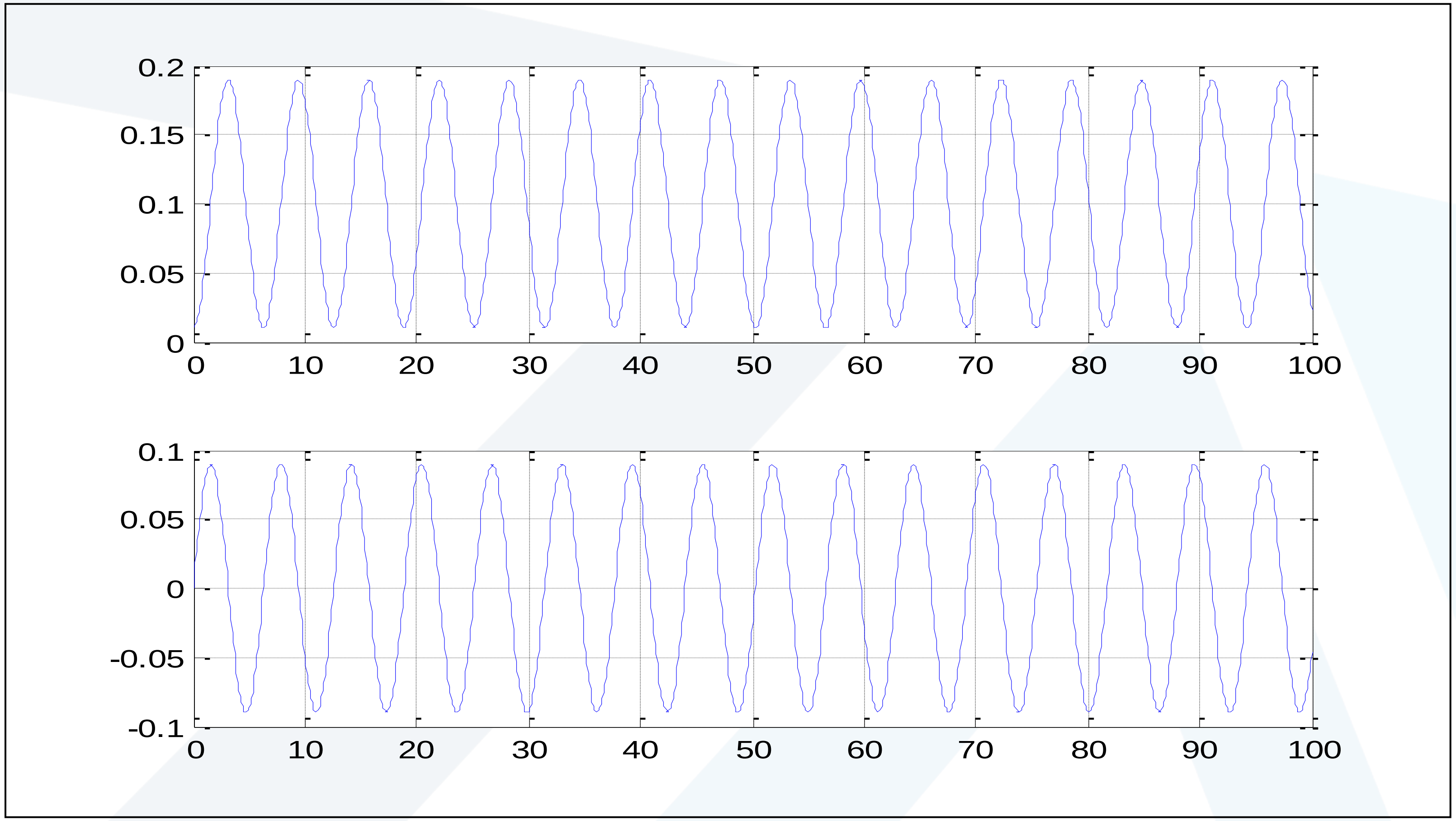
$$e^{\mathbf{A}t} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$$

$$x(t) = e^{A(t-t_0)}x_0 + e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$x(t)=\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} 0.01 \\ 0 \end{bmatrix}+\begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}\int_{t_0}^{t}\begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}0.1\, d\tau$$

x1=0.1-0.09*cos(t);
x2=v=0.09*sin(t);

```
t=0:0.1:100;
x=0.1-0.09*cos(t);
v=0.09*sin(t);
subplot(211);
plot(t,x);
grid
subplot(212);
plot(t,v)
grid
```



Multiple (Repeated) Eigenvalues

In this case, we will assume that the polynomial of $\det[A - \lambda I] = 0$ has n roots, and m of these roots are equal. In other words, the roots are

$$\lambda_1 = \lambda_2 = \lambda_3 \ldots = \lambda_m, \ \lambda_{m+1}, \ \lambda_n$$

The coefficients a_i of the state transition matrix

$$e^{At} = a_0 I + a_1 A + a_2 A^2 + \ldots + a_{n-1} A^{n-1}$$

are found from the simultaneous solution of the system of equations of

$$
\begin{aligned}
a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1} &= e^{\lambda_1 t} \\
\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d}{d\lambda_1}e^{\lambda_1 t} \\
\frac{d^2}{d\lambda_1^2}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^2}{d\lambda_1^2}e^{\lambda_1 t} \\
&\ldots \\
\frac{d^{m-1}}{d\lambda_1^{m-1}}(a_0 + a_1\lambda_1 + a_2\lambda_1^2 + \ldots + a_{n-1}\lambda_1^{n-1}) &= \frac{d^{m-1}}{d\lambda_1^{m-1}}e^{\lambda_1 t} \\
a_0 + a_1\lambda_{m+1} + a_2\lambda_{m+1}^2 + \ldots + a_{n-1}\lambda_{m+1}^{n-1} &= e^{\lambda_{m+1} t} \\
&\ldots \\
a_0 + a_1\lambda_n + a_2\lambda_n^2 + \ldots + a_{n-1}\lambda_n^{n-1} &= e^{\lambda_n t}
\end{aligned}
$$

Example

Compute the state transition matrix e^{At} given that

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

Solution:

1. We first find the eigenvalues λ of the matrix A and these are found from the polynomial of $\det[A - \lambda I] = 0$.

$$\det[A - \lambda I] = \det\begin{bmatrix} -1-\lambda & 0 \\ 2 & -1-\lambda \end{bmatrix} = 0 \qquad (-1-\lambda)(-1-\lambda) = 0 \qquad (\lambda+1)^2 = 0$$

$$\lambda_1 = \lambda_2 = -1$$

2. Since A is a 2×2 matrix, we only need the first two terms of the state transition matrix, that is,

$$e^{At} = a_0 I + a_1 A$$

3. We find a_0 and a_1

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$
$$\frac{d}{d\lambda_1}(a_0 + a_1\lambda_1) = \frac{d}{d\lambda_1}e^{\lambda_1 t}$$

$$a_0 + a_1\lambda_1 = e^{\lambda_1 t}$$
$$a_1 = te^{\lambda_1 t}$$

and by substitution with $\lambda_1 = \lambda_2 = -1$, we obtain

$$a_0 - a_1 = e^{-t}$$
$$a_1 = te^{-t}$$

Simultaneous solution of the last two equations yields

$$a_0 = e^{-t} + te^{-t}$$
$$a_1 = te^{-t}$$

4. By substitution, we obtain

$$e^{At} = (e^{-t} + te^{-t})\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t}\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 2te^{-t} & e^{-t} \end{bmatrix}$$

We can use the MATLAB **eig(x)** function to find the eigenvalues of an $n \times n$ matrix. To find out how it is used, we invoke the **help eig** command.

```
A= [-2 1; 0 -1];
lambda=eig(A)
```

```
lambda =
    -2
    -1
```

```
B = [5 7 -5; 0 4 -1; 2 8 -3];
lambda=eig(B)
```

```
lambda =
    1.0000
    3.0000
    2.0000
```

```
C = [-1 0; 2 -1];
lambda=eig(C)
```

```
lambda =
    -1
    -1
```

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