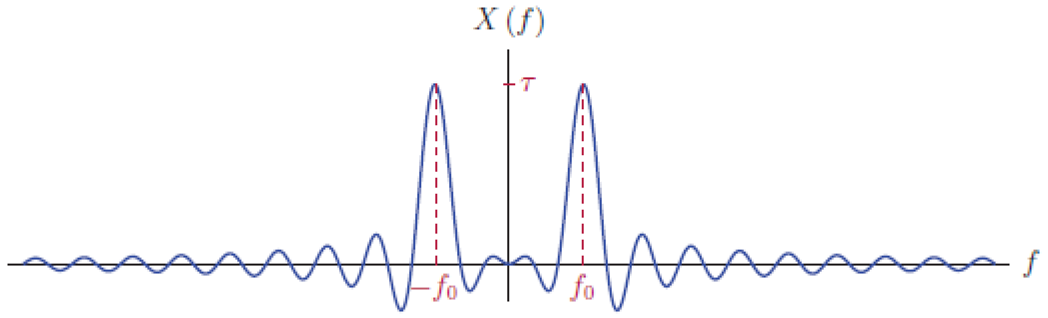
CECC507: Signals and Systems

Lecture Notes 7: Fourier Analysis for Discrete Time Signals and Systems



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Chapter 5

Fourier Analysis for Discrete Time Signals and Systems

1. Introduction

- DTLTI system can be represented by means of a constant coefficient linear difference equation, or alternatively by means of an impulse response.
- The output signal of a DTLTI system can be determined by solving the corresponding difference equation or by using the convolution operation.

2. Analysis of Non-Periodic Discrete-Time Signals

Discrete-time Fourier transform (DTFT)

1. Synthesis equation: (Inverse transform) $x[n] = \mathcal{F}^{-1}\{X(\Omega)\} = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n} d\Omega$

2. Analysis equation: (Forward transform) $X(\Omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

Existence of the DTFT

- A sufficient condition for the convergence of DTFT is that the signal $x[n]$ be absolute summable,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Alternatively, it is also sufficient for $x[n]$ to be square-summable: $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

DTFT of some signals

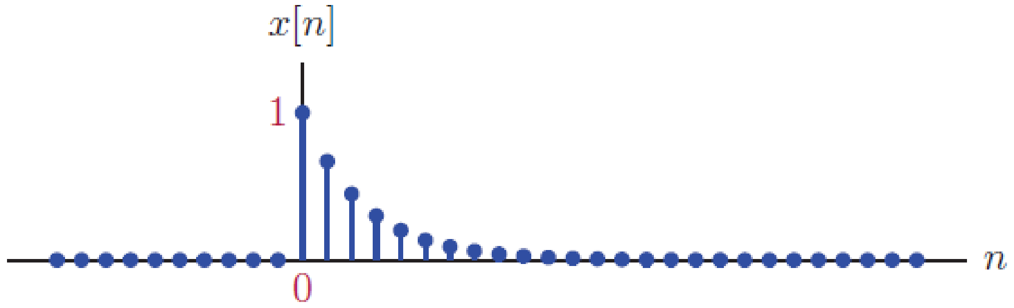
- Example 1: DTFT of unit-impulse signal

$$\mathcal{F}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$$

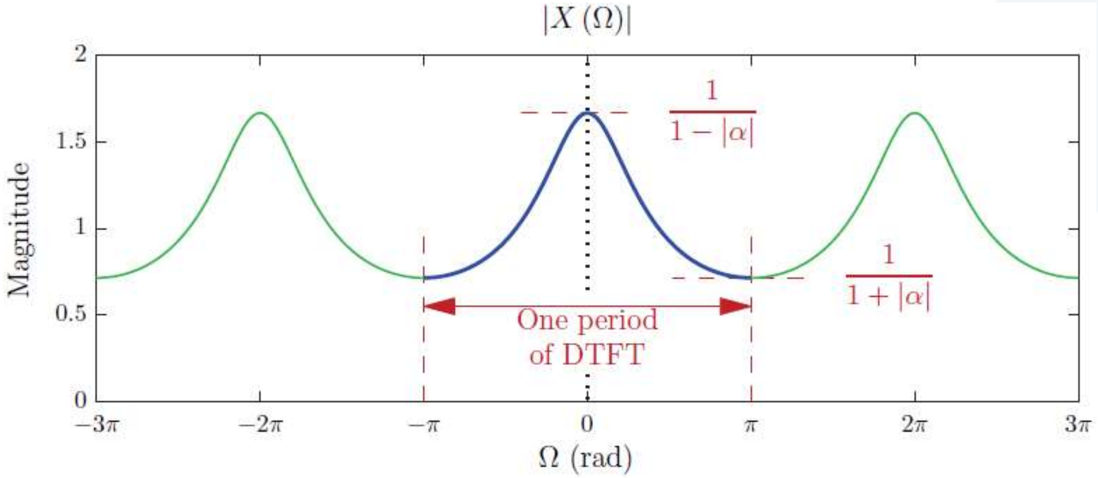
- Example 2: DTFT of right-sided exponential signal

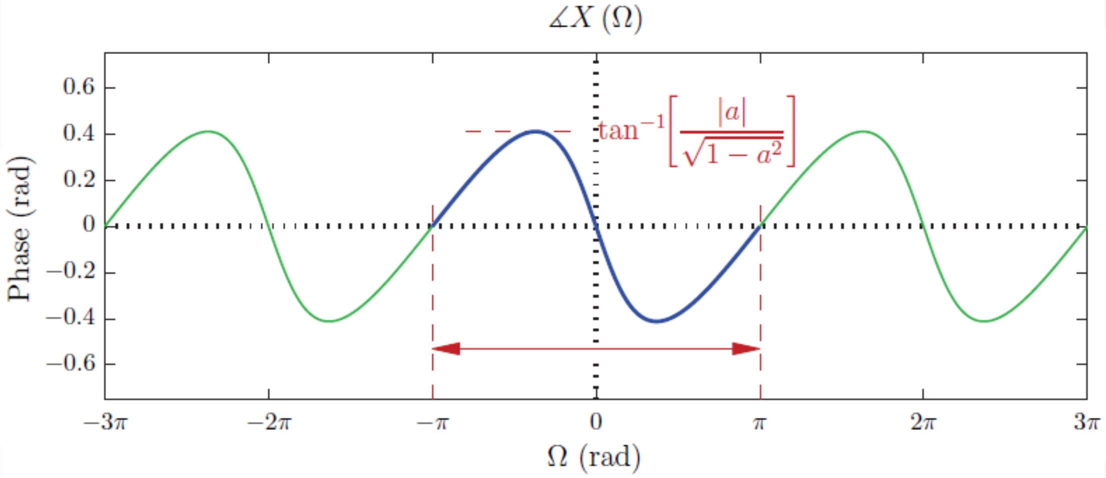
$$x[n] = \alpha^n u[n], \quad |\alpha| < 1$$

$$X(\Omega)=\sum_{n=0}^{\infty}\alpha^{n}\,e^{-j\Omega n}=\frac{1}{1-\alpha\,e^{-j\Omega}}$$



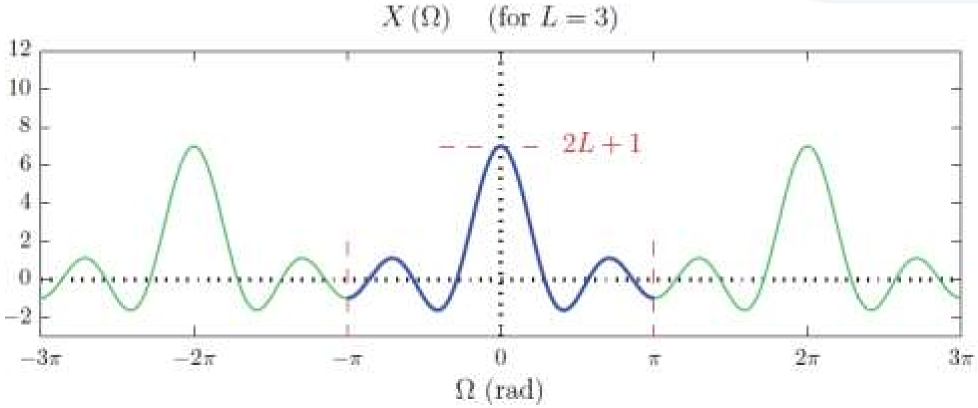
$$\left|X(\Omega)\right|=\frac{1}{\sqrt{1+\alpha^{2}-2\alpha\cos(\Omega)}},\qquad \measuredangle X(\Omega)=-\tan\frac{\alpha\sin(\Omega)}{1-\alpha\cos(\Omega)}$$

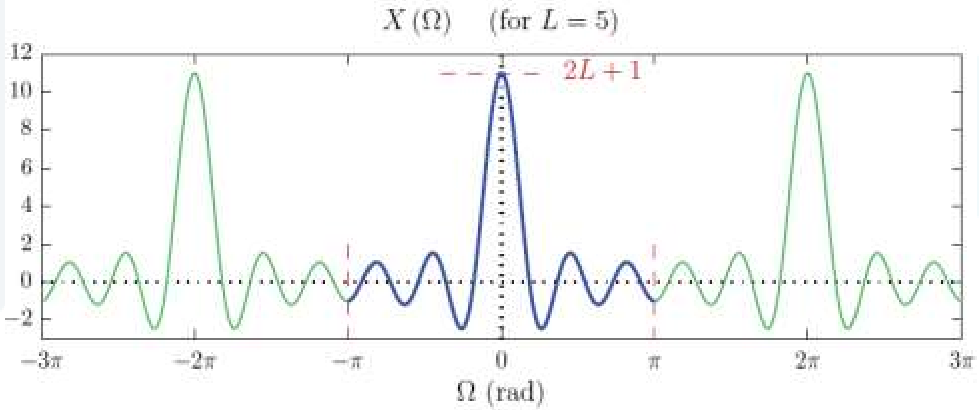




- Example 3: DTFT for discrete-time pulse

$$x[n] = \begin{cases} 1, & -L \le n \le L \\ 0, & \text{otherwise} \end{cases} \qquad X(\Omega) = \sum_{n=-L}^{L} (1)\, e^{-j\Omega n} = \frac{e^{j\Omega L} - e^{-j\Omega(L+1)}}{1 - e^{-j\Omega}} = \frac{\sin\left(\frac{\Omega}{2}(2L+1)\right)}{\sin\left(\frac{\Omega}{2}\right)}$$





Properties of the DTFT

Periodicity

$X(\Omega + 2\pi r) = X(\Omega)$ for all integer r

Linearity

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(\Omega) \quad \text{and} \quad x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(\Omega)$$

$$\Rightarrow \alpha_1 x_1[n] + \alpha_2 x_2[n] \xleftrightarrow{\mathcal{F}} \alpha_1 X_1(\Omega) + \alpha_2 X_2(\Omega)$$

Time shifting $\quad x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow x[n-m] \xleftrightarrow{F} X(\Omega)e^{-j\Omega m}$

Time reversal and Conjugation $\quad x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow x[-n] \xleftrightarrow{F} X(-\Omega)$

$$x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow x^*[n] \xleftrightarrow{F} X^*(-\Omega)$$

Frequency shifting $\quad x[n] \xleftrightarrow{F} X(\Omega) \quad \Rightarrow \quad x[n]e^{j\Omega_0 n} \xleftrightarrow{F} X(\Omega - \Omega_0)$

Modulation property $\quad x[n] \xleftrightarrow{\mathcal{F}} X(\Omega) \quad \Rightarrow$

$$x[n]\cos(\Omega_0 n) \xleftrightarrow{\mathcal{F}} \tfrac{1}{2}\left[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)\right]$$

$$x[n]\sin(\Omega_0 t) \xleftrightarrow{\mathcal{F}} \tfrac{1}{2}\left[X(\Omega - \Omega_0)e^{-j\pi/2} + X(\Omega + \Omega_0)e^{j\pi/2}\right]$$

Differentiation in the frequency domain $\quad x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow n^m x[n] \xleftrightarrow{F} j^m \dfrac{d^m X(\Omega)}{d\Omega^m}$

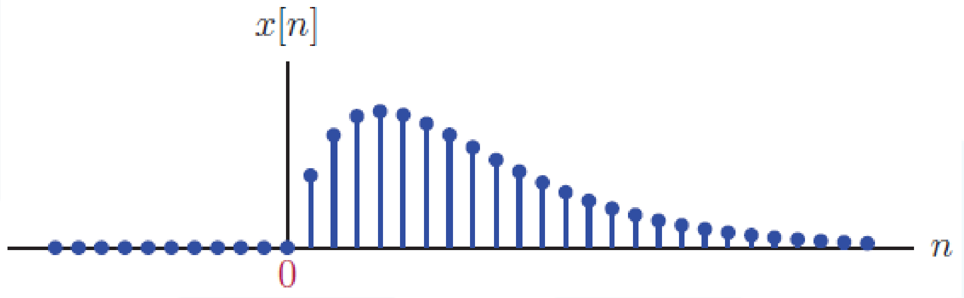
- Example 4: Use of differentiation in frequency property

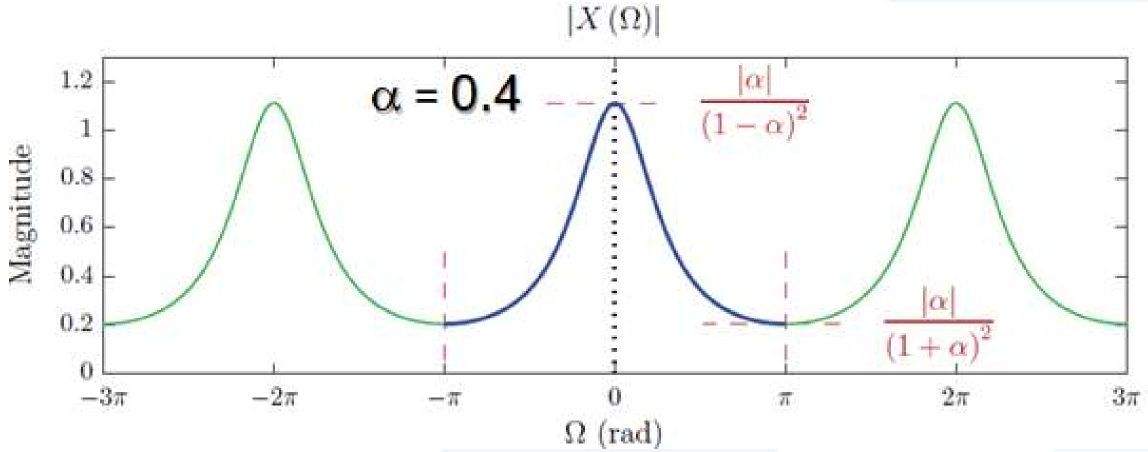
$$x[n] = n\alpha^n u[n].\ |\alpha| < 1$$

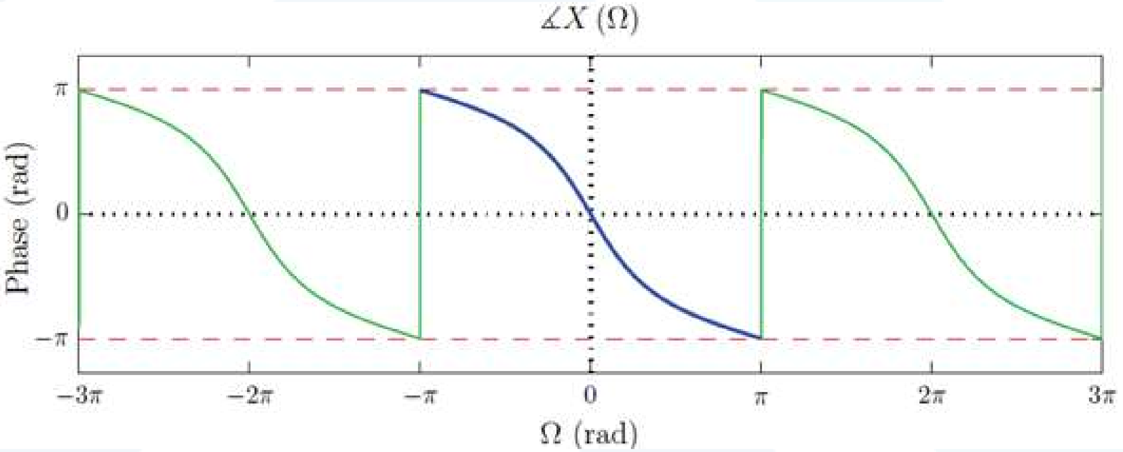
$$\alpha^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-\alpha\, e^{-j\Omega}}$$

$$n\alpha^n u[n] \overset{F}{\longleftrightarrow} j\,\frac{d}{d\Omega}\left[\frac{1}{1-\alpha\, e^{-j\Omega}}\right] = \frac{\alpha e^{-j\Omega}}{(1-\alpha\, e^{-j\Omega})^2}$$

$$X(\Omega) = \frac{\alpha e^{-j\Omega}}{(1-\alpha\, e^{-j\Omega})^2}$$







Convolution property $x_1[n] \overset{F}{\longleftrightarrow} X_1(\Omega)$ and $x_2[n] \overset{F}{\longleftrightarrow} X_2(\Omega)$

$$\Rightarrow \quad x_1[n] * x_2[n] \overset{F}{\longleftrightarrow} X_1(\Omega)X_2(\Omega)$$

- **Example 5:** Convolution using the DTFT

  $h[n] = (2/3)^n u[n]$, $x[n] = (3/4)^n u[n]$. Determine $y[n] = h[n] * x[n]$ using the DTFT

$$H(\Omega) = \frac{1}{1 - \frac{2}{3} e^{-j\Omega}}, \qquad X(\Omega) = \frac{1}{1 - \frac{3}{4} e^{-j\Omega}}$$

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{1}{\left(1 - \frac{2}{3} e^{-j\Omega}\right)\left(1 - \frac{3}{4} e^{-j\Omega}\right)} = \frac{-8}{1 - \frac{2}{3} e^{-j\Omega}} + \frac{9}{1 - \frac{3}{4} e^{-j\Omega}}$$

$$y[n] = -8(2/3)^n u[n] + 9(3/4)^n u[n]$$

Multiplication of two signals $x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(\Omega)$ and $x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(\Omega)$

$$\Rightarrow x_1[n]x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\tau)X_2(\Omega - \tau)d\tau$$

3. Energy and Power in the Frequency Domain

Parseval's theorem

- For a non-periodic energy signal $x[n]$ with DTFT, $X(\Omega)$:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\Omega)|^2 \, d\Omega$$

Let the function $G_x(\Omega)$ be defined as:

$G_x = |X(\Omega)|^2$ energy spectral density (ESD) of the signal $x[n]$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} G_x(\Omega) \, d\Omega$$

$$E_x \text{ in } (-\Omega_0, \Omega_0) = \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} G_x(\Omega) \, d\Omega$$

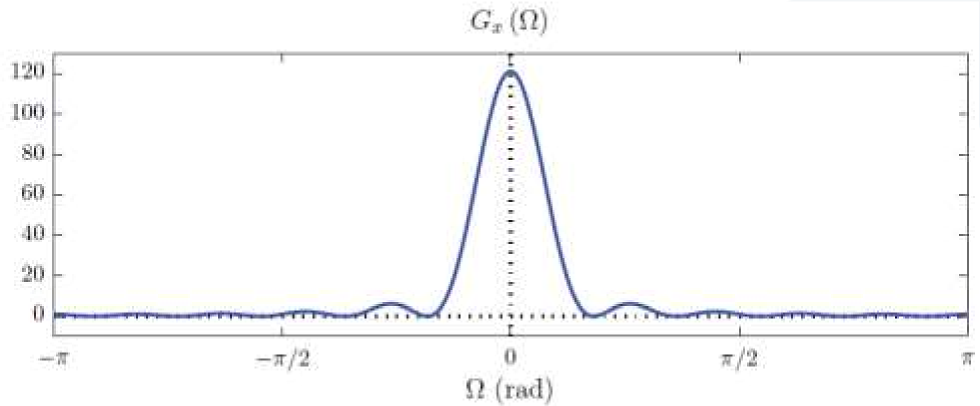
- Example 6: Energy spectral density of a discrete-time pulse

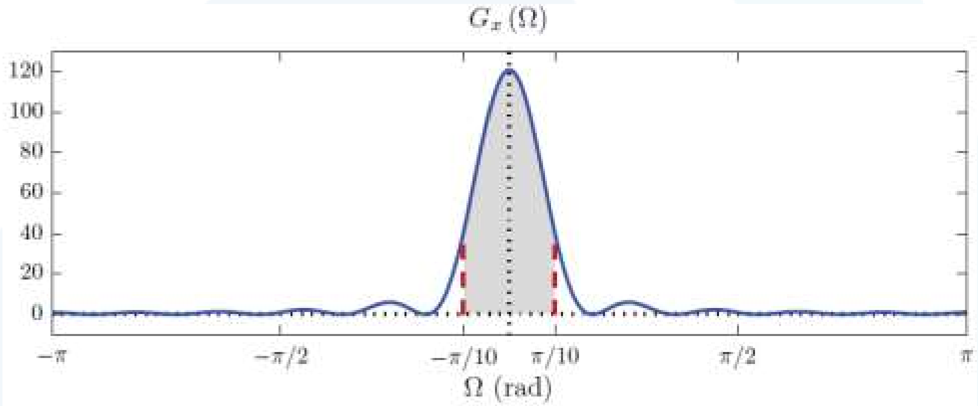
Determine the ESD of the rectangular pulse $x[n] = \begin{cases} 1, & n = -5, \cdots, 5 \\ 0, & \text{otherwise} \end{cases}$

The energy of the signal in the frequency interval $-\pi/10 < \Omega < \pi/10$:

$$X(\Omega) = \frac{\sin(11\Omega/2)}{\sin(\Omega/2)} \Rightarrow G_x(\Omega) = |X_x(\Omega)|^2 = \frac{\sin^2(11\Omega/2)}{\sin^2(\Omega/2)}$$

$$E_x \text{ in}(-\pi/10, \pi/10) = \frac{1}{2\pi}\int_{-\pi/10}^{\pi/10} G_x(\Omega)\, d\Omega \approx 8.9309$$

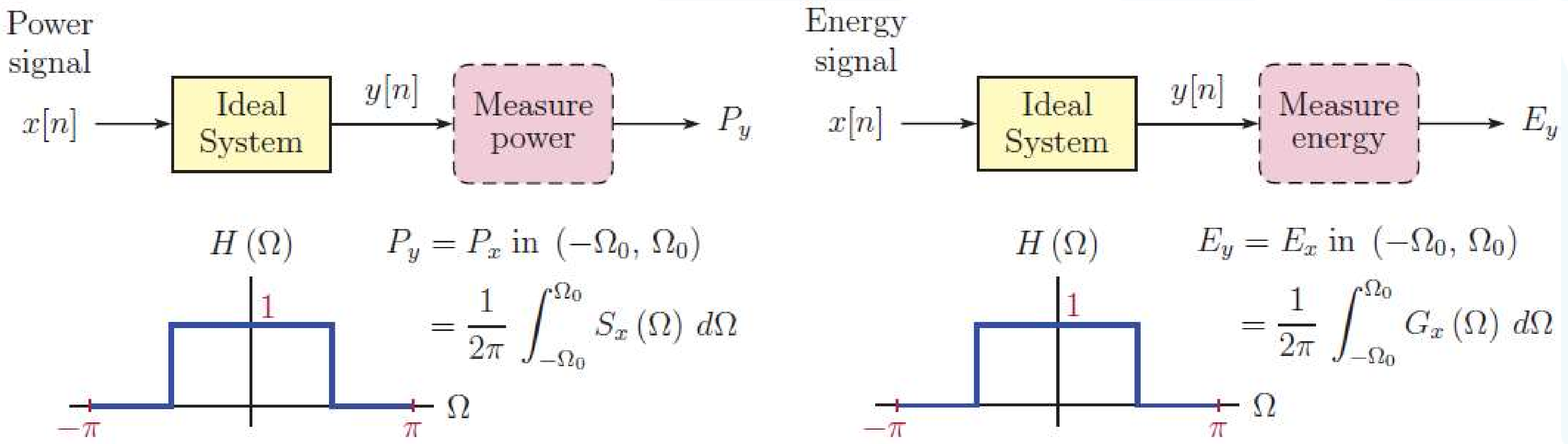




Energy or power in a frequency range

- The power/energy of $x[n]$ in the frequency range $-\Omega_0 < \Omega < \Omega_0$ is the same as the power/energy of the output signal of a system with transfer function:

$$H(\Omega) = \begin{cases} 1, & |\Omega| < \Omega_0 \\ 0, & \Omega_0 < |\Omega| < \pi \end{cases}$$



Autocorrelation

- For an energy signal $x[n]$ the autocorrelation function is defined as:

$$r_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m]$$

- For an energy signal, the energy spectral density is the DTFT of the autocorrelation function, that is, $\mathcal{F}\{r_{xx}[m]\} = G_x(\Omega)$

Properties of the autocorrelation function

- $r_{xx}[0] = E_x \geq |r_{xx}[m]|$ for all m.
- $r_{xx}[-m] = r_{xx}[m]$ for all m, the autocorrelation function has even symmetry.

4. Transfer Function Concept

- In time-domain analysis of systems two distinct descriptions for DTLTI systems:
  1. A linear constant-coefficient difference equation that describes the relationship between the input and the output signals.
  2. An impulse response which can be used with the convolution operation for determining the response of the system to an arbitrary input signal.
- The concept of Transfer function will be introduced as the third method for describing the characteristics of a system.

$$H(\Omega) = \mathcal{F}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega t}$$

- The transfer function concept is valid for LTI systems only.
- In general, $H(\Omega)$ is a complex function of Ω, $H(\Omega) = |H(\Omega)|e^{j\Theta(\omega)}$.

Obtaining the transfer function from the difference equation

$$y[n] = h[n] * x[n] \xleftrightarrow{F} Y(\Omega) = H(\Omega)X(\Omega) \Rightarrow H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

$$y[n-m] \xleftrightarrow{F} e^{-j\Omega m}\, Y(\Omega), \quad m = 0, 1, \cdots$$

$$x[n-m] \xleftrightarrow{F} e^{-j\Omega m}\, X(\Omega), \quad m = 0, 1, \cdots$$

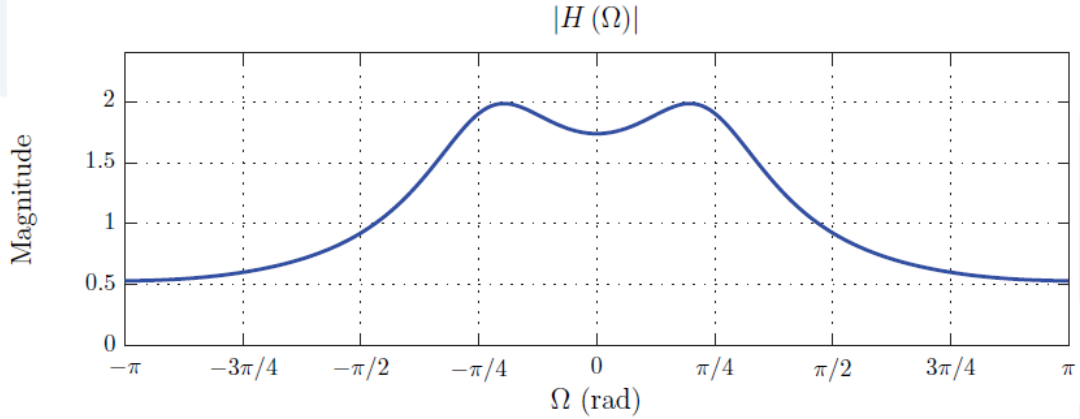
- Example 7: Finding the transfer function from the difference equation

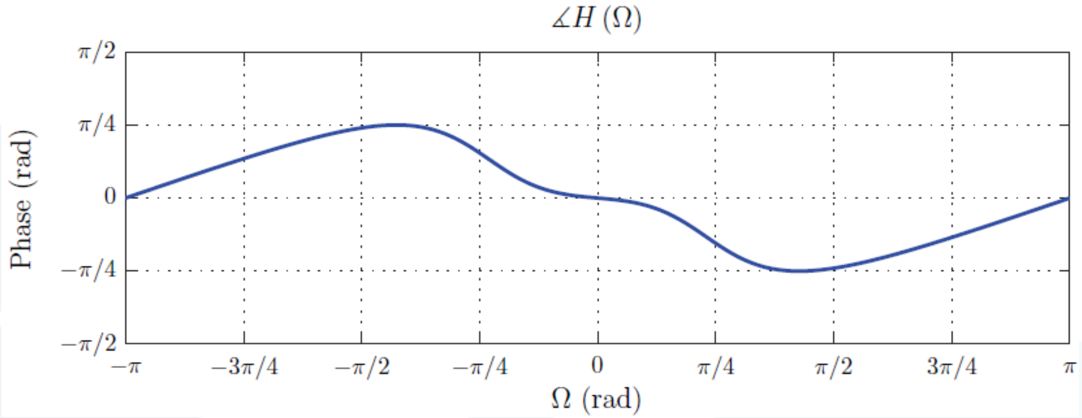
$$y[n] - 0.9y[n-1] + 0.36y[n-2] = x[n] - 0.2x[n-1]$$

$$Y(\Omega) - 0.9\,Y(\Omega)e^{-j\Omega} + 0.36\,Y(\Omega)e^{-j2\Omega} = X(\Omega) - 0.2\,X(\Omega)e^{-j\Omega}$$

$$Y(\Omega)[1 - 0.9e^{-j\Omega} + 0.36e^{-j2\Omega}] = X(\Omega)[1 - 0.2e^{-j\Omega}]$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 - 0.2e^{-j\Omega}}{1 - 0.9e^{-j\Omega} + 0.36e^{-j2\Omega}}$$



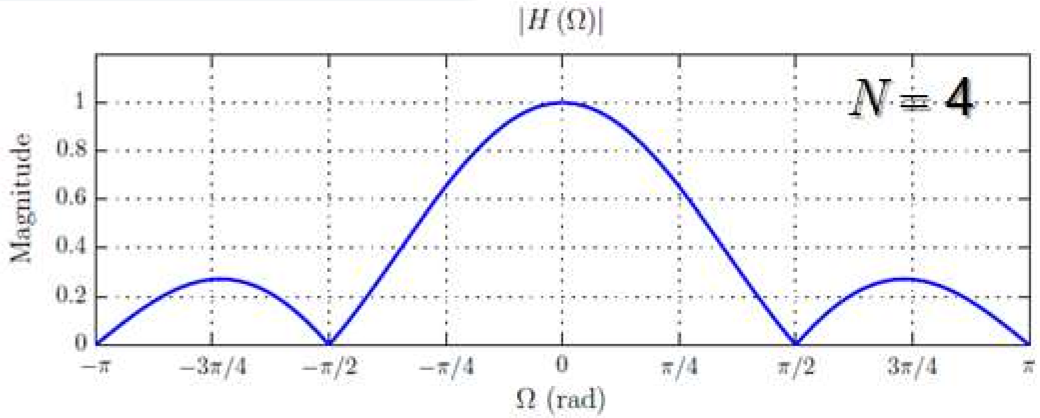


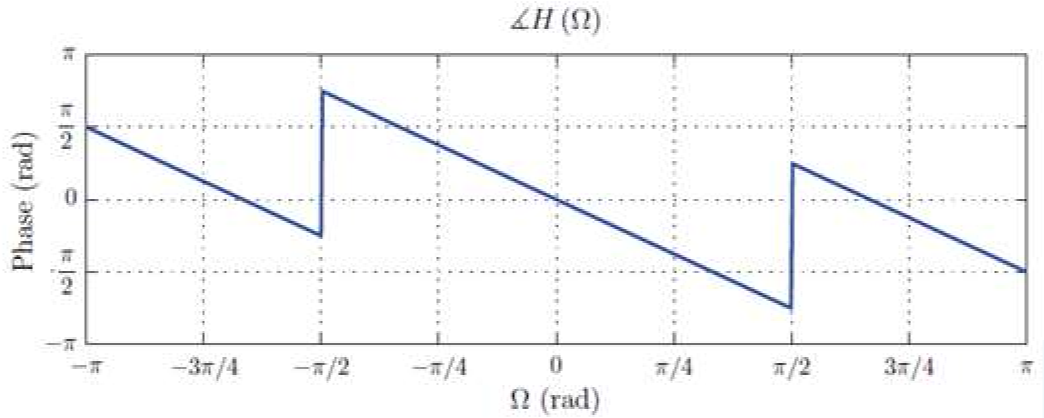
- Example 8: Transfer function for length-N moving average filter

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

$$Y(\Omega) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\Omega k} X(\Omega) \Rightarrow H(\Omega) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\Omega k} = \frac{1}{N}\frac{1-e^{-j\Omega N}}{1-e^{-j\Omega}}$$

$$H(\Omega) = \frac{\sin(\Omega N/2)}{N\sin(\Omega/2)} e^{j\Omega(N-1)/2}$$





5. DTLTI Systems with Non Periodic Input Signals

For a non-periodic signal $x[n]$ as input to a DTLTI system. The output of the system $y[n]$ is given by: $y[n] = h[n] * x[n]$

- Let us assume that The system is stable ensuring that $H(\Omega)$ converges, and the DTFT of the input signal also converges.

$$Y(\Omega) = H(\Omega)X(\Omega) \qquad |Y(\Omega)| = |H(\Omega)||X(\Omega)|, \quad \measuredangle Y(\Omega) = \measuredangle X(\Omega) + \Theta(\Omega)$$

6. Discrete Fourier Transform

- The DTFT of a DT signal $x[n]$ is a transform $X(\Omega)$ which, if it exists, is a 2π-periodic function of the continuous variable $\Omega \Rightarrow$ Storing the DTFT of a signal on a digital computer is impractical because of the continuous nature of Ω.
- Sometimes it is desirable to have a transform that is also discrete. This can be accomplished through the use of the discrete Fourier transform (DFT) provided that the signal under consideration is finite-length.

Discrete Fourier Transform (DFT):

1. Analysis equation (Forward transform): $X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \; k = 0, \ldots, N-1$

2. Synthesis equation (Inverse transform): $x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N}, \; n = 0, \ldots, N-1$

- Example 9: DFT of simple signal

$$\tilde{x}[n] = \{ \underset{\substack{\uparrow \\ n=0}}{1}, -1, 2\}$$

$$X[k] = e^{-j(2\pi/3)k(0)} - e^{-j(2\pi/3)k(1)} + e^{-j(2\pi/3)k(2)} = 1 - e^{-j2\pi k/3} + e^{-j4\pi k/3}$$

$$X[0] = 2, \quad X[1] = 0.5 + j2.5981, \quad X[2] = 0.5 - j2.5981$$

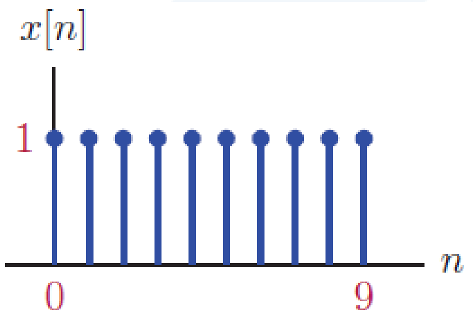
- Example 10: DFT of discrete-time pulse

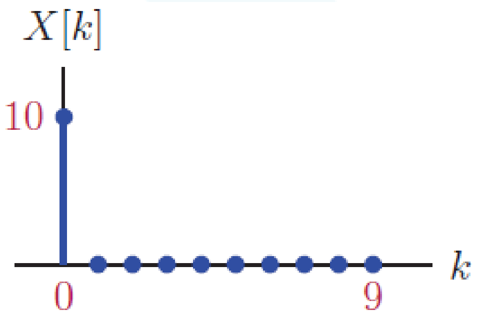
$$x[n] = u[n] - u[n-10]$$

$$\tilde{x}[n] = \{ \underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$$

$$X[k] = \sum_{n=0}^{9} e^{-j(2\pi/10)kn}, \quad k = 0, \ldots, 9$$

$$X[k] = \frac{1 - e^{-j2\pi k}}{1 - e^{-j2\pi k/10}} = \begin{cases} 10, & k = 0 \\ 0, & k = 1, \ldots, 9 \end{cases}$$





Relationship of the DFT to the DTFT

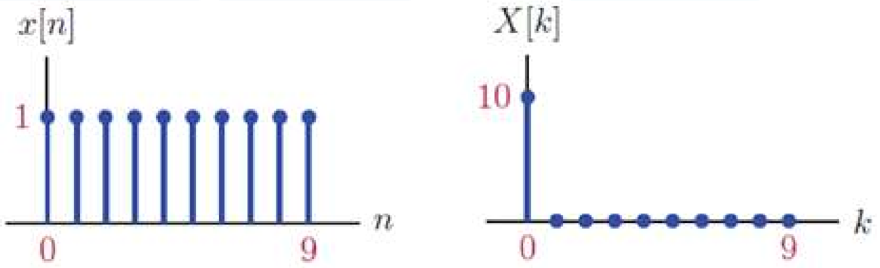
- The DFT of a length-N signal is equal to its DTFT evaluated at a set of N angular frequencies equally spaced in the interval $[0, 2\pi)$. Let an indexed set of angular frequencies be defined as: $\Omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$.

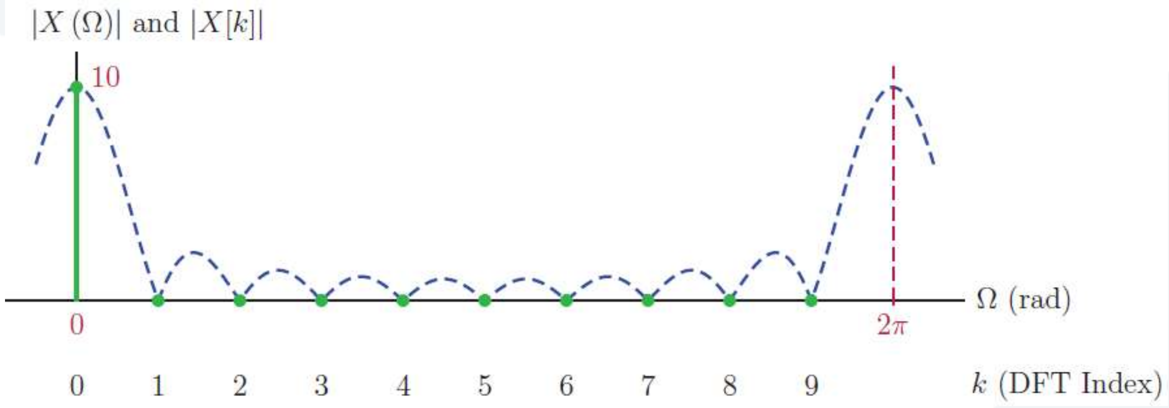
The DFT of the signal is written as $X[k] = X(\Omega_k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$

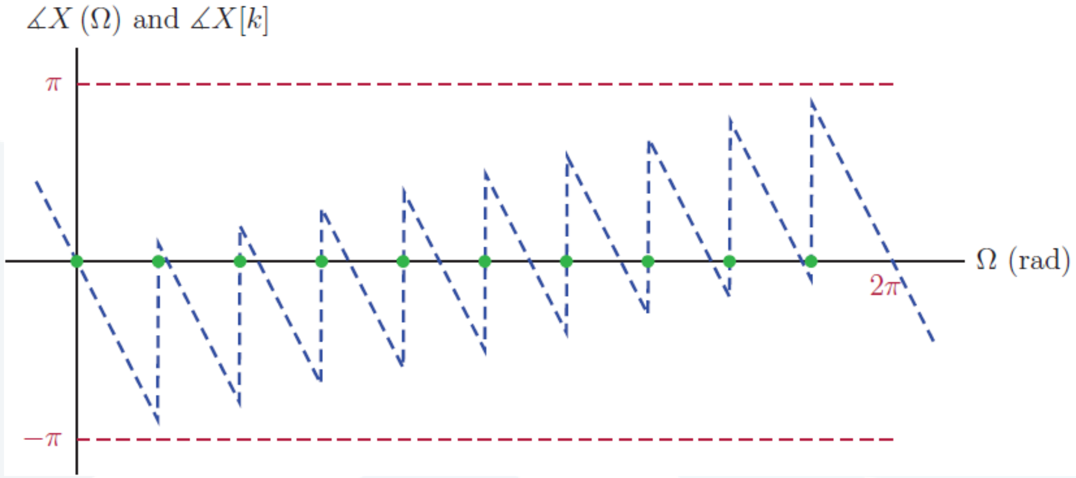
- Example 11: DFT of discrete-time pulse revisited

$$x[n] = u[n] - u[n-10]$$

$$X(\Omega) = \sum_{n=0}^{9} e^{-j\Omega n} = \frac{\sin(5\Omega)}{\sin(0.5\Omega)} e^{-j4.5\Omega}$$







Properties of the DFT

- The properties of the DFT are similar to those of DTFT with one significant difference: Any shifts in the time domain or the transform domain are circular shifts rather than linear shifts.
- Also, any time reversals used in conjunction with the DFT are circular time reversals rather than linear ones.

1. Obtain periodic extension $\tilde{x}[n]$ from $x[n]$: $\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$

2. Apply a time shift to $\tilde{x}[n]$ to obtain $\tilde{x}[n - m]$. The amount of the time shift may be positive or negative.

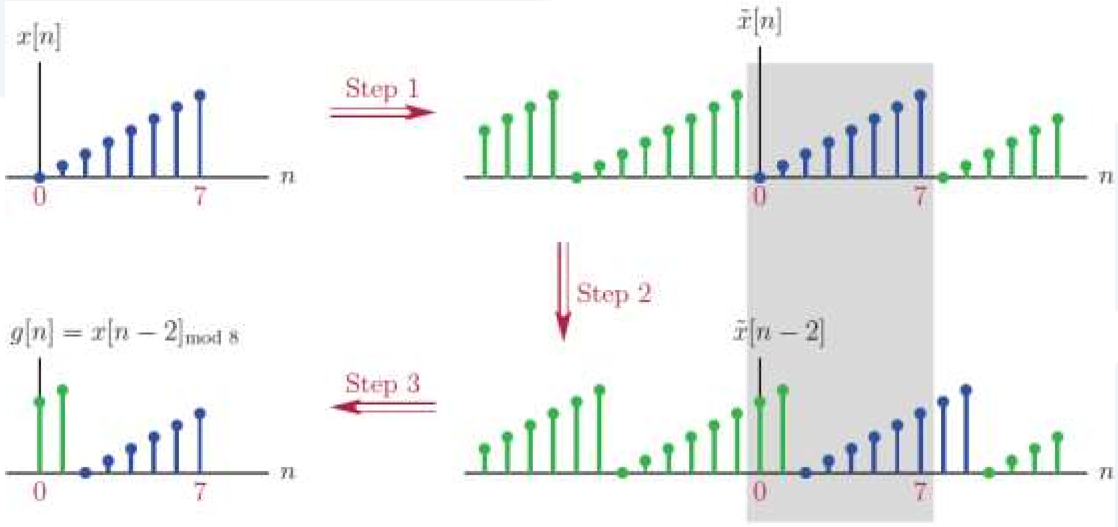
3. Obtain an length-N signal $g[n]$ by extracting the main period of $\tilde{x}[n - m]$.

$$g[n] = \begin{cases} \tilde{x}[n-m], & n = 0, 1, ..., N-1 \\ 0, & \text{otherwise} \end{cases}$$

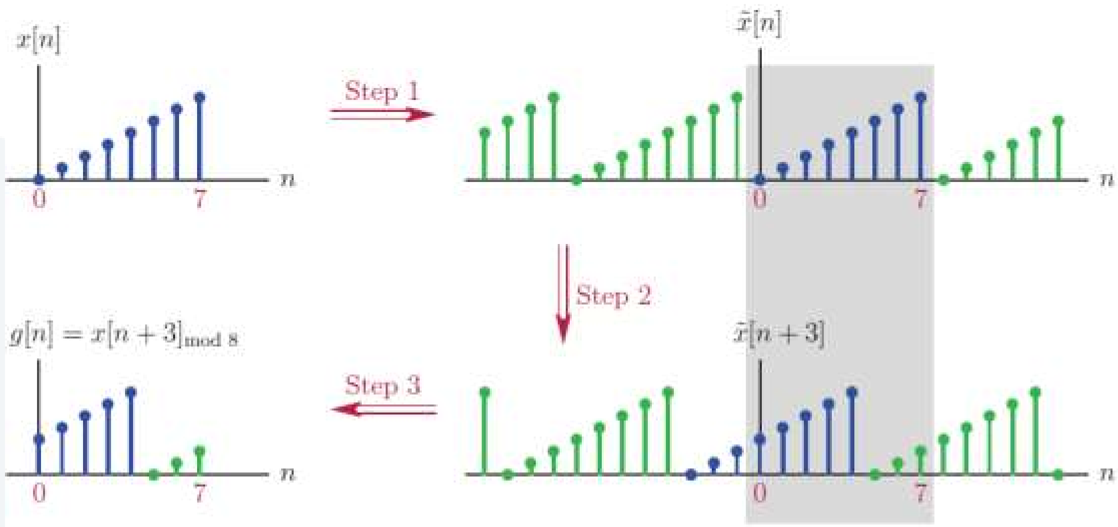
The resulting signal $g[n]$ is a circularly shifted version of $x[n]$, $g[n] = x[n-m]_{\text{mod } N}$

- For the time reversal operation:

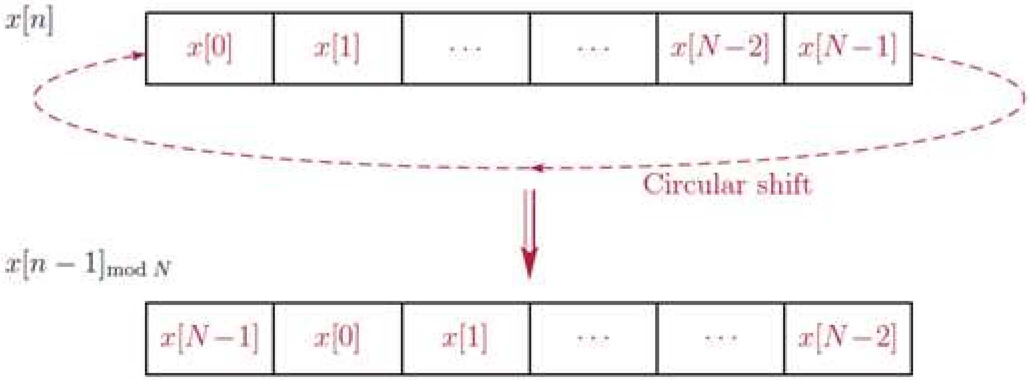
$$\tilde{x}[n - m] \to \tilde{x}[-n] \text{ and } g[n] = x[n-m]_{\text{mod } N} \to g[n] = x[-n]_{\text{mod } N}$$



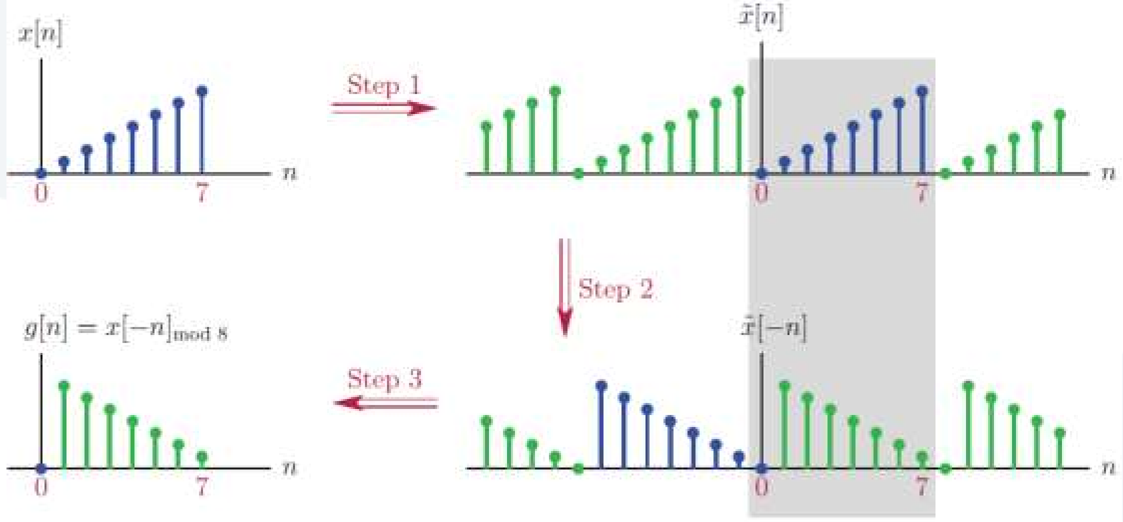
Circular shift to the right by two samples



Circular shift to the left by three samples



Circular shifting a length-N signal



Circular time reversal of a length-8 signal

- A length-N signal $x[n]$ is circularly conjugate symmetric if it satisfies $x^*[n] = x[-n]_{\text{mod } N}$ or circularly conjugate antisymmetric if it satisfies $x^*[n] = -x[-n]_{\text{mod } N}$
- Every signal can be decomposed into two components such that one is circularly conjugate symmetric $x_E[n]$ and the other is circularly conjugate antisymmetric $x_O[n]$: $x[n] = x_E[n] + x_O[n]$.

$$x_E[n] = \frac{1}{2}\{x[n] + x^*[-n]_{\text{mod } N}\}, \quad x_O[n] = \frac{1}{2}\{x[n] - x^*[-n]_{\text{mod } N}\}$$

Linearity $x_1[n] \xleftrightarrow{DFT} X_1[k]$ and $x_1[n] \xleftrightarrow{DFT} X_2[k]$

$$\Rightarrow \alpha_1 x_1[n] + \alpha_2 x_2[n] \xleftrightarrow{DFT} \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

Time shifting $x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x[n-m]_{\text{mod } N} \xleftrightarrow{DFT} X[k]e^{-j(2\pi/N)km}$

Time reversal $x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x[-n]_{\text{mod } N} \xleftrightarrow{DFT} X[-k]_{\text{mod } N}$

Conjugation property $x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x^*[n] \xleftrightarrow{DFT} X^*[-k]_{\text{mod } N}$

Symmetry of the DFT $x[n]$: Real, $\text{Im}\{x[n]\} = 0 \Rightarrow X^*[k] = X[-k]_{\text{mod } N}$

$x[n]$: Imag, $\text{Re}\{x[n]\} = 0 \Rightarrow X^*[k] = -X[-k]_{\text{mod } N}$

$x^*[n] = x[-n]_{\text{mod } N} \Rightarrow X[k]$: Real $\quad x^*[n] = -x[-n]_{\text{mod } N} \Rightarrow X[k]$: Imag

Frequency shifting $x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x[n]e^{j(2\pi/N)mn} \xleftrightarrow{DFT} X[k-m]_{\text{mod } N}$

Circular convolution $y[n] = x[n] \otimes h[n] = \sum_{k=0}^{N-1} x[k]h[n-k]_{\text{mod } N}, \quad n = 0, 1, \ldots, N-1$

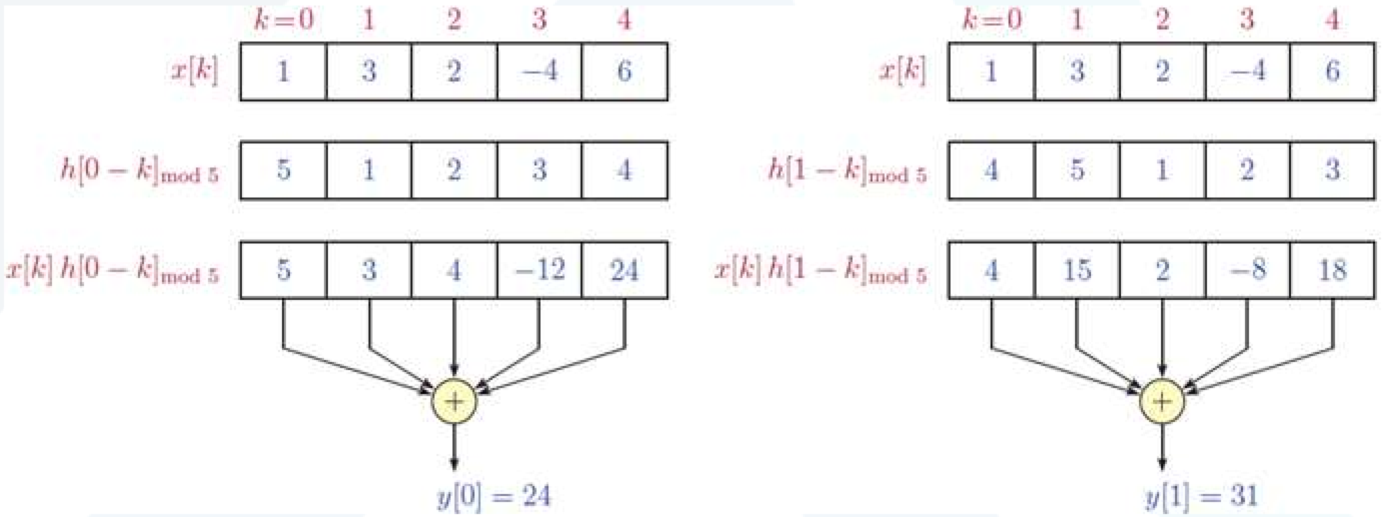
- Example 12: Circular convolution of two signals

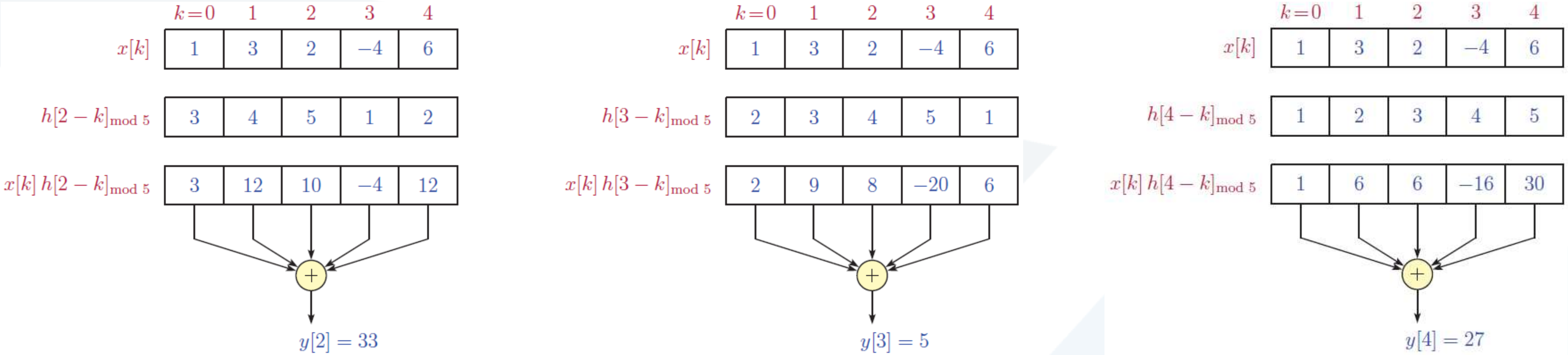
$$x[n] = \{\underset{\uparrow}{1}, 3, 2, -4, 6\} \qquad h[n] = \{\underset{\uparrow}{5}, 4, 3, 2, 1\}$$

$$y[n] = x[n] \otimes h[n]$$

$$= \sum_{k=0}^{4} x[k]h[n-k]_{\text{mod } 5}, \; n = 0, \ldots, 4$$

$$y[n] = \{\underset{\uparrow}{24}, 31, 33, 5, 27\}$$





- The circular convolution property of the discrete Fourier transform:

$$x[n] \xleftrightarrow{DFT} X[k] \quad \text{and} \quad h[n] \xleftrightarrow{DFT} H[k]$$

$$\Rightarrow x[n] \otimes h[n] \xleftrightarrow{DFT} X[k]\,H[k]$$

- Example 13: Circular convolution through DFT

  Verify the circular convolution property of example 12

k	$X[k]$	$H[k]$	$Y[k]$
0	$8.0000+j\,0.0000$	$15.0000+j\,0.0000$	$120.0000+j\,0.0000$
1	$5.3992+j\,0.6735$	$2.5000+j\,3.4410$	$11.1803+j\,20.2622$
2	$-6.8992\text{-}j\,7.4697$	$2.5000+j\,0.8123$	$-11.1803-j\,24.2784$
3	$-6.8992+j\,7.4697$	$2.5000-j\,0.8123$	$-11.1803+j\,24.2784$
4	$5.3992-j\,0.6735$	$2.5000-j\,3.4410$	$11.1803-j\,20.2622$

$$Y[k] = X[k]\,H[k]$$

Obtaining circular convolution $y[n] = x[n] \otimes h[n]$

1. Compute the DFTs: $X[k] = \text{DFT}\{x[n]\}$, and $H[k] = \text{DFT}\{h[n]\}$.
2. Multiply the two DFTs to obtain $Y[k]$: $Y[k] = X[k]\ H[k]$.
3. Compute $y[n]$ through inverse DFT: $y[n] = \text{DFT}^{-1}\{Y[k]\}$.

- The output signal of a DTLTI system is equal to the linear convolution of its impulse response with the input signal.

- Example 14: Linear vs. circular convolution

$$x[n] = \{\underset{\uparrow}{1}, 3, 2, -4, 6\} \qquad h[n] = \{\underset{\uparrow}{5}, 4, 3, 2, 1\}$$

$$y[n] = x[n] \otimes h[n] = \{\underset{\uparrow}{24}, 31, 33, 5, 27\}$$

$$y_l[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \{5, 19, 25, -1, 27, 19, 12, 8, 6\}$$ linear convolution

How does $y_l[n]$ relate to $y[n]$?

- The most obvious difference between the two results $y_l[n]$ and $y[n]$ is the length of each (9 vs. 5).

	$n=0$	1	2	3	4
	$y_\ell[0]$	$y_\ell[1]$	$y_\ell[2]$	$y_\ell[3]$	$y_\ell[4]$
	$y_\ell[5]$	$y_\ell[6]$	$y_\ell[7]$	$y_\ell[8]$	
Totals:	$y[0]$	$y[1]$	$y[2]$	$y[3]$	$y[4]$

	$n=0$	1	2	3	4
	5	19	25	-1	27
	19	12	8	6	
Totals:	24	31	33	5	27

Relationship between linear and circular convolution

Computing linear convolution using the DFT:

Given two finite length signals with N_x and N_h samples respectively:

$$x[n],\ n = 0, \ldots, N_x - 1 \text{ and } h[n],\ n = 0, \ldots, N_h - 1$$

1. Anticipating the length of the linear convolution result to be $N_y = N_x + N_h - 1$, extend the length of each signal to N_y through zero padding:

$$x_p[n] = \begin{cases} x[n], & n = 0, \cdots, N_x - 1 \\ 0, & n = N_x, \cdots, N_y - 1 \end{cases} \qquad h_p[n] = \begin{cases} h[n], & n = 0, \cdots, N_h - 1 \\ 0, & n = N_h, \cdots, N_y - 1 \end{cases}$$

2. Compute the DFTs of the zero-padded signals $x_p[n]$ and $h_p[n]$:

$$X_p[k] = \text{DFT}\{x_p[n]\} \text{ , and } H_p[k] = \text{DFT}\{h_p[n]\}$$

3. Multiply the two DFTs to obtain $Y_p[k]$: $Y_p[k] = X_p[k]\ H_p[k]$.

4. Compute $y_p[n]$ through inverse DFT: $y_p[n] = \text{DFT}^{-1}\{Y_p[k]\}$:

$$y_p[n] = y_l[n] \text{ for } n = 0, \ldots, N_y - 1$$