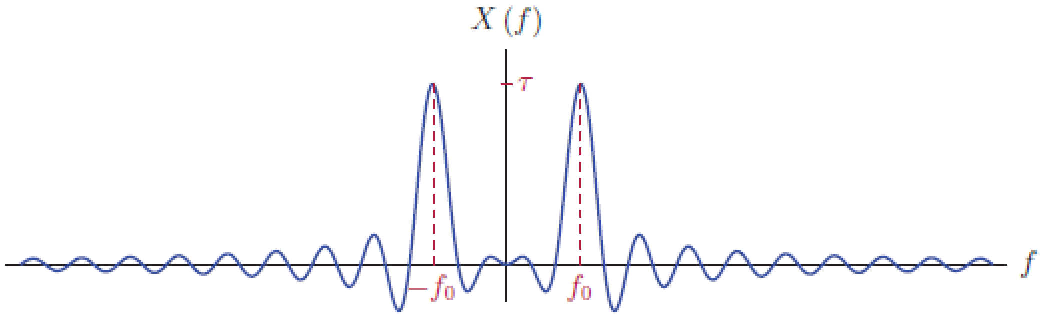
CECC507: Signals and Systems

Lecture Notes 5: Fourier Analysis for Continuous Time Signals and Systems: Part A



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Chapter 4

Fourier Analysis for Continuous Time Signals and Systems

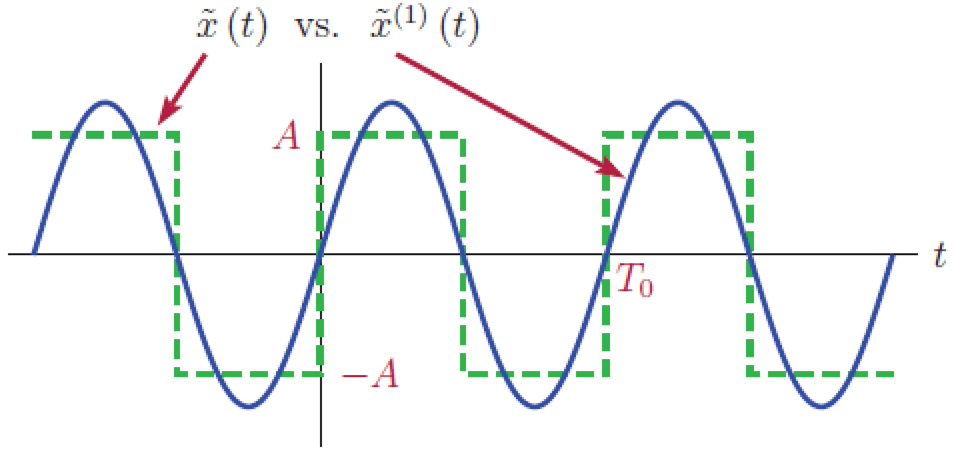
1. Introduction

- Fourier analysis leads to the frequency spectrum of a continuous-time signal.
- The frequency spectrum displays the various sinusoidal components that make up the signal.
- In the frequency domain linear systems are described by linear algebraic equations that can be easily solved, in contrast to the time-domain representation, where they are described by linear differential equations.
- A weighted summation of Sines and Cosines of different frequencies can be used to represent periodic (Fourier Series), or non-periodic (Fourier Transform) functions.

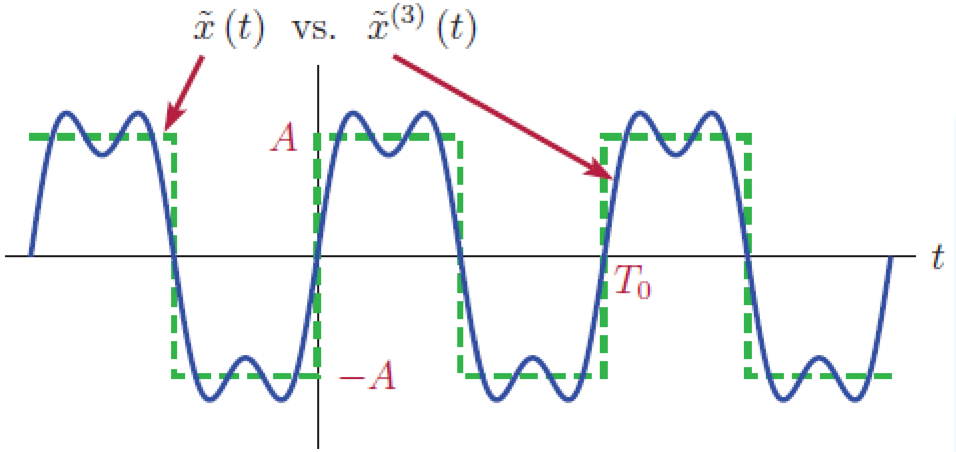
2. Analysis of Periodic Continuous-Time Signals

- We will study methods of expressing periodic continuous-time signals in two different but equivalent formats, namely the trigonometric Fourier series (TFS) and the exponential Fourier series (EFS).

Approximating a periodic signal with trigonometric functions



$\tilde{x}^{(1)}(t) = b_1 \sin(\omega_0 t)$



$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$

Trigonometric Fourier series (TFS)

$$\tilde{x}(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + \ldots + a_k\cos(k\omega_0 t) + \ldots$$
$$+ b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + \ldots + b_k\sin(k\omega_0 t) + \ldots$$

- In a compact notation (trigonometric Fourier Series TFS of the periodic signal $\tilde{x}(t)$):

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k\sin(k\omega_0 t)$$

where $\omega_0 = 2\pi f_0$ is the fundamental frequency in rad/s.

- The set of orthogonal basis functions:

$$\phi_k(t) = \cos(k\omega_0 t), \;\; \psi_k(t) = \sin(k\omega_0 t) \;\; k = 0, 1, 2, \ldots, \infty$$

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k\phi_k(t) + \sum_{k=1}^{\infty} b_k\psi_k(t)$$

- We call the frequencies that are integer multiples of the fundamental frequency the harmonics.
- The frequencies $2\omega_0$, $3\omega_0$, ..., $k\omega_0$ are the second, the third, and the k-th harmonics of the fundamental frequency respectively.
- We need to determine the coefficients: a_0, a_k, and b_k.

$$\int_{t_0}^{t_0+T_0} \cos(m\omega_0 t)\cos(k\omega_0 t)\,dt = \begin{cases} T_0/2, & m = k \\ 0, & m \neq k \end{cases}$$

$$\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t)\sin(k\omega_0 t)\,dt = \begin{cases} T_0/2, & m = k \\ 0, & m \neq k \end{cases}$$

$$\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t)\cos(k\omega_0 t)\,dt = 0$$

Trigonometric Fourier series (TFS)

1. Synthesis equation:

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

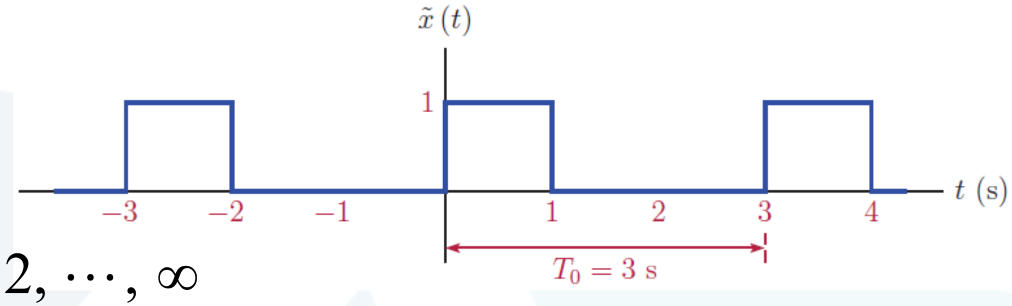
2. Analysis equation:

$$a_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\,dt \qquad \text{(dc component)}$$

$$a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\cos(k\omega_0 t)\,dt, \qquad \text{for } k = 1, 2, \cdots, \infty$$

$$b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\sin(k\omega_0 t)\,dt, \qquad \text{for } k = 1, 2, \cdots, \infty$$

- **Example 1:** Trigonometric Fourier series of a periodic pulse train



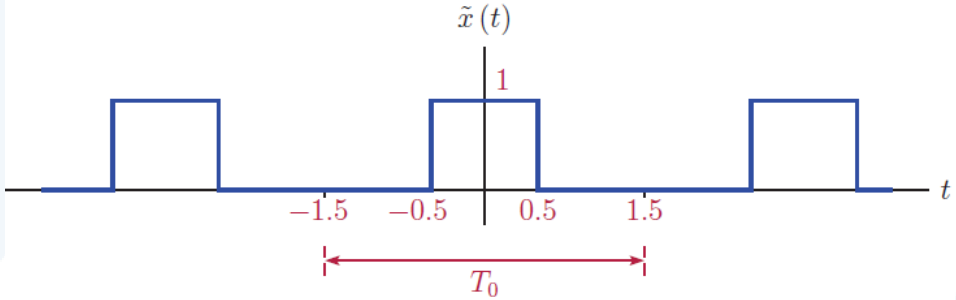
$$a_0 = \frac{1}{3}\int_0^3 (1)\,dt = \frac{1}{3}$$

$$a_k = \frac{2}{3}\int_0^3 (1)\cos(\frac{2\pi kt}{3})dt = \frac{\sin(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \cdots, \infty$$

$$b_k = \frac{2}{3}\int_0^3 (1)\sin(\frac{2\pi kt}{3})dt = \frac{1-\cos(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \cdots, \infty$$

$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty}\frac{\sin(2\pi k/3)}{\pi k}\cos(\frac{2\pi kt}{3}) + \sum_{k=1}^{\infty}\frac{1-\cos(2\pi k/3)}{\pi k}\sin(\frac{2\pi kt}{3})$$

- **Example 2:** Periodic pulse train



$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty}\frac{2\sin(\pi k/3)}{\pi k}\cos(\frac{2\pi kt}{3})$$

Exponential Fourier series (EFS)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Single-tone signals:

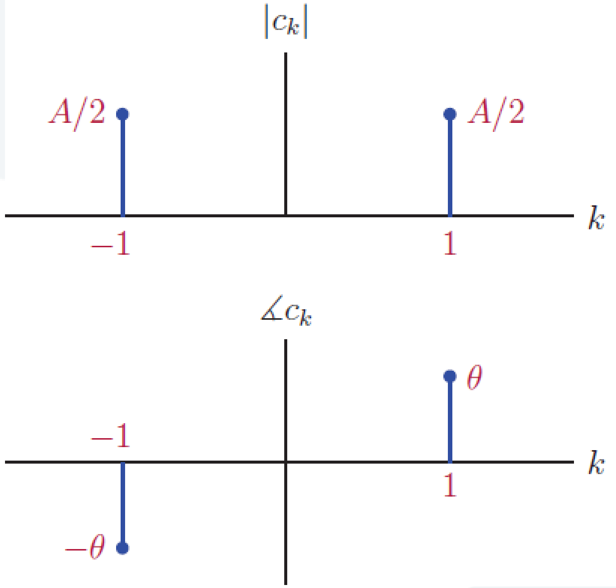
$$\tilde{x}(t) = A\cos(\omega_0 t + \theta) = \frac{A}{2} e^{j(\omega_0 t + \theta)} + \frac{A}{2} e^{-j(\omega_0 t + \theta)} = \frac{A}{2} e^{j\theta} e^{j\omega_0 t} + \frac{A}{2} e^{-j\theta} e^{-j\omega_0 t}$$

$$c_1 = \frac{A}{2} e^{j\theta}, \quad c_{-1} = \frac{A}{2} e^{-j\theta}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

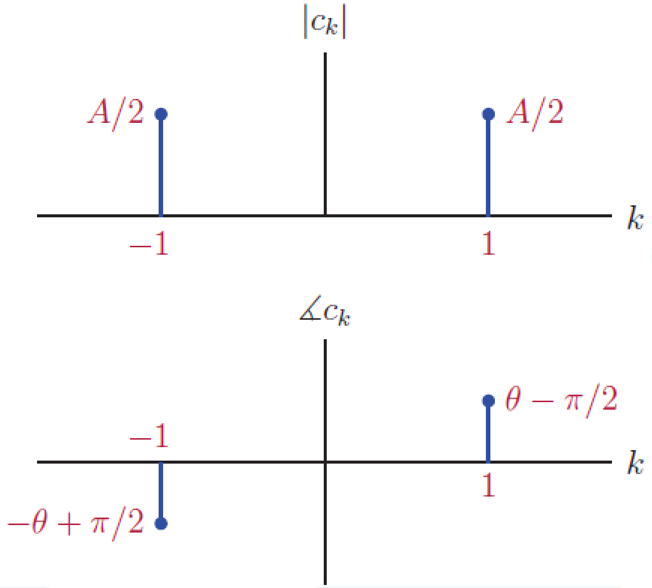
$$\tilde{x}(t) = A\sin(\omega_0 t + \theta) = \frac{A}{2} e^{j(\theta - \pi/2)} e^{j\omega_0 t} + \frac{A}{2} e^{-j(\theta - \pi/2)} e^{-j\omega_0 t}$$

$$c_1 = \tfrac{A}{2} e^{j(\theta - \pi/2)}, \quad c_{-1} = \tfrac{A}{2} e^{-j(\theta - \pi/2)}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

- The EFS representations of the two signals are shown graphically, in the form of a line spectrum.



$\tilde{x}(t) = A\cos(\omega_0 t + \theta)$



$\tilde{x}(t) = A\sin(\omega_0 t + \theta)$

The general case:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

$$c_0 = a_0$$

$$c_k + c_{-k} = a_k \quad \text{and} \quad j(c_k - c_{-k}) = b_k, \quad \text{for } k = 1, \cdots, \infty$$

$$c_k = \tfrac{1}{2}(a_k - jb_k) \quad \text{and} \quad c_{-k} = \tfrac{1}{2}(a_k + jb_k), \quad \text{for } k = 1, \cdots, \infty$$

- What if we would like to compute the EFS coefficients of a signal without first having to obtain the TFS coefficients? The exponential basis functions also form an orthogonal set.

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} e^{-jk\omega_0 t} dt = \begin{cases} T_0, & m = k \\ 0, & m \neq k \end{cases}$$

Exponential Fourier series (EFS):

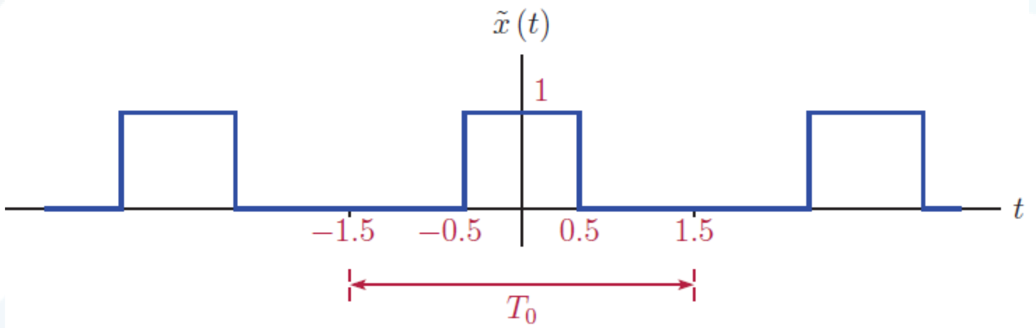
1. Synthesis equation: $$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

2. Analysis equation: $$c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t)\, e^{-jk\omega_0 t} dt$$

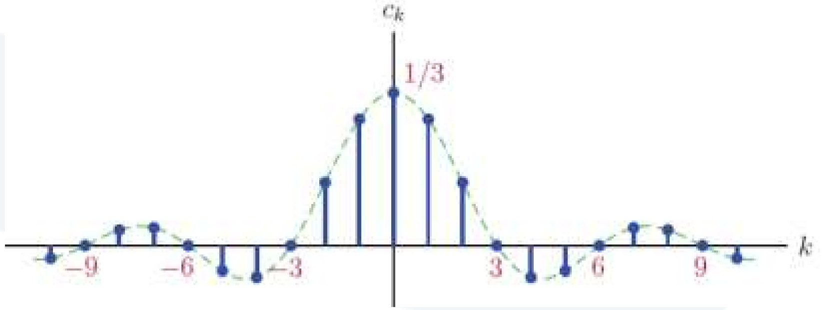
- In general, the coefficients of the EFS representation of a periodic signal $\tilde{x}(t)$ are complex valued.
- They can be graphed in the form of a line spectrum if each coefficient is expressed in polar complex form with its magnitude and phase: $c_k = |c_k| e^{j\theta_k}$
- Example 3: Exponential Fourier series for periodic pulse train

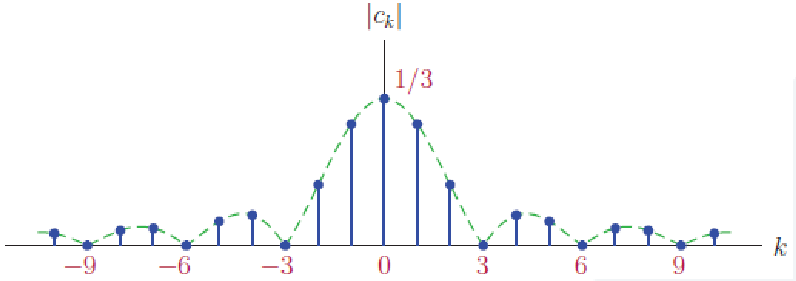
$$c_k = \frac{1}{3}\int_{-0.5}^{0.5} (1)\, e^{-j2\pi kt/3} dt = \frac{\sin(\pi k/3)}{\pi k}$$

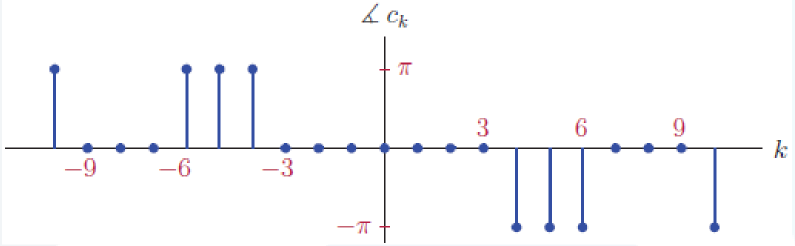
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/3)}{\pi k} e^{j2\pi kt/3}$$



- A line graph of the set of coefficients c_k is useful for illustrating the make-up of the signal $\tilde{x}(t)$ in terms of its harmonics.







Properties of Fourier series

Linearity

$$\alpha_1 \tilde{x}(t) + \alpha_2 \tilde{y}(t) = \sum_{k=-\infty}^{\infty} [\alpha_1 c_k + \alpha_2 d_k] e^{jk\omega_0 t}$$

Symmetry of Fourier series

$$\tilde{x}(t)\text{: real, } \mathrm{Im}\{\tilde{x}(t)\} = 0 \Rightarrow c_{-k} = c_k^*, \qquad \tilde{x}(t)\text{: imag, } \mathrm{Re}\{\tilde{x}(t)\} = 0 \Rightarrow c_{-k} = -c_k^*$$

3. Analysis of Non-Periodic Continuous-Time Signals

- Consider the non-periodic signal $x(t)$.

Fourier transform for continuous-time signals:

1. Synthesis equation: (Inverse transform)

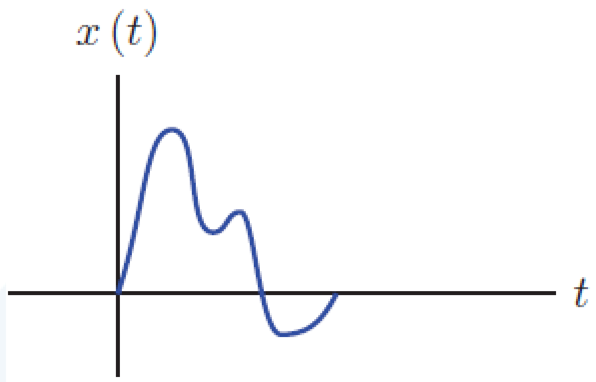
$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft} df \quad \text{(using } f\text{)}$$

2. Analysis equation: (Forward transform)

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \qquad \text{(using } f\text{)}$$

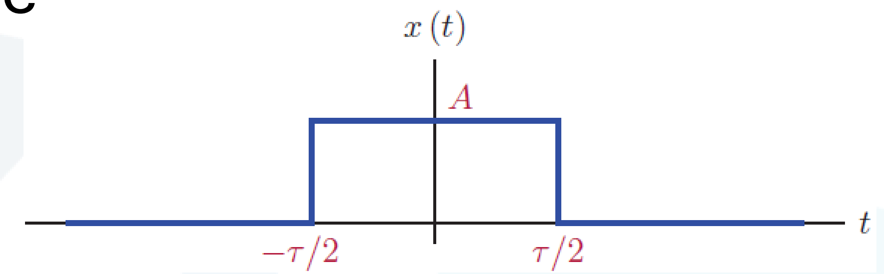


What frequencies are contained in this signal?

Fourier transforms of some signals

Example 4: Fourier transform of a rectangular pulse

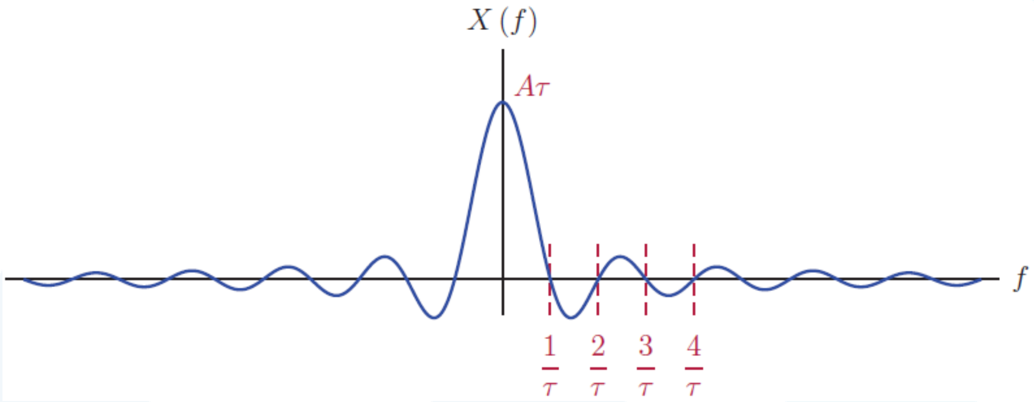
$$x(t) = A\Pi\left(\frac{t}{\tau}\right)$$



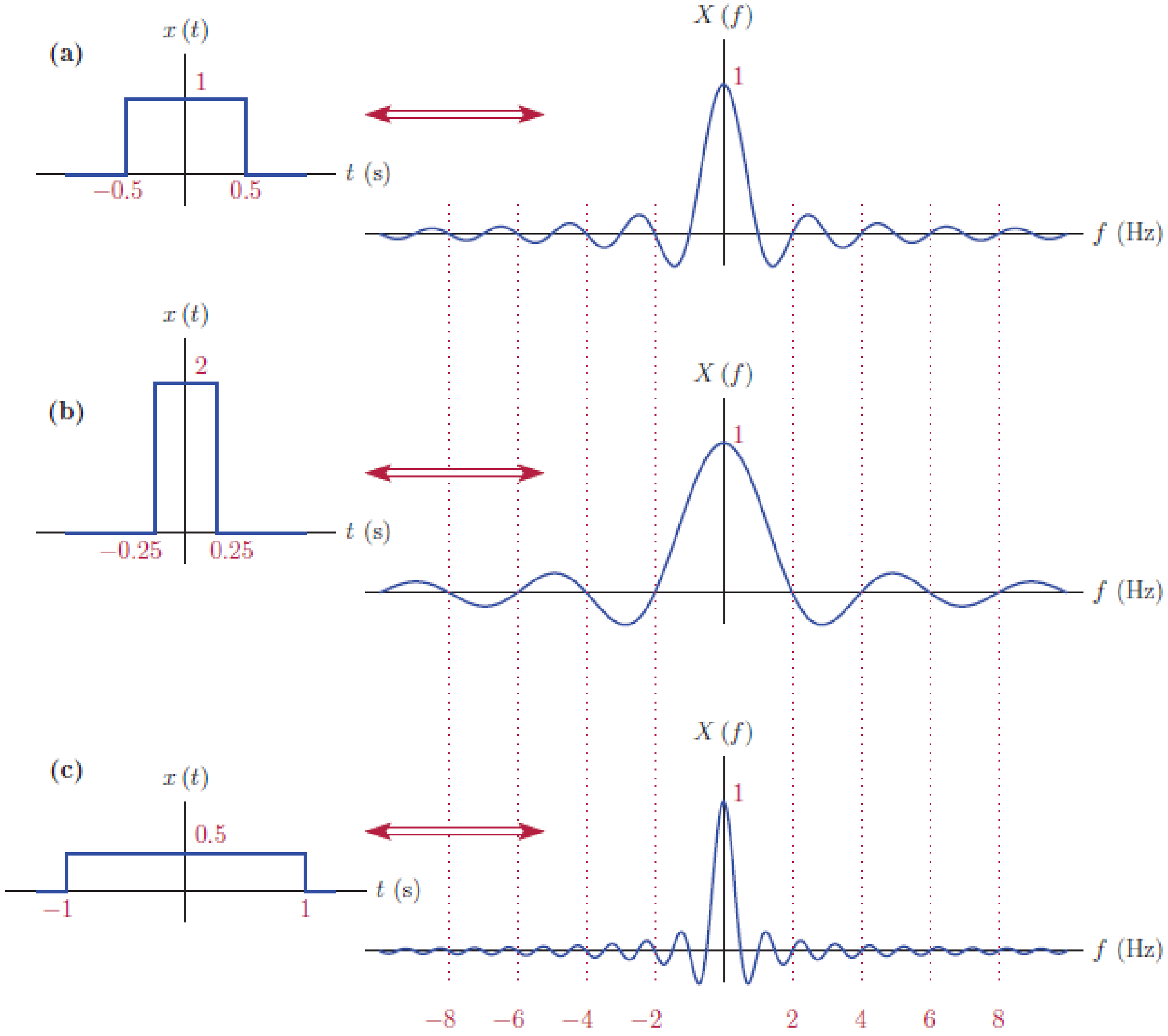
$$X(\omega) = \int_{-\tau/2}^{\tau/2} (A)\, e^{-j\omega t} dt = A \frac{e^{-j\omega t}}{-j\omega}\bigg|_{-\tau/2}^{\tau/2} = \frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$X(f) = A\tau \operatorname{sinc}(f\tau)$$



- Effects of changing the pulse width on the frequency spectrum:


(a)
x (t)
1
−0.5
0.5
t (s)
X (f)
1
f (Hz)
(b)
x (t)
2
−0.25
0.25
t (s)
X (f)
1
f (Hz)
(c)
x (t)
0.5
−1
1
t (s)
X (f)
1
f (Hz)
−8
−6
−4
−2
2
4
6
8

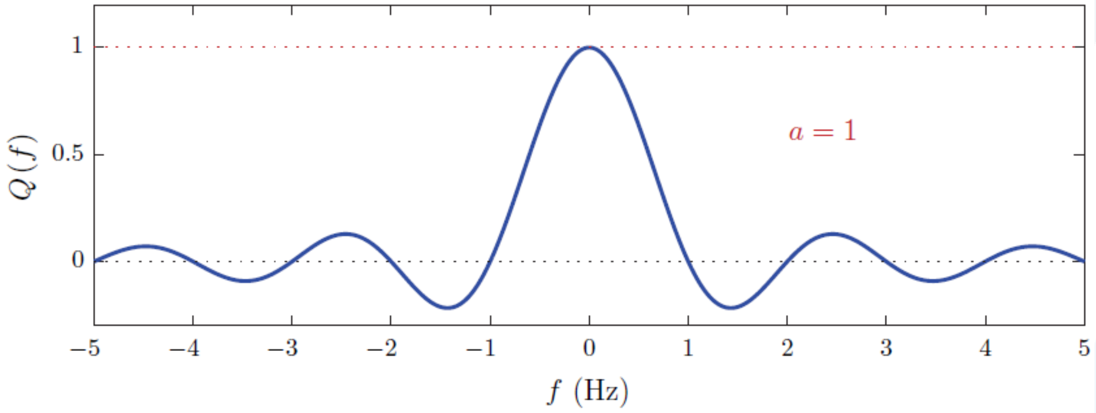
Example 5: Transform of the unit-impulse function

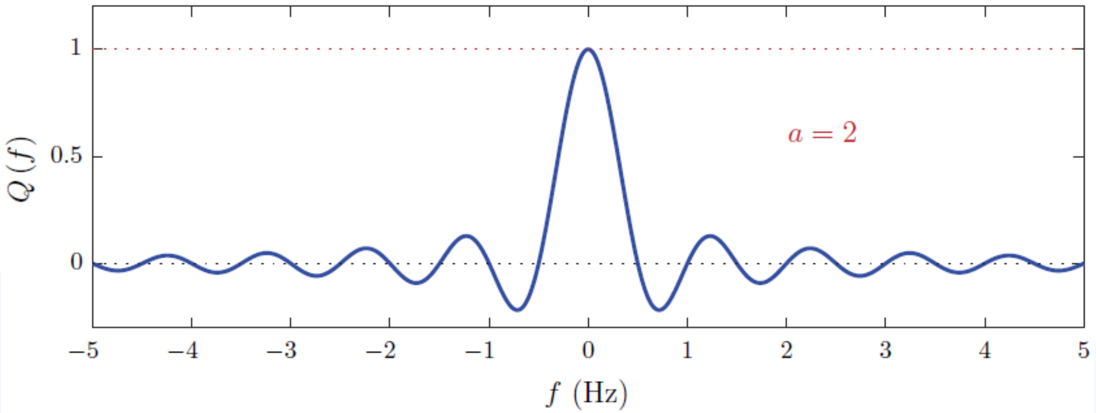
$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

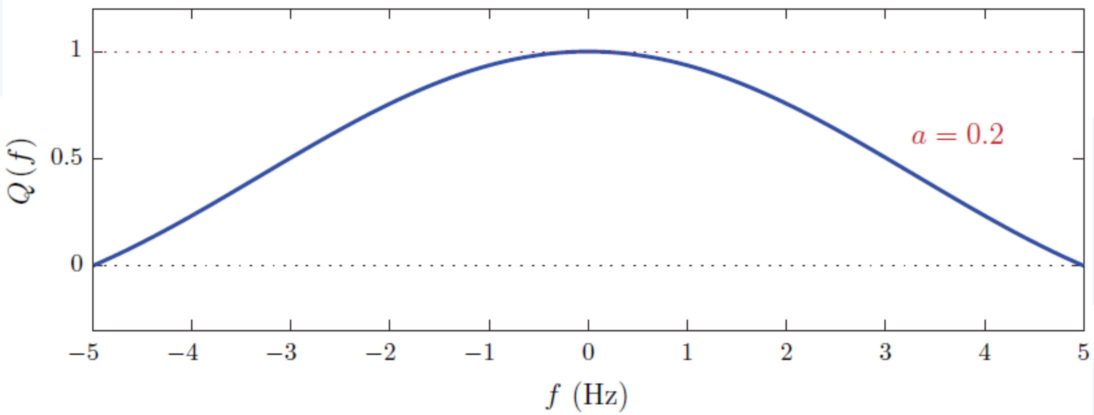
$$q(t) = \frac{1}{a}\Pi\left(\frac{t}{a}\right) \Rightarrow \delta(t) = \lim_{a\to 0} q(t)$$

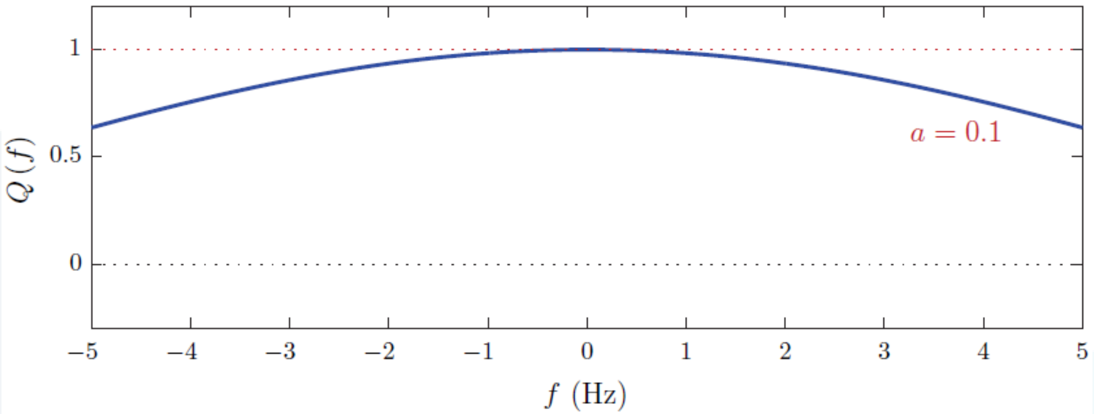
$$Q(f) = \mathcal{F}\{q(t)\} = \text{sinc}\,(fa)$$

$$\mathcal{F}\{\delta(t)\} = \lim_{a\to 0}\{Q(f)\} = \lim_{a\to 0}\{\text{sinc}(fa)\} = 1$$



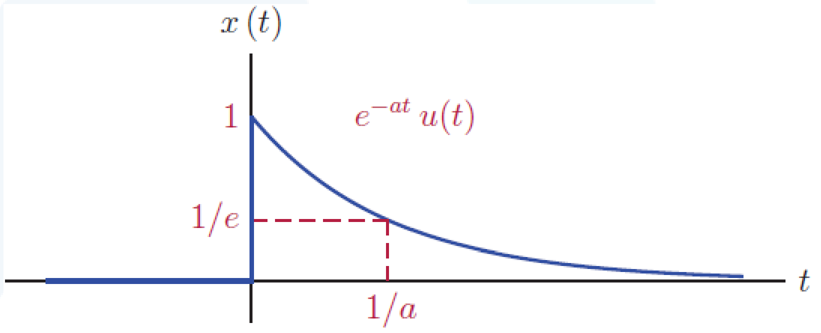






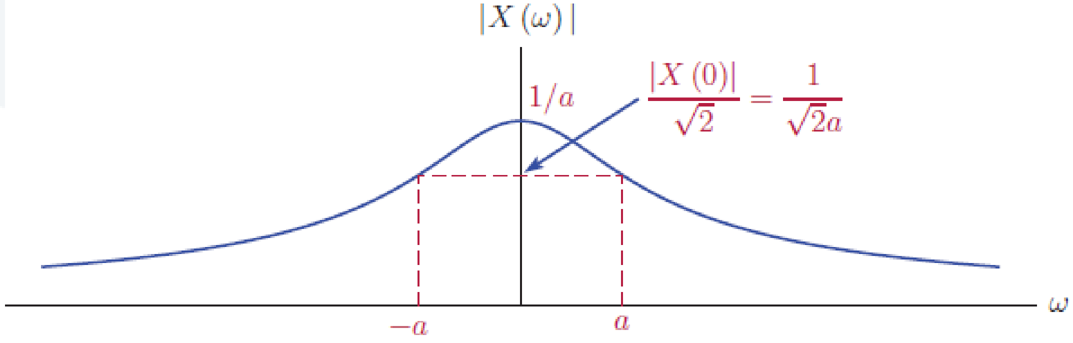
Example 6: Fourier transform of a right-sided exponential signal

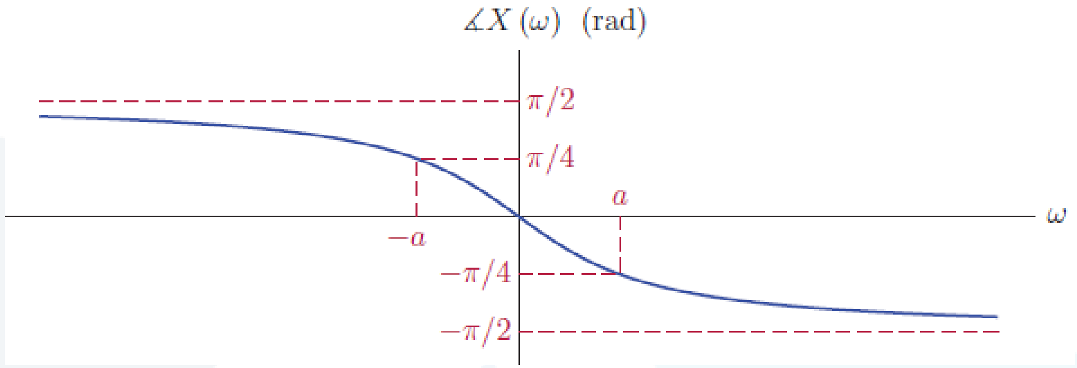
$$x(t) = e^{-at}u(t),\ a > 0$$



$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt = \frac{1}{a + j\omega}$$

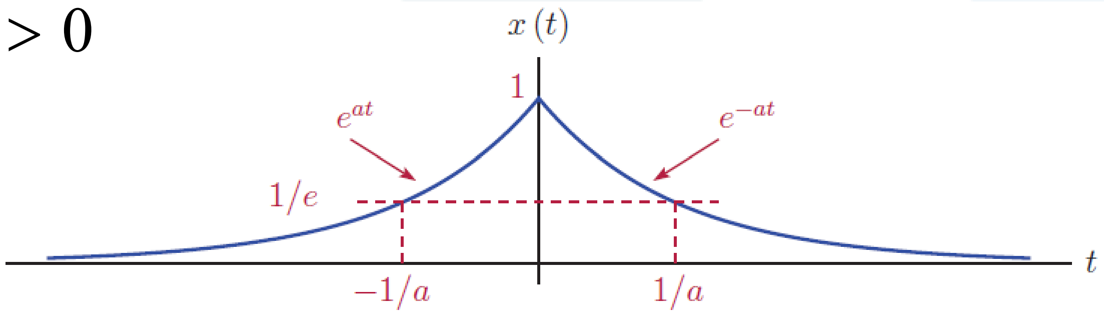
$$|X(\omega)| = \left|\frac{1}{a + j\omega}\right| = \frac{1}{\sqrt{a^2 + \omega^2}}, \qquad \theta(\omega) = -\tan^{-1}(\omega/a)$$

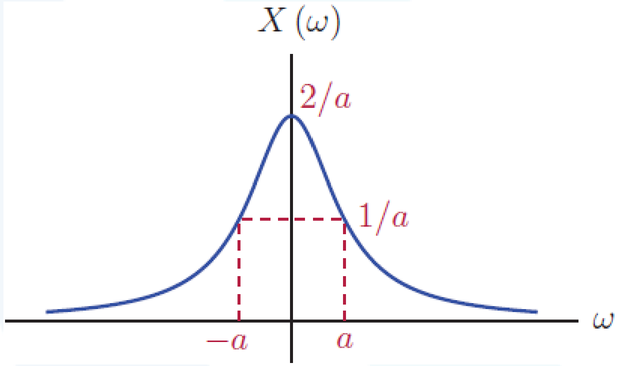




- Example 7: Fourier transform of a two-sided exponential signal

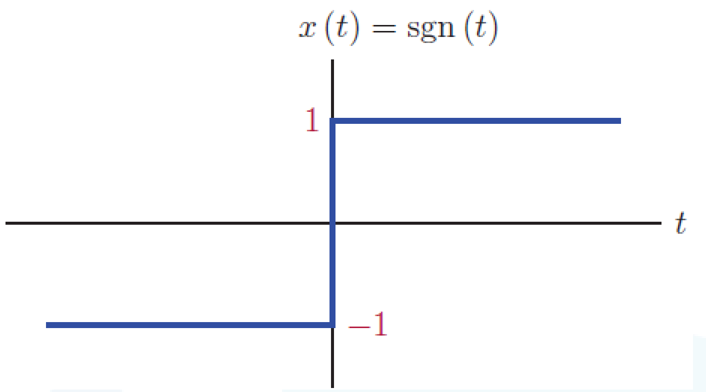
$$x(t) = e^{-a|t|},\ a > 0$$





$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at}\, e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt = \frac{2a}{a^2 + \omega^2}$$

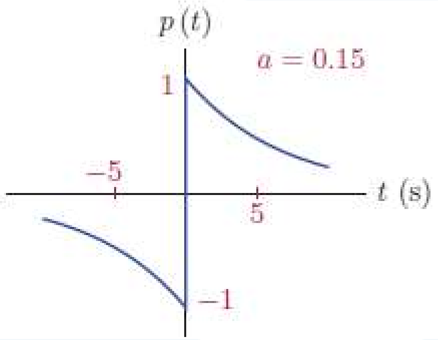
- **Example 8:** Fourier transform of the signum function

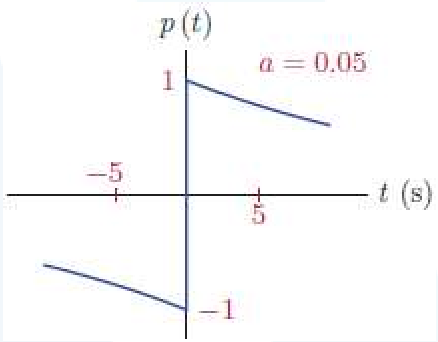


$$x(t) = \text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{0} (-1)\, e^{-j\omega t} dt + \int_{0}^{\infty} (1)\, e^{-j\omega t} dt$$

The two integrals cannot be evaluated. Instead, we will define an intermediate signal $p(t)$ as:





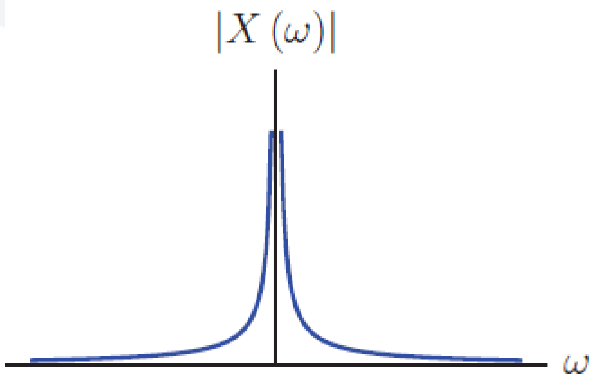
$$p(t) = \begin{cases} -e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}, \text{ where } a \geq 0$$

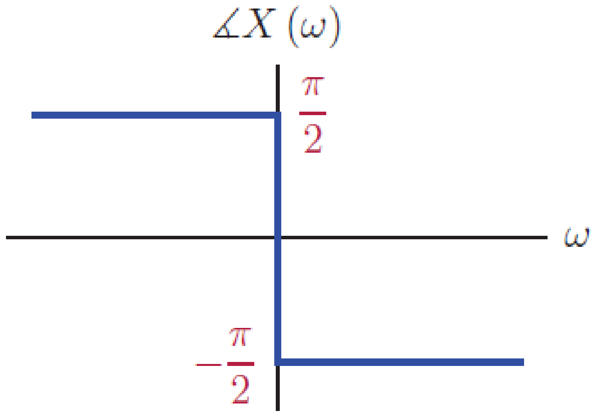
$$P(\omega) = \int_{-\infty}^{0} (-e^{at})\, e^{-j\omega t} dt + \int_{0}^{\infty} (e^{-at})\, e^{-j\omega t} dt = \frac{-j2\omega}{a^2 + \omega^2}$$

$$X(\omega) = \mathcal{F}\{\text{sgn}(t)\} = \lim_{a \to 0} \frac{-j2\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$

$$|X(\omega)| = \frac{2}{|\omega|}$$

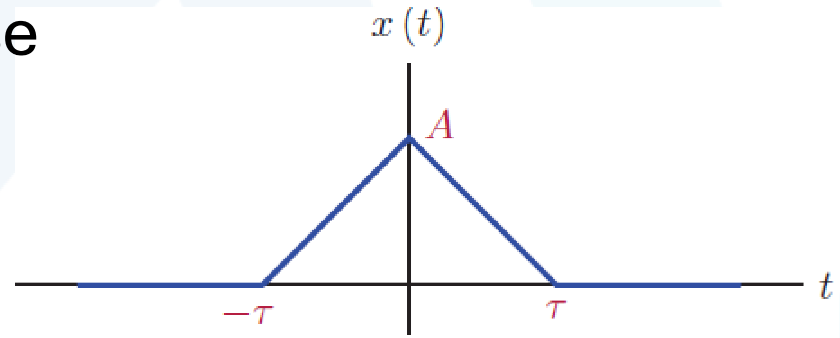
$$\theta(\omega) = \begin{cases} \frac{\pi}{2}, & \omega < 0 \\ -\frac{\pi}{2}, & \omega > 0 \end{cases}$$





- Example 9: Fourier transform of a triangular pulse

$$x(t) = A\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} A + At/\tau, & -\tau < t < 0 \\ A - At/\tau, & 0 < t < \tau \\ 0, & |t| \geq \tau \end{cases}$$



$$X(\omega) = \int_{-\tau}^{0} (A + At/\tau)\, e^{-j\omega t} dt + \int_{0}^{\tau} (A - At/\tau)\, e^{-j\omega t} dt = \frac{2A}{\omega^2 \tau}[1 - \cos(\omega\tau)]$$

$$\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) = \frac{\sin(\omega\tau/2)}{\omega\tau/2} = \frac{2}{\omega\tau}\sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right) \qquad X(f) = A\tau \operatorname{sinc}^2(f\tau)$$