Lecture 5: Vector Spaces

CECC122: Linear Algebra and Matrix Theory

Manara University

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4.1 Vectors in R^n

4.2 Vector Spaces

4.3 Subspaces of Vector Spaces

4.4 Spanning Sets and Linear Independence

4.2 Vector Spaces

- Vector spaces:

Let V be a set on which two operations (vector addition and scalar multiplication) are defined. If the following axioms are satisfied for every $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ in V and every scalar c and d, then V is called a vector space.

Addition:

(1) $\boldsymbol{u}+\boldsymbol{v}$ is in V — Closure under addition

(2) $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$ — Commutative property

(3) $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})=(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}$ — Associative property

(4) V has a zero vector $\mathbf{0}$: for every $\boldsymbol{u}$ in V, $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$ — Additive identity

(5) For every $\boldsymbol{u}$ in V, there is a vector in V denoted by $-\boldsymbol{u}$: $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$ — Scalar identity

Scalar multiplication:

(6) $c\boldsymbol{u}$ is a vector in V — Closure under scalar multiplication

(7) $c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$ — Distributive property

(8) $(c+d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$ — Distributive property

(9) $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$ — Associative property

(10) $1(\boldsymbol{u}) = \boldsymbol{u}$ — Scalar identity

- Notes:

(1) A vector space $(V, +, .)$ consists of four entities:

a nonempty set V of vectors, a set of scalars, and two operations $(+, .)$

(2) $V = \{\mathbf{0}\}$ zero vector space

- Examples of vector spaces:

(1) n-tuple space: $V = R^n$

$$(u_1,u_2,\dots,u_n)+(v_1,v_2,\dots,v_n)=(u_1+v_1,u_2+v_2,\dots,u_n+v_n)$$ vector addition

$$k(u_1,u_2,\cdots,u_n) = (ku_1,ku_2,\cdots,ku_n)$$ scalar multiplication

(2) Matrix space: $V = M_{mxn}$ (the set of all $m \times n$ matrices with real values)

Ex: $(m = n = 2)$

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ vector addition

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ scalar multiplication

- **Theorem 4.3: (Properties of scalar multiplication)**

Let $\mathbf{v}$ any element of a vector space V, and let c be any scalars. Then the following properties are true:

(1) $0\mathbf{v} = \mathbf{0}$

(2) $c\,\mathbf{0} = \mathbf{0}$

(3) If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$

(4) $(-1)\mathbf{v} = -\mathbf{v}$

- **Theorem 4.4: (Test for a subspace)**

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold:

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W.

- **Notes:**

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, c and d are any scalars, then $c\mathbf{u}+d\mathbf{v}$ is in W.

$\Rightarrow W$ is a subspace of V

(2) If W is a subspace of a vector space V, then W contains the zero vector $\mathbf{0}$ of V

4.3 Subspaces of Vector Spaces

- Subspace:

$(V, +, .)$: a vector space

$\left.\begin{array}{l} W \neq \emptyset \\ W \subseteq V \end{array}\right\}$: a nonempty subset

$(W, +, .)$: a vector space (under the operations of addition and scalar multiplication defined in V)

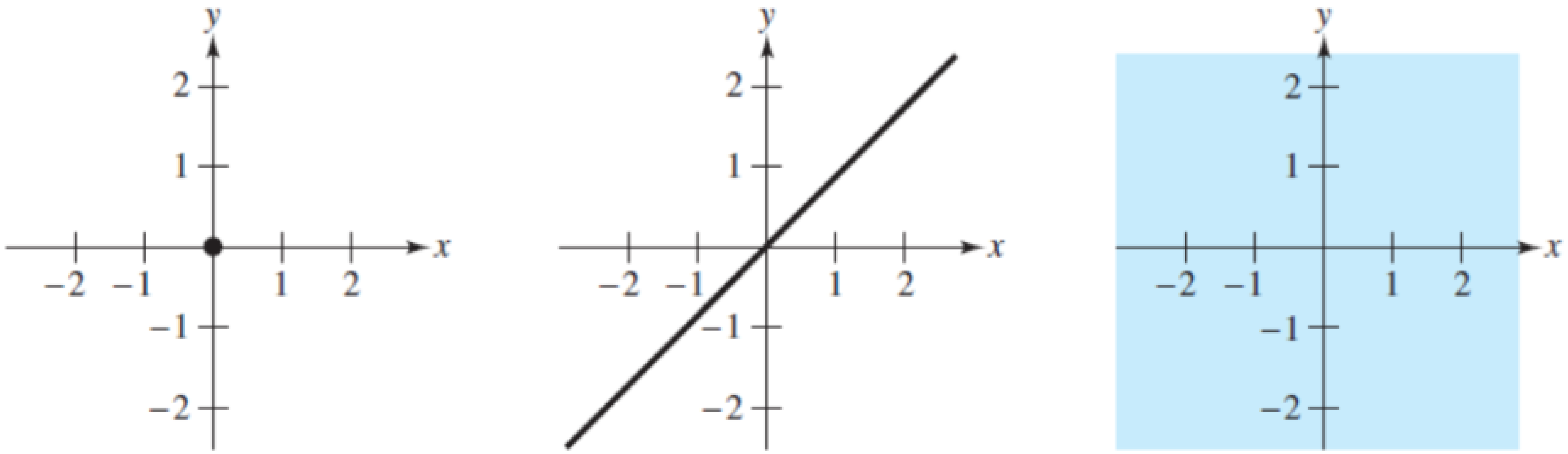
$\Rightarrow W$ is a subspace of V

- Trivial subspace: Every vector space V has at least two subspaces

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V.

(2) V is a subspace of V.

- Ex 1: Subspace of R^2



$W = \{(0, 0)\}$ | W = all points on a line passing through the origin | $W = R^2$

(1) $\{\mathbf{0}\}$ $\quad\quad \mathbf{0} = (0, 0)$

(2) Lines through the origin

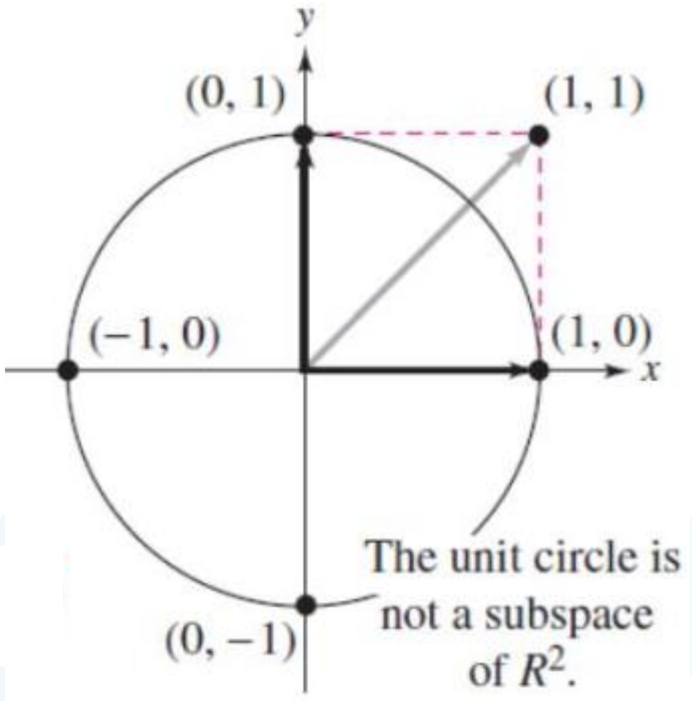
(3) R^2

- **Ex 2: (A Subset of R^2 That Is Not a Subspace)**

Show that the subset of R^2 consisting of all points on $x^2 + y^2 = 1$ is not a subspace

Sol:

points (1, 0) and (0, 1) are in the subset, but their sum (1, 0) + (0, 1) = (1, 1) is not. (not closed under addition)



- **Ex 3:** Subspace of R^3

(1) $\{\mathbf{0}\}$ $\qquad \mathbf{0} = (0, 0, 0)$

(2) Lines through the origin

(3) Planes through the origin

(4) R^3

- Ex 4: (Determining subspaces of R^2)

Which of the following two subsets is a subspace of R^2?

(*a*) The set of points on the line given by $x + 2y = 0$. Yes

(*b*) The set of points on the line given by $x + 2y = 1$. No

- Theorem 4.5: (The intersection of two subspaces is a subspace)

If V and W are both subspaces of a vector space U, then the intersection of V and W (denoted by $V \cap W$) is also a subspace of U.

4.4 Spanning Sets and Linear Independence

- Linear combination:

A vector $\boldsymbol{v}$ in a vector space V is called a linear combination of the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ in V if $\boldsymbol{v}$ can be written in the form

$$\boldsymbol{v} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k \qquad c_1, c_2, \ldots, c_k\text{: scalars}$$

- Ex 1: (Finding a linear combination)

$$\boldsymbol{v}_1 = (1, 2, 3), \quad \boldsymbol{v}_2 = (0, 1, 2), \quad \boldsymbol{v}_3 = (-1, 0, 1)$$

Prove (*a*) $\boldsymbol{w} = (1, 1, 1)$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$

(*b*) $\boldsymbol{w} = (1, -2, 2)$ is not a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$

Sol: (a) $\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3}$

$$(1, 1, 1) = c_1(1, 2, 3) + c_2(0, 1, 2) + c_3(-1, 0, 1)$$

$$= (c_1 - c_3, 2c_1 + c_2, 2c_2 + c_3)$$

$$\Rightarrow \begin{array}{rcrcrcl} c_1 & & & - & c_3 & = & 1 \\ 2c_1 & + & c_2 & & & = & 1 \\ 3c_1 & + & 2c_2 & + & c_3 & = & 1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow c_1 = 1 + t, c_2 = -1 - 2t, c_3 = t$ (this system has infinitely many solutions)

$t = 1 \Rightarrow \boldsymbol{w} = 2\boldsymbol{v_1} - 3\boldsymbol{v_2} + \boldsymbol{v_3}$

$(b)\ \mathbf{w} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ this system has no solution $(0 \neq 7)$

$\Rightarrow \mathbf{w} \neq c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$

- A spanning set of a vector space:

If every vector in a given vector space can be written as a linear combination of vectors in a given set S, then S is called a spanning set of the vector space.

- **Ex 2: (A spanning set for R^3)**

  The set $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ spans R^3 because any vector $\boldsymbol{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\boldsymbol{u} = u_1(1, 0, 0) + u_2(0, 1, 0) + u_3(0, 0, 1) = (u_1, u_2, u_3)$$

- **The span of a set: span (S)**

  If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S,

$$\text{span}(S) = \{c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_k\boldsymbol{v}_k \mid \forall c_i \in R\}$$

  (the set of all linear combinations of the vectors in S)

- **Linear Independent (L.I.) and Linear Dependent (L.D.):**

$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0$$

(1) If the equation has only the trivial solution ($c_1 = c_2 = \ldots c_k = 0$), then S is called linearly independent.

(2) If the equation has a non trivial solution (i.e. not all zeros), then S is called linearly dependent.

- Notes

(1) $\varnothing$ is linearly independent.

(2) $\mathbf{0} \in S \Rightarrow S$ is linearly dependent.

(3) $\boldsymbol{v} \neq \mathbf{0} \Rightarrow \{\boldsymbol{v}\}$ is linearly independent.

(4) $S_1 \subseteq S_2$

S_1 is linearly dependent $\Rightarrow S_2$ is linearly dependent

S_2 is linearly independent $\Rightarrow S_1$ is linearly independent

- Ex 3: (Testing for linearly independent)

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{(1, 2, 3), (0, 1, 2), (-2, 0, 1)\}$$

$$\quad\quad v_1 \quad\quad v_2 \quad\quad v_3$$

Sol:

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = 0 \Rightarrow \begin{aligned} c_1 \quad\quad - 2c_3 &= 0 \\ 2c_1 + c_2 \quad\quad &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow c_1 = c_2 = c_3 = 0$ (only the trivial solution) $\Rightarrow S$ is linearly independent

- **Independence of two vectors:**

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in a vector space V are linearly dependent if and only if one is a scalar multiple of the other.

- **Ex 4: (Testing for linear dependent of 2 Vectors)**

(1) $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \{(1, 2, 0), (-2, 2, 1)\}$ is L.I. because $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are not scalar multiples of each other.

(2) $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2\} = \{(4, -4, -2), (-2, 2, 1)\}$ is L.D. because $\boldsymbol{v}_1 = -2\boldsymbol{v}_2$

4.5 Basis and Dimension

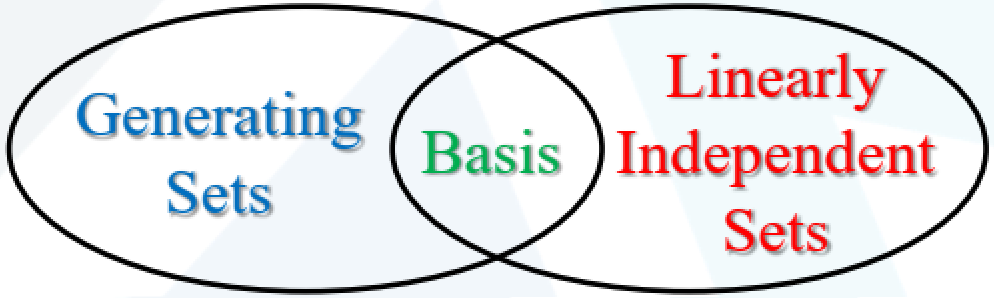
- Basis:

V: a vector space $\qquad S=\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}\subseteq V$

(a) S spans V (i.e., span(S) = V)

(b) S is linearly independent

$\Rightarrow$ S is called a basis for V



- Notes:

(1) Ø is a basis for $\{\mathbf{0}\}$

(2) the standard basis for R^3:

$\{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \quad \mathbf{i}=(1, 0, 0),\ \mathbf{j}=(0, 1, 0),\ \mathbf{k}=(0, 0, 1)$

(3) the standard basis for R^n :

$\{e_1, e_2, \ldots, e_n\}$ $e_1 = (1,0,\ldots,0), e_2 = (0,1,\ldots,0), e_n = (0,0,\ldots,1)$

Ex: R^4 $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$

- Finite dimensional:

A vector space V is called finite dimensional, if it has a basis consisting of a finite number of elements.

- Dimension:

The dimension of a finite dimensional vector space V is defined to be the number of vectors in a basis for V.

V: a vector space, S: a basis for V $\Rightarrow \dim(V) = \#(S)$ (the number of vectors in S)

- Notes:

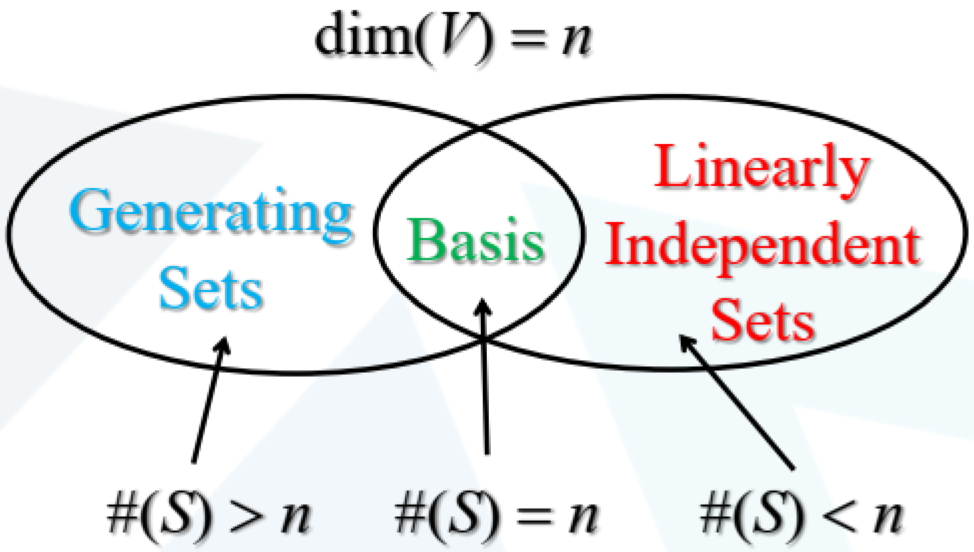
(1) $\dim(\{\mathbf{0}\}) = 0 = \#(\emptyset)$

(2) $\dim(V) = n,\ S \subseteq V$

S: a L.I. set $\Rightarrow \#(S) \le n$

S: a generating set $\Rightarrow \#(S) \ge n$

S: a basis $\Rightarrow \#(S) = n$



4.6 Rank and Nullity of a Matrix

- **Rank of a Matrix:**

The **rank** of an *mxn* matrix A, denoted by rank(A), is the maximum number of linearly independent row vectors in A or the maximum number of linearly independent column vectors in A

- **Nullity of a Matrix:**

The **nullity** of an *mxn* matrix A, denoted by nullity(A), is the dimension of the solution space of the linear system $A\mathbf{x} = \mathbf{0}$

- **Theorem 4.6:**

If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$

- Notes:

  (1) The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix

  (2) The number of leading 1's in the reduced row-echelon form of A is equal to the rank of A

  (3) The number of free variables in the reduced row-echelon form of A is equal to the nullity of A

- Theorem 4.7: (Consistency of $A\boldsymbol{x} = \boldsymbol{b}$)

  If $\text{rank}([A|\boldsymbol{b}]) = \text{rank}(A)$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

- Note:

A linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$ is consistent iff the rank of A is the same as the rank of the augmented matrix of the system $[A|\boldsymbol{b}]$

- Notes:

(1) If $\text{rank}(A) = \text{rank}(A|\boldsymbol{b}) = n$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique sol.

(2) If $\text{rank}(A) = \text{rank}(A|\boldsymbol{b}) < n$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ has ∞-many sols.

(3) If $\text{rank}(A) < \text{rank}(A|\boldsymbol{b})$, then the system $A\boldsymbol{x} = \boldsymbol{b}$ is inconsistent.

- Ex 1: (Rank by Row Reduction)

$$A=\begin{bmatrix}1 & 1 & -1 & 3\\ 2 & -2 & 6 & 8\\ 3 & 5 & -7 & 8\end{bmatrix}\xrightarrow{\text{Gauss Elimination}}\begin{bmatrix}1 & 1 & -1 & 3\\ 0 & 1 & -2 & -\frac{1}{2}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

rank(A) = 2 (2 non-zero rows = 2 non-zero rows)

nullity(A) = 2 (2 free variables)

- Ex 2: (Finding the solution set of a nonhomogeneous system)

$$\begin{aligned} x_1 + x_2 - x_3 &= -1\\ x_1 \qquad + x_3 &= 3\\ 3x_1 + 2x_2 - x_3 &= 1 \end{aligned}$$

Sol:

$$A=\begin{bmatrix}1 & 1 & -1\\ 1 & 0 & 1\\ 3 & 2 & -1\end{bmatrix} \xrightarrow{\text{Gauss-Jordan Elimination}} \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -2\\ 0 & 0 & 0\end{bmatrix}$$

$$[A \,\vdots\, \boldsymbol{b}]=\left[\begin{array}{ccc:c}1 & 1 & -1 & -1\\ 1 & 0 & 1 & 3\\ 3 & 2 & -1 & 1\end{array}\right] \xrightarrow{\text{Gauss-Jordan Elimination}} \left[\begin{array}{ccc:c}1 & 0 & 1 & 3\\ 0 & 1 & -2 & -4\\ 0 & 0 & 0 & 0\end{array}\right]$$

$$\begin{aligned} x_1 \quad\quad + x_3 &= 3 \\ x_2 - 2x_3 &= -4 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 3 - x_3 \\ x_2 &= -4 + 2x_3 \end{aligned}$$

letting $x_3 = t$, then the solutions are: $\{(3-t,\ -4+2t,\ t) \mid t \in R\}$

So the system has infinitely many solutions (consistent)

- **Check:** $\text{rank}(A) = \text{rank}([A \,\vdots\, \boldsymbol{b}]) = 2$

- **Theorem 4.8 (Dimension Theorem for Matrices)**

If A is a matrix with n columns, then rank(A) + nullity(A) = n

- **Ex 3: (Rank and nullity of a matrix)**

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 3 & 9 & 0 & -12 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank(A) = 3 (the number of nonzero rows in B)

nullity(A) = n – rank(A) = 5 – 3 = 2

- Summary of equivalent conditions for square matrices:

If A is an nxn matrix, then the following conditions are equivalent:

(1) A is invertible

(2) $A\mathbf{x} = \boldsymbol{b}$ has a unique solution for any $n\times 1$ matrix $\boldsymbol{b}$.

(3) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution

(4) A is row-equivalent to I_n

(5) $|A| \neq 0$

(6) $\text{rank}(A) = n$

(7) The n row vectors of A are linearly independent.

(8) The n column vectors of A are linearly independent.

4.7 Coordinates and Change of Basis

- Coordinate representation relative to a basis

Let $B = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be an ordered basis for a vector space V and let $\boldsymbol{x}$ be a vector in V such that $\boldsymbol{x} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n$

The scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\boldsymbol{x}$ relative to the basis B. The coordinate matrix (or coordinate vector) of x relative to B is the column matrix in R^n whose components are the coordinates of $\boldsymbol{x}$.

$$[\boldsymbol{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

- Ex 1: (Coordinates and components in R^n)

  Find the coordinate matrix of $\boldsymbol{x} = (-2, 1, 3)$ in R^3 relative to the standard basis $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

Sol:

$$\boldsymbol{x} = (-2, 1, 3) = -2(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1)$$

$$[\boldsymbol{x}]_S = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

- Ex 2: (Finding a coordinate matrix relative to a nonstandard basis)

  Find the coordinate matrix of $\boldsymbol{x} = (1, 2, -1)$ in R^3 relative to the (nonstandard) basis

  $B' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \{(1, 0, 1), (0, -1, 2), (2, 3, -5)\}$

Sol:

$$\boldsymbol{x} = c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + c_3\boldsymbol{u}_3 \Rightarrow (1,2,-1) = c_1(1,0,1) + c_2(0,-1,2) + c_3(2,3,-5)$$

$$\Rightarrow \begin{aligned} c_1 \quad\quad\quad\quad + 2c_3 &= 1 \\ -c_2 + 3c_3 &= 2 \\ c_1 + 2c_2 - 5c_3 &= -1 \end{aligned} \quad \text{i.e.} \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & -1 & 3 & 2 \\ 1 & 2 & -5 & -1 \end{array}\right] \xrightarrow{\text{G. J. Elimination}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & -2 \end{array}\right] \Rightarrow [\boldsymbol{x}]_{B'} = \begin{bmatrix} 5 \\ -8 \\ -2 \end{bmatrix}$$