Exercises 5: Vector Spaces

CECC122: Linear Algebra and Matrix Theory

Manara University

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Determine whether W is a subspace of the vector space V

① $W = \{(x, y): x = 2y\}$, $V = R^2$

W is nonempty and $W \subset R^2$, W is closed under addition and scalar multiplication $\Rightarrow$ W is a subspace of R^2

② $W = \{(x, 2x, 3x): x \text{ is a real number}\}$, $V = R^3$

W is nonempty and $W \subset R^3$, W is closed under addition and scalar multiplication $\Rightarrow$ W is a subspace of R^3

③ $W = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 0\}$

$W = \{\mathbf{0}\}$, W is a subspace of R^3

④ $W = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 1\}$

W is not closed under addition or scalar multiplication, so it is not a subspace of R^3.
$(1, 0, 0) \in W$, and yet $2(1, 0, 0) = (2, 0, 0) \notin W$

Write each vector as a linear combination of the vectors in S (if possible)

① $S = \{(2, -1, 3), (5, 0, 4)\}$

(a) $\boldsymbol{u} = (1, 1, -1)$

(b) $\boldsymbol{v} = (8, -1/4, 27/4)$

(a) $\boldsymbol{u} = (1, 1, -1) = c_1(2, -1, 3) + c_2(5, 0, 4)$

$$\begin{aligned} 2c_1 + 5c_2 &= 1 \\ -c_1 \qquad &= 1 \\ 3c_1 + 4c_2 &= -1 \end{aligned}$$

This system has no solution. So, $\boldsymbol{u}$ cannot be written as a linear combination of vectors in S

(b) $\boldsymbol{v} = (8, -1/4, 27/4) = c_1(2, -1, 3) + c_2(5, 0, 4)$

$$2c_1 + 5c_2 = 8$$
$$-c_1 = -1/4$$
$$3c_1 + 4c_2 = 27/4$$

The solution to this system is $c_1 = 1/4$ and $c_2 = 3/2$. So, $\boldsymbol{v}$ can be written as a linear combination of vectors in S

(2) $S = \{(2, 0, 7), (2, 4, 5), (2, -12, 13)\}$

(a) $\boldsymbol{u} = (-1, 5, -6)$

(b) $\boldsymbol{v} = (-3, 15, 18)$

(a) $\boldsymbol{u} = (-1, 5, -6) = c_1(2, 0, 7) + c_2\,(2, 4, 5) + c_3\,(2, -12, 13)$

$$
\begin{aligned}
2c_1 + 2c_2 + 2c_3 &= -1 \\
4c_2 - 12c_3 &= 5 \\
7c_1 + 5c_2 + 13c_3 &= -6
\end{aligned}
$$

One solution to this system is $c_1 = -7/4$, $c_2 = 5/4$ and $c_3 = 0$. So, $\boldsymbol{u}$ can be written as a linear combination of vectors in S

(b) $\boldsymbol{v} = (-3, 15, 18) = c_1(2, 0, 7) + c_2\,(2, 4, 5) + c_3\,(2, -12, 13)$

$$
\begin{aligned}
2c_1 + 2c_2 + 2c_3 &= -3 \\
4c_2 - 12c_3 &= 15 \\
7c_1 + 5c_2 + 13c_3 &= 18
\end{aligned}
$$

This system has no solution. So, $\boldsymbol{v}$ cannot be written as a linear combination of vectors in S

Determine whether the set S is linearly independent or linearly dependent

① $S = \{(-2, 2), (3, 5)\}$

$(-2, 2)$ is not a scalar multiple of $(3, 5)$. So the set S is linearly independent

② $S = \{(0, 0), (1, -1)\}$

$\mathbf{0} \in S \Rightarrow S$ is linearly dependent

③ $S = \{(3, -6), (-1, 2)\}$

$(3, -6) = -3\,(-1, 2) \Rightarrow S$ is linearly dependent

④ $S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$

These vectors are multiples of each other, the set S is linearly dependent

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⑤ $S = \{(-2, 1, 3), (2, 9, -3), (2, 3, -3)\}$

$$c_1(-2, 1, 3) + c_2(2, 9, -3) + c_3(2, 3, -3) = \mathbf{0} = (0, 0, 0)$$

$$-2c_1 + 2c_2 + 2c_3 = 0$$

$$c_1 + 9c_2 + 3c_3 = 0$$

$$3c_1 - 3c_2 - 3c_3 = 0$$

The system has many solutions. One solution is (3, –2, 5), so S is linearly dependent

⑥ $S = \{(-4, -3, 4), (1, -2, 3), (6, 0, 0)\}$

$$c_1(-4, -3, 4) + c_2(1, -2, 3) + c_3(6, 0, 0) = \mathbf{0} = (0, 0, 0)$$

$$-4c_1 + c_2 + 6c_3 = 0$$

$$-3c_1 - 2c_2 = 0$$

$$4c_1 + 3c_2 = 0$$

This system has only the trivial solution $c_1 = c_2 = c_3 = 0$. So S is linearly independent

(7) $S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}$

$(1, -5, 3) = (1, 0, 0) + 5/4(0, 4, 0) + 1/2(0, 0, -6)$

The fourth vector is a linear combination of the first three. So S is linearly dependent

Let v_1, v_2, and v_3 be three linearly independent vectors in a vector space V. Is the set $\{v_1 - 2v_2,\ 2v_2 - 3v_3,\ 3v_3 - v_1\}$ linearly dependent or linearly independent? Explain

To see if the given set is linearly independent, solve the equation

$$c_1(\mathbf{v}_1 - 2\mathbf{v}_2) + c_2(2\mathbf{v}_2 - 3\mathbf{v}_3) + c_3(3\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$$

$$(c_1 - c_3)\mathbf{v}_1 + (-2c_1 + 2c_2)\mathbf{v}_2 + (-3c_2 + 3c_3)\mathbf{v}_3 = \mathbf{0}$$

v_1, v_2, and v_3 be three linearly independent vectors ⇒

$$\begin{aligned} c_1 \quad\quad\quad - c_3 &= 0 \\ -2c_1 + 2c_2 \quad\quad &= 0 \\ -3c_2 + 3c_3 &= 0 \end{aligned}$$

This system has infinitely many solutions, so $\{v_1 - 2v_2, 2v_2 - 3v_3, 3v_3 - v_1\}$ is linearly dependent

Let v_1, v_2, and v_3 be three linearly independent vectors in a vector space V. Is the set $\{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$ linearly dependent or linearly independent? Explain

To see if the given set is linearly independent, solve the equation

$$c_1(v_1 + v_2) + c_2(v_2 + v_3) + c_3(v_3 + v_1) = \mathbf{0}$$
$$(c_1 + c_3)v_1 + (c_1 + c_2)v_2 + (c_2 + c_3)v_3 = \mathbf{0}$$

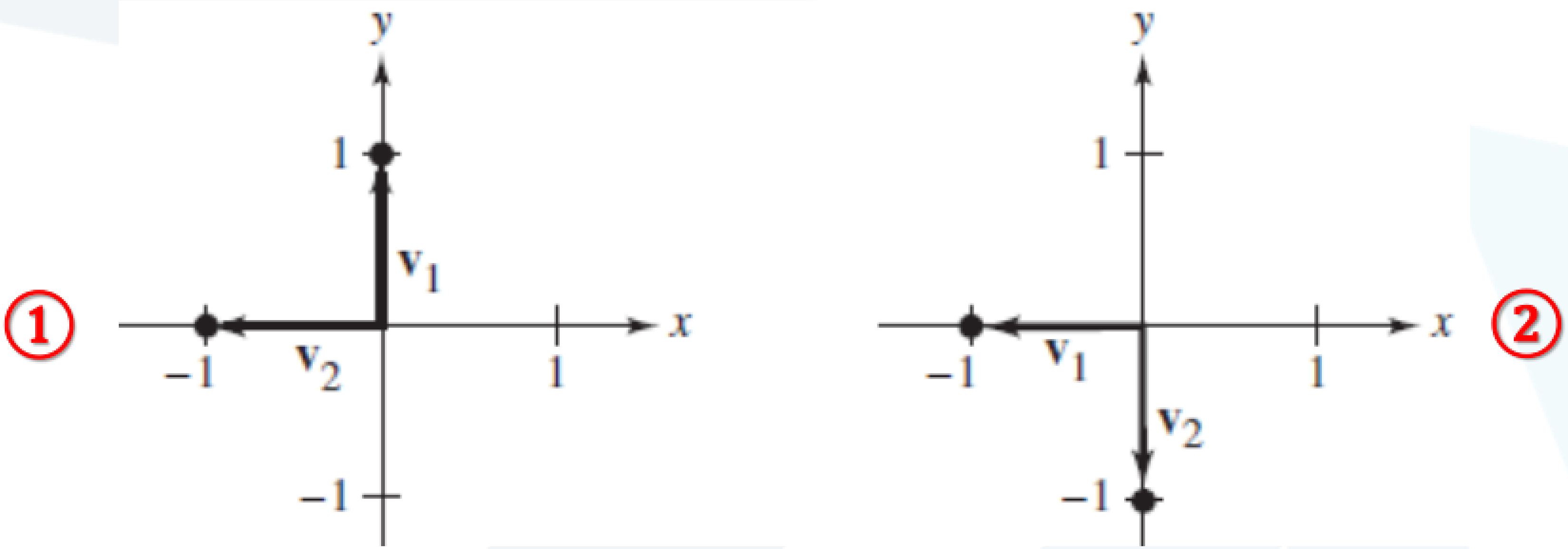
$\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ be three linearly independent vectors $\Rightarrow$

$$\begin{aligned} c_1 \quad\quad + c_3 &= 0 \\ c_1 + c_2 \quad\quad &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

This system has only the trivial solution, so

$\{\boldsymbol{v}_1 - 2\boldsymbol{v}_2, 2\boldsymbol{v}_2 - 3\boldsymbol{v}_3, 3\boldsymbol{v}_3 - \boldsymbol{v}_1\}$ is linearly dependent

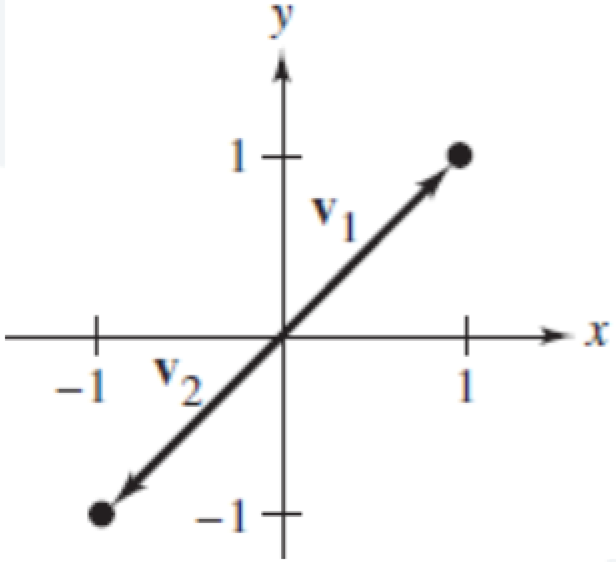
Determine whether the set $\{v_1, v_2\}$ is a basis for R^2

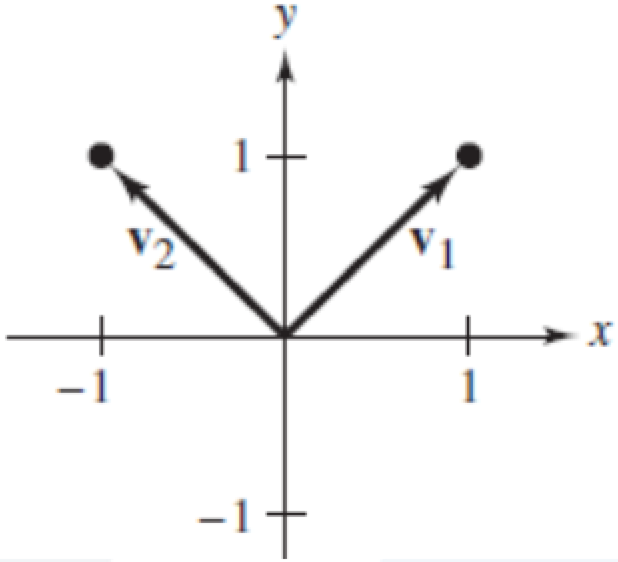


1. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2

2. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2

3

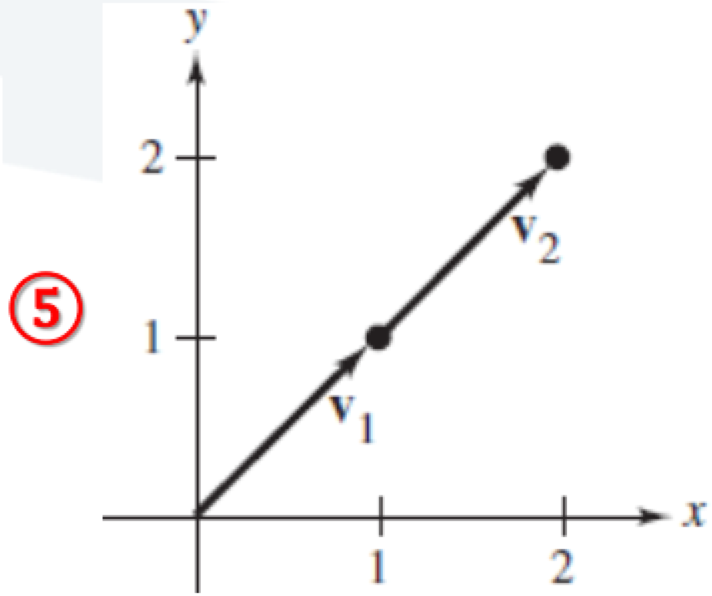


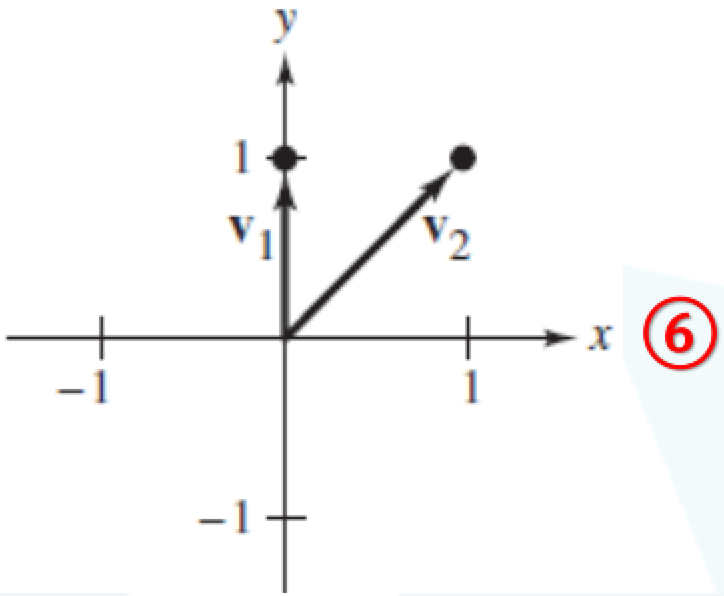


4

3. v_1 and v_2 are multiplies of each other, they do not form a basis for R^2

4. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2





5. v_1 and v_2 are multiplies of each other, they do not form a basis for R^2

6. $\{v_1, v_2\}$ consists of exactly two linearly independent vectors, it is a basis for R^2

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Determine whether S is a basis for the given vector space

① $S = \{(4, -3), (5, 2)\}$ for R^2

S consists of exactly two linearly independent vectors, it is a basis for R^2

② $S = \{(1, 2), (1, -1), (-1, 2)\}$ for R^2

S consists of more than two vectors, so S is linearly dependent, and it is not a basis for R^2

③ $S = \{(1, 5, 3), (0, 1, 2), (0, 0, 6)\}$ for R^3

To determine if the vectors in S are linearly independent, find the solution to

$c_1(1, 5, 3) + c_2 (0, 1, 2) + c_3(0, 0, 6) = (0, 0, 0)$

Which corresponds to the solution of

$$
\begin{aligned}
c_1 \qquad\qquad\qquad\quad &= 0 \\
5c_1 + c_2 \qquad\qquad &= 0 \\
3c_1 + 2c_2 + 6c_3 &= 0
\end{aligned}
$$

This system has only the trivial solution. So, S consists of exactly three linearly independent vectors, and is, therefore, a basis for R^3

④ $S = \{(2, 1, 0), (0, -1, 1)\}$ for R^3

S does not span R^3 (consists of less than three vectors), although it is linearly independent $\Rightarrow S$ is not a basis for R^3

⑤ $S = \{(0, 3, -2), (4, 0, 3), (-8, 15, -16)\}$ for R^3

To determine if the vectors in S are linearly independent, find the solution to

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$$c_1(0, 3, -2) + c_2(4, 0, 3) + c_3(-8, 15, -16) = (0, 0, 0)$$

which corresponds to the solution of

$$\begin{aligned} 4c_2 - 8c_3 &= 0 \\ 3c_1 + 15c_3 &= 0 \\ -2c_1 + 3c_2 - 16c_3 &= 0 \end{aligned}$$

This system has nontrivial solutions (for instance, $c_1 = -5$, $c_2 = 2$ and $c_3 = 1$), so the vectors are linearly dependent, and S is not a basis for R^3

(6) $S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$ for R^3

This set contains the zero vector, and is, therefore, linearly dependent. So, S is not a basis for R^3

Determine whether the set, (a) is linearly independent, and (b) is a basis for R^3

(1) $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$

(a) $c_1(1, -5, 4) + c_2\,(11, 6, -1) + c_3(2, 3, 5) = (0, 0, 0)$

$$\begin{aligned} c_1 + 11c_2 + 2c_3 &= 0 \\ -5c_1 + 6c_2 + 3c_3 &= 0 \\ 4c_1 - c_2 + 5c_3 &= 0 \end{aligned}$$

This system has only the trivial solution. So, S is linearly independent.

(b) S consists of exactly three linearly independent vectors, and is, therefore, a basis for R^3

② $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$

(a) S is linearly dependent because the 4th vector is a linear combination of the first three $(-1, 2, -3) = -1(1, 0, 0) + 2(0, 1, 0) - 3(0, 0, 1)$

(b) S is not a basis because it is not linearly independent

Find the rank and nullity of the matrix A

① $A = \begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & -3 & -6 & -4 \\ 1 & 5 & -3 & 11 \\ 2 & 7 & -6 & 16 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$,

$\text{nullity}(A) = 4 - 2 = 2$

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & -1 & -18 \\ -1 & 3 & 10 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow{\text{G.J. Elimination}} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$,
$\text{nullity}(A) = 3 - 2 = 1$

Given the coordinate matrix of x relative to a (nonstandard) basis B for R^n, find the coordinate matrix of x relative to the standard basis

① $B = \{(1, 1), (-1, 1)\}$, $[\boldsymbol{x}]_B = [3 \;\; 5]^T$

$\boldsymbol{x} = 3(1, 1) + 5(-1, 1) = (-2, 8)$

$(-2, 8) = -2(1, 0) + 8(0, 1)$, the coordinate vector of $\boldsymbol{x}$ relative to the standard basis is $[\boldsymbol{x}]_S = [-2 \;\; 8]^T$

② $B = \{(1, 0, 0), (1, 1, 0), (0, 1, 1)\}$, $[\boldsymbol{x}]_B = [2 \;\; 0 \;\; -1]^T$

$$\boldsymbol{x} = 2(1, 0, 0) + 0(1, 1, 0) - 1(0, 1, 1) = (2, -1, -1)$$

$(-2, -1, -1) = 2(1, 0, 0) - 1(0, 1, 0) - 1(0, 0, 1)$, the coordinate vector of $\boldsymbol{x}$ relative to the standard basis is $[\boldsymbol{x}]_B = [2 \;\; -1 \;\; -1]^T$

Find the coordinate matrix of x in R^n relative to the basis B'

① $B' = \{(5, 0), (0, -8)\}$, $\boldsymbol{x} = (2, 2)$

$$c_1(5, 0) + c_2(0, -8) = (2, 2)$$

The resulting system of linear equations is

$$\begin{aligned} 5c_1 \quad\quad &= 2 \\ -8c_2 &= 2 \end{aligned}$$

So $c_1 = 2/5$, $c_2 = -1/4 \Rightarrow [\boldsymbol{x}]_{B'} = [2/5 \;\; -1/4]^T$

② $B' = \{(1, 2, 3), (1, 2, 0), (0, -6, 2)\}, \boldsymbol{x} = (3, -3, 0)$

$$c_1(1, 2, 3) + c_2(1, 2, 0) + c_3(0, -6, 2) = (3, -3, 0)$$

$$\begin{aligned} c_1 + c_2 \qquad\qquad &= 3 \\ 2c_1 + 2c_2 - 6c_3 &= -3 \\ 3c_1 \qquad + 2c_3 &= 0 \end{aligned}$$

The solution is $c_1 = -1$, $c_2 = 4$ and $c_3 = 3/2 \Rightarrow [\boldsymbol{x}]_{B'} = [-1 \;\; 4 \;\; 3/2]^T$

Determine whether W is a subspace of the vector space V

1. $W = \{(x, y): x - y = 1\}$, $V = R^2$

2. Which of the subsets of R^3 is a subspace of R^3?

(a) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 0\}$

(b) $W = \{(x_1, x_2, x_3): x_1 + x_2 + x_3 = 1\}$

Write v as a linear combination of u_1, u_2, and u_3, if possible

1. $\boldsymbol{v} = (3, 0, -6)$, $\boldsymbol{u}_1 = (1, -1, 2)$, $\boldsymbol{u}_2 = (2, 4, -2)$, $\boldsymbol{u}_3 = (1, 2, -4)$

2. $\boldsymbol{v} = (4, 4, 5)$, $\boldsymbol{u}_1 = (1, 2, 3)$, $\boldsymbol{u}_2 = (-2, 0, 1)$, $\boldsymbol{u}_3 = (1, 0, 0)$

Determine whether the set S is linearly independent or linearly dependent

1. $S = \{(1, -5, 4), (11, 6, -1), (2, 3, 5)\}$

2. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$

3. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$

4. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 2, -3)\}$

5. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$

Determine whether the set, (a) is linearly independent, and (b) is a basis for R^3

1. $S = \{(4, 0, 1), (0, -3, 2), (5, 10, 0)\}$

2. $S = \{(-1/2, 3/4, -1), (5, 2, 3), (-4, 6, -8)\}$

3. $S = \{(2, 0, 1), (2, -1, 1), (4, 2, 0)\}$

4. $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (2, -1, 0)\}$

Given the coordinate matrix of x relative to a (nonstandard) basis B for R^n, find the coordinate matrix of x relative to the standard basis

1. $B = \{(2, 4), (-1, 1)\}, [\mathbf{x}]_B = [4 \;\; -7]^T$

2. $B = \{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}, [\mathbf{x}]_B = [4 \;\; 0 \;\; 2]^T$

Find the rank and nullity of the matrix A

① $$A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 4 & -2 & 4 & -2 \\ -2 & 0 & 1 & 3 \end{bmatrix}$$

② $$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 4 & 0 & 3 \\ -2 & 3 & 0 & 2 \\ 1 & 2 & 6 & 1 \end{bmatrix}$$

Find the coordinate matrix of x in R^n relative to the basis B'

1. $B' = \{(2, 2), (0, -1)\}, \boldsymbol{x} = (-1, 2)$

2. $B' = \{(1, 0, 0), (0, 1, 0), (1, 1, 1)\}, \boldsymbol{x} = (4, -2, 9)$