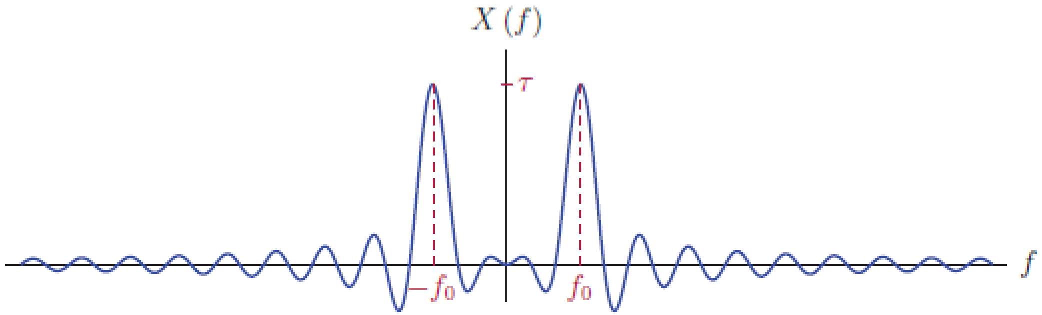
CECC507: Signals and Systems

Lecture Notes 6: Fourier Analysis for Continuous Time Signals and Systems: Part B



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Chapter 4

Fourier Analysis for Continuous Time Signals and Systems

Properties of Fourier transform

Linearity of the Fourier transform: $\mathcal{F}\{\alpha_1 x(t) + \alpha_2 y(t)\} = \alpha_1 \mathcal{F}\{x(t)\} + \alpha_2 \mathcal{F}\{y(t)\}$

Duality property: $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad X(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$

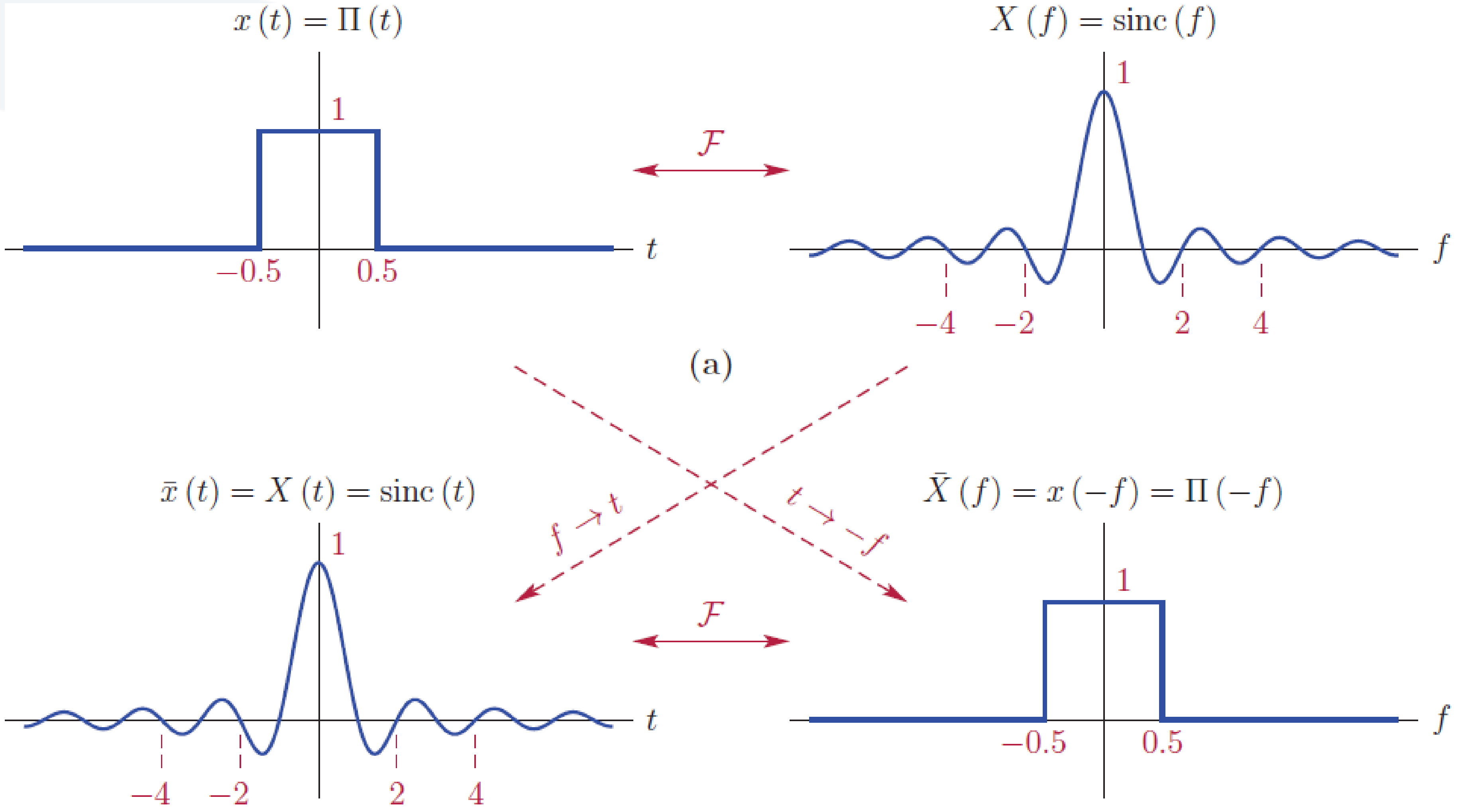
Duality property (using f): $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f) \quad \Rightarrow \quad X(t) \stackrel{\mathcal{F}}{\longleftrightarrow} x(-f)$

- Example 1: Fourier transform of the sinc function

$$\mathcal{F}\left\{\frac{1}{2\pi}\Pi\left(\frac{t}{2\pi}\right)\right\} = \text{sinc}(\omega) \Rightarrow$$

$$\mathcal{F}\{\text{sinc}(t)\} = \Pi\left(\frac{-\omega}{2\pi}\right) = \Pi\left(\frac{\omega}{2\pi}\right)$$

$$\mathcal{F}\{\text{sinc}(t)\} = \Pi(f)$$



$x(t) = \Pi(t)$
1
-0.5
0.5
t
$\mathcal{F}$
$X(f) = \mathrm{sinc}(f)$
1
-4
-2
2
4
f
(a)
$\bar{x}(t) = X(t) = \mathrm{sinc}(t)$
$f \to t$
$t \to -f$
$\bar{X}(f) = x(-f) = \Pi(-f)$
1
-4
-2
2
4
t
$\mathcal{F}$
1
-0.5
0.5
f

- Example 2: Transform of a constant-amplitude signal

$$F\{\delta(t)\} = 1 \text{ , all } \omega \quad \Rightarrow \quad F\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega), \quad F\{1\} = \delta(f) \quad \text{(duality)}$$

- Example 3: Fourier transform of the unit-step function

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{0}^{\infty} e^{-j\omega t} dt$$ could not be evaluated

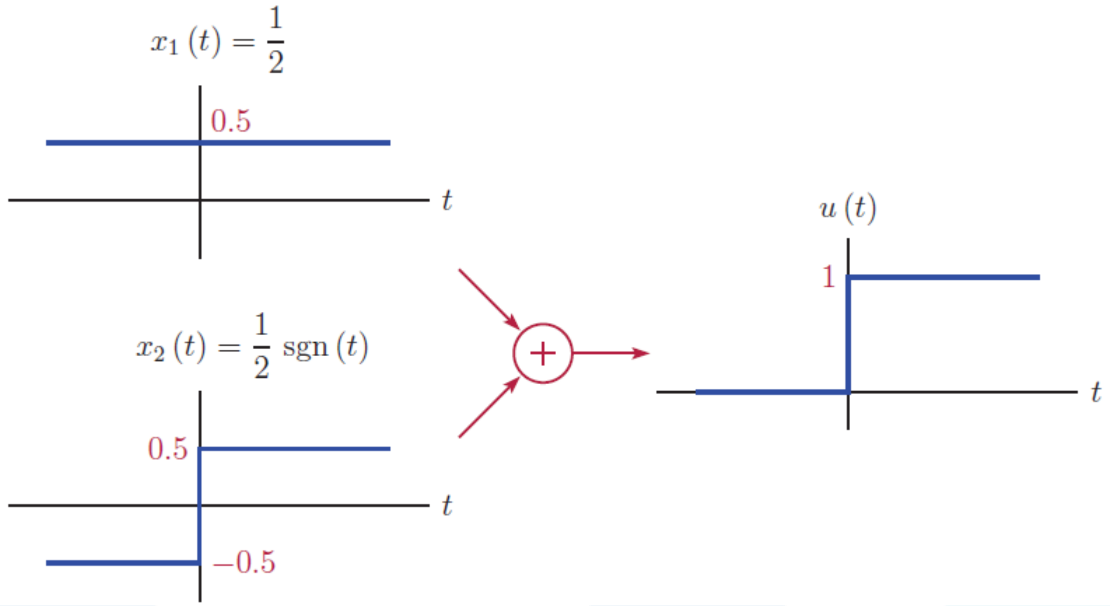
$$u(t) = ½ + ½ \text{ sgn}(t)$$

$$\mathcal{F}\{u(t)\} = \mathcal{F}\{½ + ½ \text{ sgn}(t)\}$$

$$= ½F\{1\} + ½\mathcal{F}\{\text{ sgn}(t)\}$$

$$\mathcal{F}\{u(t)\} = \pi\,\delta(\omega) + \frac{1}{j\omega}$$

$$\mathcal{F}\{u(t)\} = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$



Symmetry of the Fourier transform

$$x(t)\text{: real, } \mathrm{Im}\{x(t)\} = 0 \Rightarrow X^*(\omega) = X(-\omega)$$

$$\tilde{x}(t)\text{: imag, } \mathrm{Re}\{\tilde{x}(t)\} = 0 \Rightarrow X^*(\omega) = -X(-\omega)$$

Transforms of even and odd signals

- If the real-valued signal $x(t)$ is an even function of time, the resulting Fourier transform $X(\omega)$ is real-valued for all ω.

$$x(-t) = x(t), \text{ for all } t \;\Rightarrow \mathrm{Im}\{X(\omega)\} = 0, \text{ for all } \omega$$

- If the real-valued signal $x(t)$ has odd-symmetry, the resulting Fourier transform $X(\omega)$ is purely imaginary.

$$x(-t) = -x(t), \text{ for all } t \;\Rightarrow \mathrm{Re}\{X(\omega)\} = 0, \text{ for all } \omega$$

Time shifting $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad x(t-\tau) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)e^{-j\omega\tau}$

Frequency shifting $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad x(t)e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)$

Modulation property $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow$

$$x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{2}[X(\omega-\omega_0)+X(\omega+\omega_0)]$$

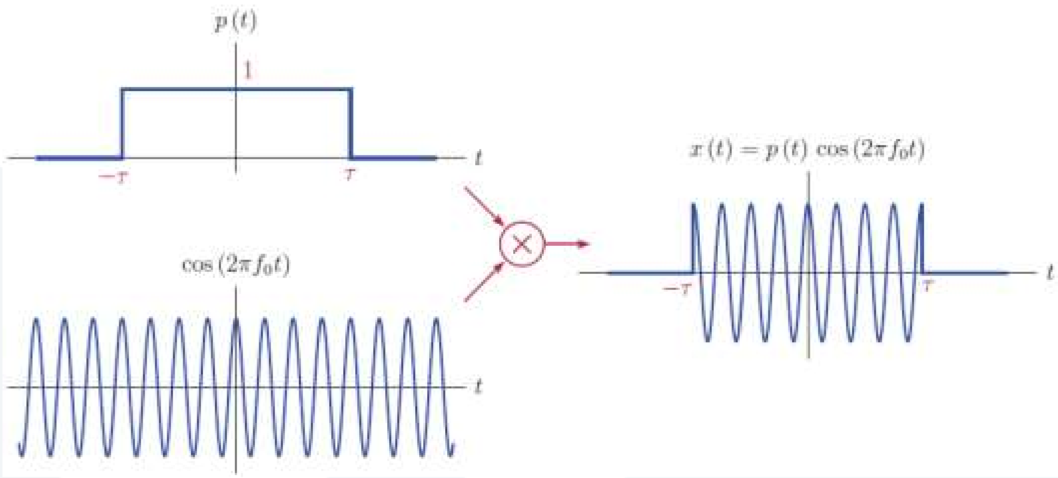
$$x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{2}\left[X(\omega-\omega_0)e^{-j\pi/2}+X(\omega+\omega_0)e^{j\pi/2}\right]$$

- Example 4: Modulated pulse

$$x(t) = \begin{cases} \cos(2\pi f_0 t), & |t| < \tau \\ 0, & |t| > \tau \end{cases}$$

Using $p(t)$, the signal $x(t)$ can be expressed as $x(t) = p(t)\cos(2\pi f_0 t)$

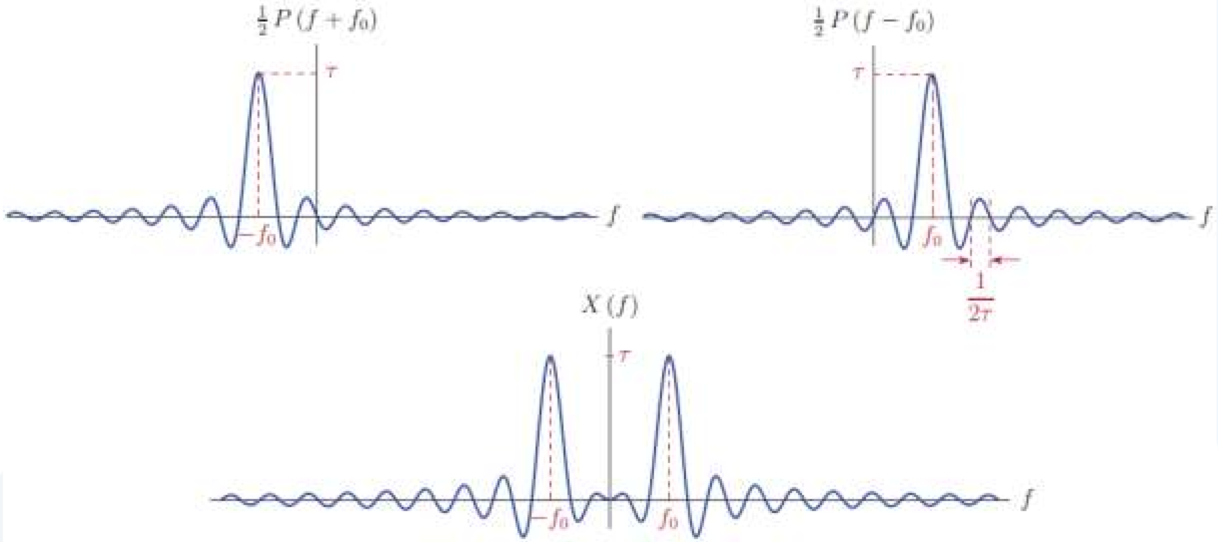
where $p(t) = \Pi\left(\frac{t}{2\tau}\right)$



$$P(f) = 2\tau\,\mathrm{sinc}\,(2\tau f)$$

$$X(f) = \tfrac{1}{2}\left[P(f - f_0) + P(f + f_0)\right]$$

$$= \tau\,\mathrm{sinc}\,(2\tau(f + f_0)) + \tau\,\mathrm{sinc}\,(2\tau(f - f_0))$$



Time and frequency scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

The parameter a is any non-zero and real-valued constant.

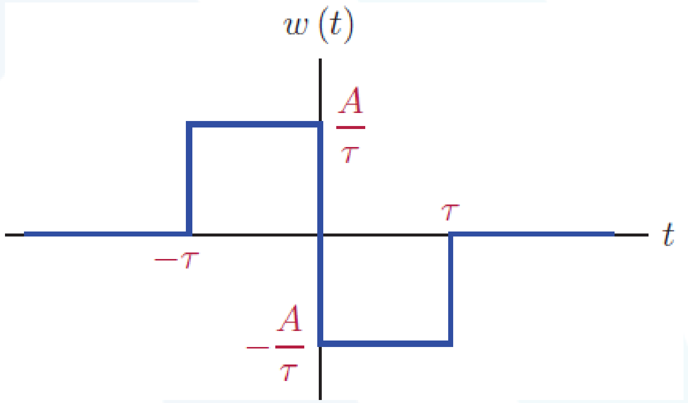
Differentiation in the time domain

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \Rightarrow \frac{d^n}{dt^n}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^n X(\omega), \quad \frac{d^n}{dt^n}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j2\pi f)^n X(f)$$

- Example 5: Triangular pulse revisited

$$x(t) = A\Lambda(t/\tau)$$

$$w(t) = \frac{dx(t)}{dt} = \frac{A}{\tau}\left[\Pi\left(\frac{t+\tau/2}{\tau}\right) - \Pi\left(\frac{t-\tau/2}{\tau}\right)\right]$$



$$W(f) = A\,\mathrm{sinc}(f\tau)e^{j2\pi f(\tau/2)} - A\,\mathrm{sinc}(f\tau)e^{-j2\pi f(\tau/2)} = 2jA\,\mathrm{sinc}(f\tau)\sin(\pi f\tau)$$

$$W(f) = (j2\pi f)X(f) \Rightarrow X(f) = \frac{W(f)}{j2\pi f} = \frac{2jA\,\mathrm{sinc}(f\tau)\sin(\pi f\tau)}{j2\pi f} = A\tau\mathrm{sinc}^2(f\tau)$$

Differentiation in the frequency domain

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad (-jt)^n x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d^n}{d\omega^n}[X(\omega)]$$

Convolution property

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

$$\Rightarrow \quad x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)X_2(\omega)$$

Multiplication of two signals

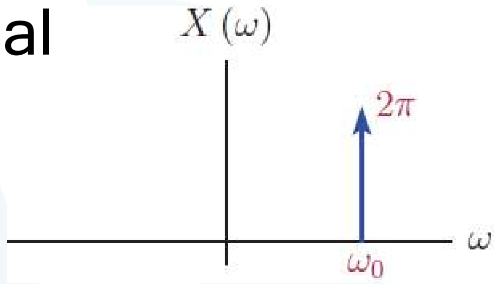
$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

$$\Rightarrow x_1(t)x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{2\pi} X_1(\omega) * X_2(\omega), \quad x_1(t)x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f) * X_2(f)$$

Integration $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad \int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Applying Fourier transform to periodic signals

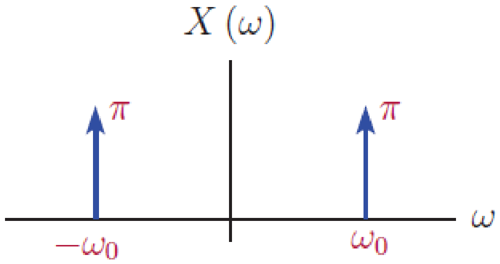
- Example 6: Fourier transform of complex exponential signal



$$x(t) = e^{j\omega_0 t}$$

$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \quad \Rightarrow \quad \mathcal{F}(e^{j\omega_0 t}) = 2\pi\delta(\omega - \omega_0)$$

- Example 7: Fourier transform of sinusoidal signal



$$x(t) = \cos(\omega_0 t)$$

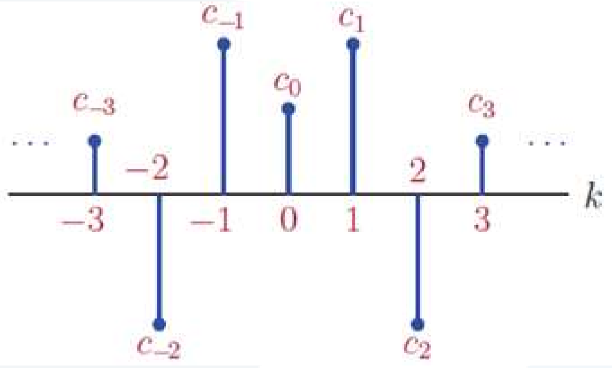
$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \Rightarrow \mathcal{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

- The idea can be generalized to apply to any periodic continuous-time signal that has an EFS representation:

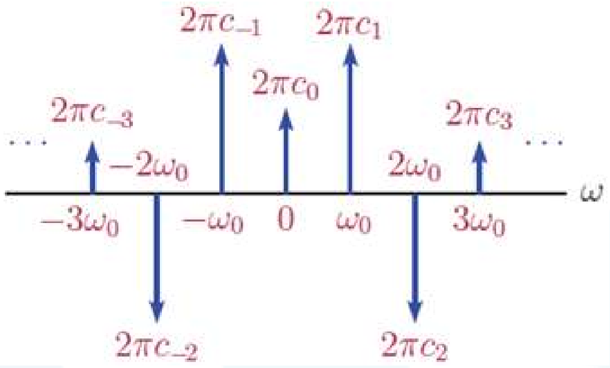
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Rightarrow X(\omega) = \int_{-\infty}^{\infty} \tilde{x}(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right] e^{-j\omega t} dt$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k \left[\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \right]$$

$$= \sum_{k=-\infty}^{\infty} c_k \left[2\pi\delta(\omega - k\omega_0) \right]$$



EFS coefficients for a signal



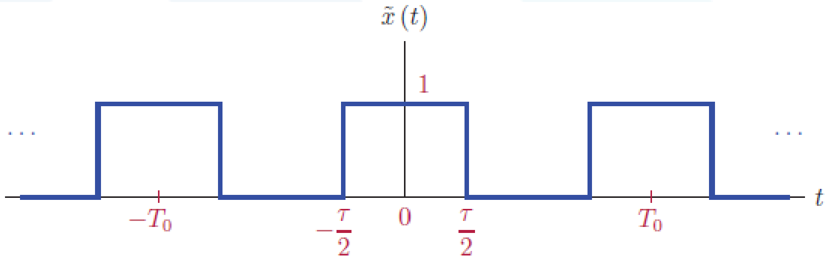
Fourier transform obtained

- Example 8: Fourier transform of periodic pulse train

  Determine the FT of the periodic pulse train with duty cycle $d = \tau / T_0$

  $c_k = d\,\text{sinc}\,(kd)$ $\quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi d\, \text{sinc}(kd)\delta(\omega - k\omega_0)$

  $\omega_0 = 1/T_0$ is the fundamental radian frequency.



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4. Energy and Power in the Frequency Domain

Parseval's theorem

- For a periodic power signal $\tilde{x}(t)$ with period T_0 and EFS coefficients $\{c_k\}$:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |\tilde{x}(t)|^2\, dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

- For a non-periodic energy signal $x(t)$ with a Fourier transform $X(f)$:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Energy and power spectral density

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_0)$$ power spectral density of the signal $x(t)$

$$\int_{-\infty}^{\infty} S_x(f)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = \sum_{k=-\infty}^{\infty} |c_k|^2$$

P_x in $(-f_0, f_0) = \int_{-f_0}^{f_0} S_x(f)\,df$

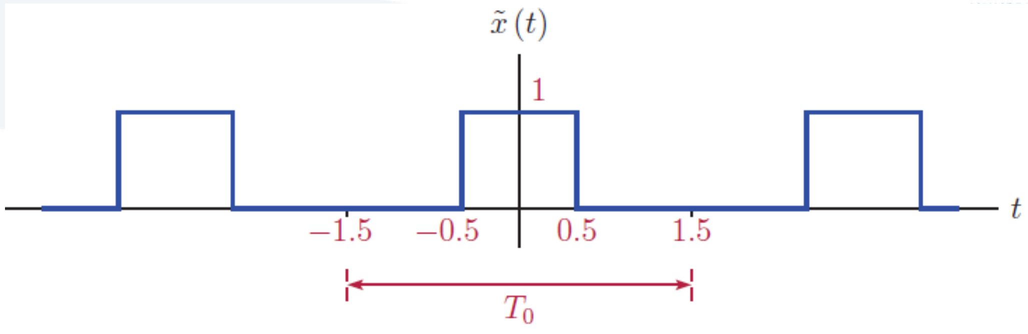
$G_x(f) = |X(f)|^2$ energy spectral density of the signal $x(t)$

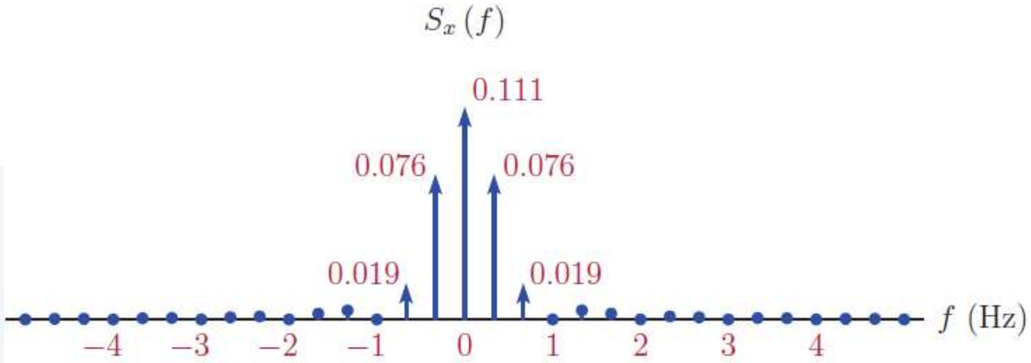
$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} G_x(f)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_x(\omega)\,d\omega$$

- Example 9: Power spectral density of a periodic pulse train

  Determine the power spectral density for $x(t)$. Also find the total power, the dc power, the power in the first three harmonics, and the power above 1 Hz.

$$c_k = \tfrac{1}{3}\operatorname{sinc}(k/3) \qquad S_x(f) = \sum_{k=-\infty}^{\infty} \left|\tfrac{1}{3}\operatorname{sinc}(k/3)\right|^2 \delta(f - k/3)$$





The total power in the signal $x(t)$: $\frac{1}{T_0}\int_{t_0}^{t_0+T_0}|\tilde{x}(t)|^2\,dt=\frac{1}{3}\int_{-0.5}^{0.5}(1)^2\,dt=\frac{1}{3}$

$$P_1=|c_{-1}|^2+|c_1|^2=\frac{3}{2\pi^2}\approx 0.1520,\quad P_2=|c_{-2}|^2+|c_2|^2=\frac{3}{8\pi^2}\approx 0.0380,\quad P_3=0$$

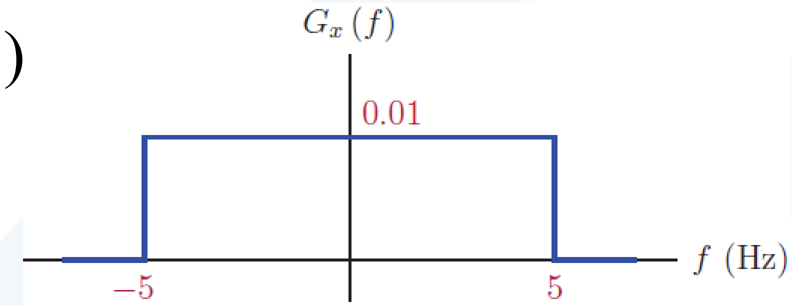
The third harmonic is at frequency $f=1$ Hz. Thus, the power above 1 Hz:

$$P_{hf}=P_x-P_{dc}-P_1-P_2-P_3=0.3333-0.1111-0.1520-0.0380-0=0.0322$$

- Example 10: Energy spectral density of the sinc function

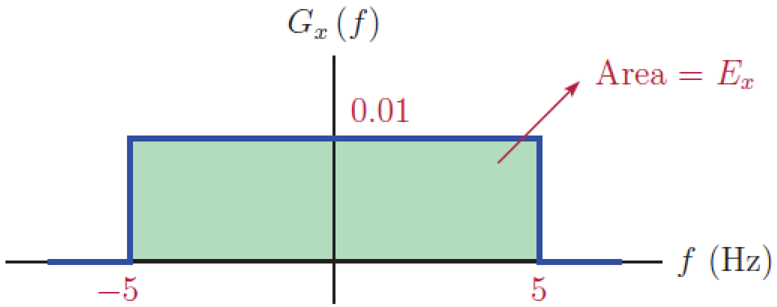
  Determine the energy spectral density of $x(t) = \text{sinc}(10t)$. Afterwards, compute the total energy, and the energy in the sinc pulse at frequencies up to 3 Hz.

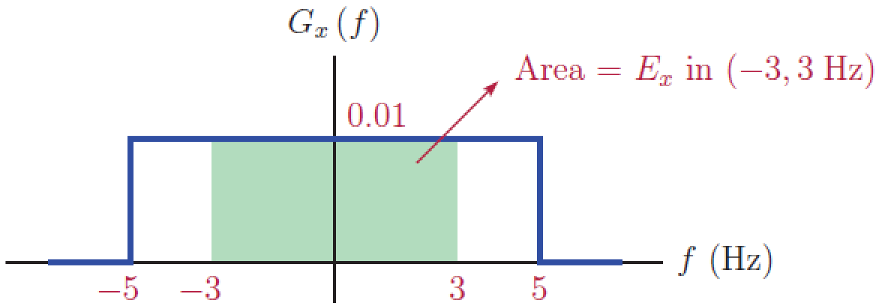
$$X(f) = \frac{1}{10}\Pi\left(\frac{f}{10}\right), \qquad G_x(f) = |X(f)|^2 = \frac{1}{100}\text{sinc}^2\left(\frac{f}{10}\right)$$



$$E_x = \int_{-\infty}^{\infty} G_x(f)\,df = \int_{-5}^{5} \frac{1}{100}\,df = 0.1$$

$$E_x \text{ in } (-3,\ 3\text{ Hz}) = \int_{-3}^{3} G_x(f)\,df = \int_{-3}^{3} \frac{1}{100}\,df = 0.06$$





Autocorrelation

- For an energy signal $x(t)$ the autocorrelation function is defined as

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

- For a periodic power signal $\tilde{x}(t)$ with period T_0, the corresponding definition of the autocorrelation function is:

$$\tilde{r}_{xx}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\tilde{x}(t+\tau)dt$$

- The energy spectral density is the FT of the autocorrelation function:

$$F\{r_{xx}(\tau)\} = G_x(f)$$

- The power spectral density is the FT of the autocorrelation function:

$$F\{\tilde{r}_{xx}(\tau)\} = S_x(f)$$

- Example 11: Power spectral density of a sinusoidal signal revisited

$$\tilde{x}(t) = 5\cos(200\pi t)$$

$$\tilde{r}_{xx}(\tau) = \frac{1}{0.01}\int_{-0.005}^{0.005} 25\cos(200\pi t)\cos(200[t+\tau])\,dt = \frac{25}{2}\cos(200\pi\tau)$$

$$S_x(f) = F\{\tilde{r}_{xx}(\tau)\} = \frac{25}{4}\delta(f+100) + \frac{25}{4}\delta(f-100)$$

Properties of the autocorrelation function

- $r_{xx}(0) \geq |r_{xx}(\tau)|$ for all τ
- $r_{xx}(-\tau) = r_{xx}(\tau)$ for all τ, that is, the autocorrelation function has even symmetry.
- If the signal $x(t)$ is periodic with period T, then its autocorrelation function $\tilde{r}_{xx}(\tau)$ is also periodic with the same period.

5. Transfer Function Concept

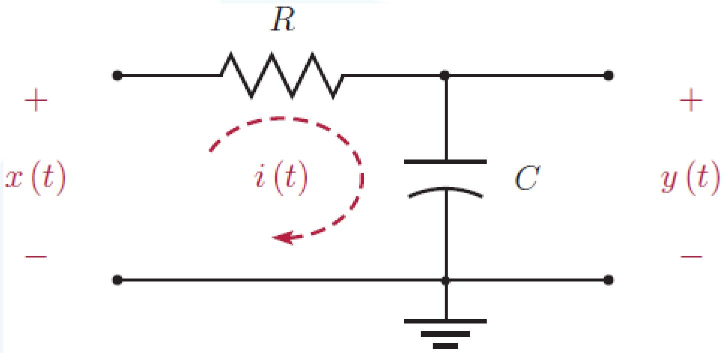
- In time-domain analysis of systems we have relied on two distinct description forms for CTLTI systems:

  1. A linear constant-coefficient differential equation that describes the relationship between the input and the output signals.
  2. An impulse response which can be used with the convolution operation for determining the response of the system to an arbitrary input signal.

- The concept of Transfer function will be introduced as the third method for describing the characteristics of a system.

$$H(\omega) = F\{h(t)\} = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt$$

- The transfer function concept is valid for LTI systems only.
- In general, $H(\omega)$ is a complex function of ω, $H(\omega) = |H(\omega)| e^{j\Theta(\omega)}$.
- Example 12: Transfer function for the simple RC circuit



$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$H(\omega) = \int_0^\infty \frac{1}{RC} e^{-t/RC} e^{-j\omega t} dt = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j(\omega/\omega_c)},$$

$$\omega_c = \frac{1}{RC}$$

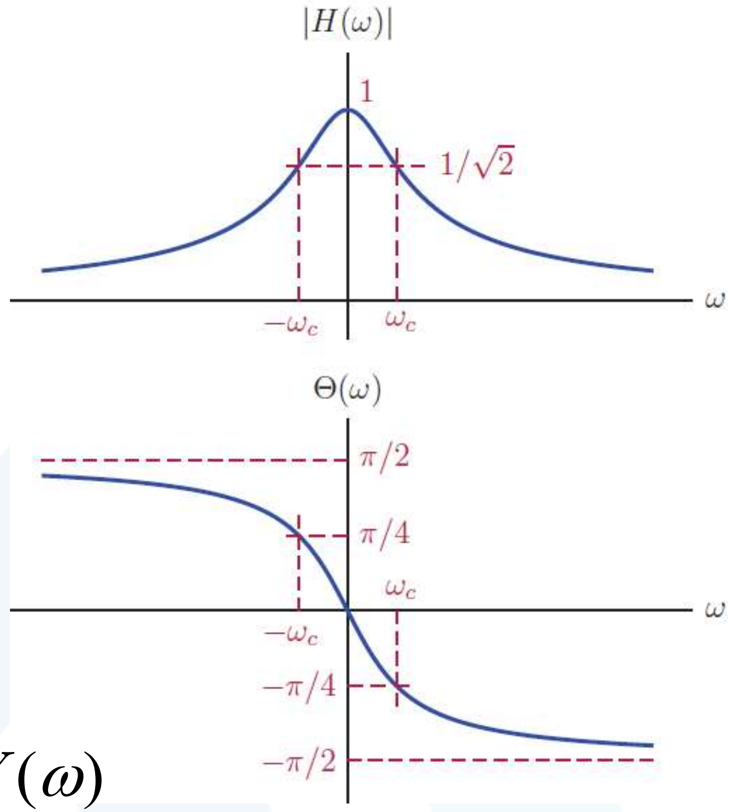
$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}, \quad \Theta(\omega) = -\tan^{-1}(\omega/\omega_c)$$

$$H(\omega_c) = \frac{1}{1 + j}, \quad |H(\omega_c)| = \frac{1}{\sqrt{2}}$$

- ω_c represents the frequency at which the magnitude of the transfer function is 3 decibels below its peak value at $\omega = 0$,

$$20\log_{10}\frac{|H(\omega_c)|}{|H(0)|} = 20\log_{10}\frac{1}{\sqrt{2}} \approx -3\text{dB}$$

- The frequency ω_c is often referred to as the 3-dB cutoff frequency of the system.



Obtaining the TF from the differential equation

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = H(\omega)X(\omega) \Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$\frac{d^k y(t)}{dt^k} \xleftrightarrow{\mathcal{F}} (j\omega)^k\, Y(\omega), \quad \frac{d^k x(t)}{dt^k} \xleftrightarrow{\mathcal{F}} (j\omega)^k\, X(\omega), \qquad k = 0, 1, \cdots$$

- Example 13: Transfer function from the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 26y(t) = x(t)$$

$$(j\omega)^2 Y(\omega) + 2(j\omega)Y(\omega) + 26Y(\omega) = X(\omega)$$

$$[(26-\omega^2) + j2\omega]Y(\omega) = X(\omega) \Rightarrow H(\omega) = \frac{1}{(26-\omega^2) + j2\omega}$$

6. CTLTI Systems with Periodic Input Signals

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k\sin(k\omega_0 t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Response of a CTLTI system to complex exponential signal

$$\tilde{x}(t) = e^{j\omega_0 t}$$

$$y(t) = h(t) * \tilde{x}(t) = \int_{-\infty}^{\infty} h(\tau)\,\tilde{x}(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\,e^{j\omega_0(t-\tau)}d\tau$$

$$= e^{j\omega_0 t}\int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega_0\tau}d\tau = e^{j\omega_0 t}H(\omega_0) = |H(\omega_0)|\,e^{j[\omega_0 t+\Theta(\omega_0)]}$$

- That is, $e^{j\omega t}$ is an eigenfunction of a LTI system and $H(\omega)$ is the corresponding eigenvalue. We refer to H as the frequency response of the system.

Response of a CTLTI system to sinusoidal signal

$$\tilde{x}(t) = \cos(\omega_0 t)$$

$$\tilde{x}(t) = \cos(\omega_0 t) = \tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t}$$

$$y(t) = \tfrac{1}{2}e^{j\omega_0 t}H(\omega_0) + \tfrac{1}{2}e^{-j\omega_0 t}H(-\omega_0)$$

$$= \tfrac{1}{2}e^{j\omega_0 t}\,|H(\omega_0)|\,e^{j\Theta(\omega_0)} + \tfrac{1}{2}e^{-j\omega_0 t}\,|H(-\omega_0)|\,e^{-j\Theta(\omega_0)}$$

If the impulse response $h(t)$ is real-valued:

$$\left|H(-\omega_0)\right| = \left|H(\omega_0)\right|, \quad \Theta(-\omega_0) = -\Theta(\omega_0)$$

$$y(t) = \frac{1}{2}\left|H(\omega_0)\right| e^{j[\omega_0 t + \Theta(\omega_0)]} + \frac{1}{2}\left|H(\omega_0)\right| e^{-j[\omega_0 t + \Theta(\omega_0)]} = \left|H(\omega_0)\right| \cos(\omega_0 t + \Theta(\omega_0))$$

Response of a CTLTI system to periodic input signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$T\{\tilde{x}(t)\} = T\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} T\left\{c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} c_k T\left\{e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}$$

7. CTLTI Systems with Non-Periodic Input Signals

$$y(t) = h(t) * x(t) \Rightarrow Y(\omega) = H(\omega)X(\omega)$$

$$\left|Y(\omega)\right| = \left|H(\omega)\right|\left|X(\omega)\right|, \quad \measuredangle Y(\omega) = \measuredangle X(\omega) + \Theta(\omega)$$

- Example 14: Pulse response of RC circuit

  Consider again the RC circuit. Let $f_c = 1/RC = 80$ Hz. Determine the FT of the response of the system to the unit-pulse input signal $x(t) = \Pi(t)$.

$$H(f) = \frac{1}{1 + j(f/f_c)}, \qquad X(f) = \text{sinc}(f),$$

$$Y(f) = \frac{1}{1 + j(f/80)} \text{sinc}(f),$$

$$|Y(f)| = \frac{1}{\sqrt{1 + (f/80)^2}} |\text{sinc}(f)|,$$

$$\measuredangle Y(f) = -\tan^{-1}(f/80) + \measuredangle[\text{sinc}(f)]$$