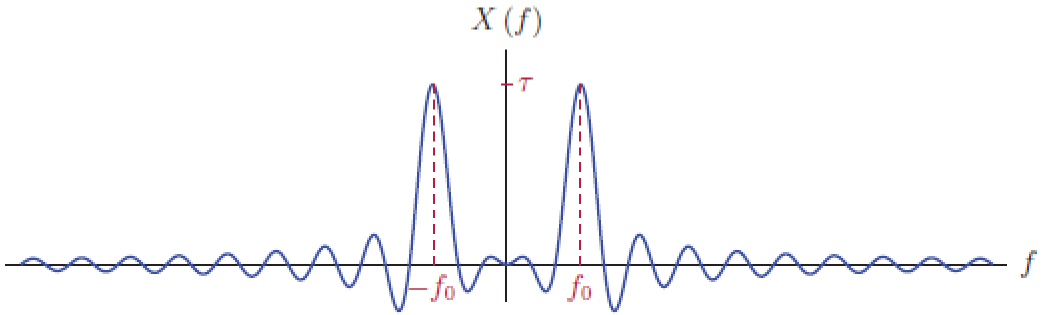
CECC507: Signals and Systems

Lecture Notes 9: Laplace Transform for Continuous-Time Signals and Systems: Part A



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Chapter 7

Laplace Transform for Continuous-Time Signals and Systems

1. Introduction

- The Laplace transform (LT) can be viewed as a generalization of the (classical) Fourier transform.
- Certain characteristics of continuous-time (CT) systems can only be studied via the Laplace transform. Such is the case of stability, transient and steady-state responses.
- The FT of a signal, if it exists, can be obtained from its Laplace transform while the reverse is not generally true.

2. Laplace Transform

- The Laplace transform of a continuous-time signal $x(t)$ is defined as:

$$L\{x(t)\} = X(s) = \int x(t)e^{-st}dt$$

where $s = \sigma + j\omega$, the independent variable of the transform. σ : damping factor, ω: frequency variable.

- There are two important variants:

Unilateral (or one-sided): $X(s) = \mathcal{L}_u\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt;$

Bilateral (or two sided): $X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt;$

Relationship Between LT and Continuous-Time FT

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt = \mathcal{F}\{e^{-\sigma t}x(t)\}$$

$$X(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]\Bigg|_{s=j\omega} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \mathcal{F}\{x(t)\}$$

- Example 1: Laplace transform of the unit impulse

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$$

- Example 2: Laplace transform of the unit-step signal

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0$$

Regions of Convergence

- We need to consider the region in the s-plane where the transform exists–or its region of convergence (ROC).
- For the Laplace transform $X(s)$ of $x(t)$ to exist we need that:

$$\left|\int_{-\infty}^{\infty} x(t)e^{-\sigma t}dt\right| = \left|\int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-jwt}dt\right| \leq \int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right|dt < \infty$$

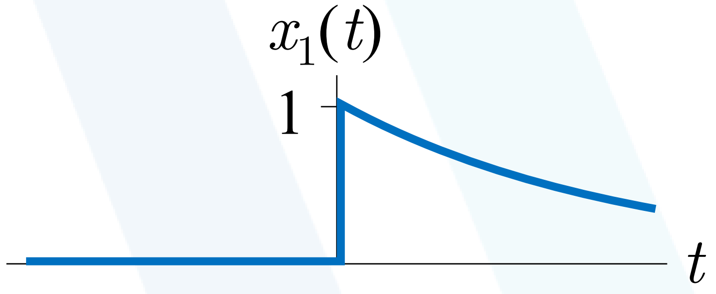
- Note: The frequency does not affect the ROC.

Poles and Zeros and the Region of Convergence

- Typically, $X(s)$ is rational, $X(s) = N(s)/D(s)$.
- For the Laplace The roots of $N(s)$ are called zeros, and the roots of $D(s)$ are called poles. The ROC is related to the poles of the transform.
- If $\{\sigma_i\}$ are the real parts of the poles of $X(s)$, the region of convergence corresponding to different types of signals is determined from its poles as follows:

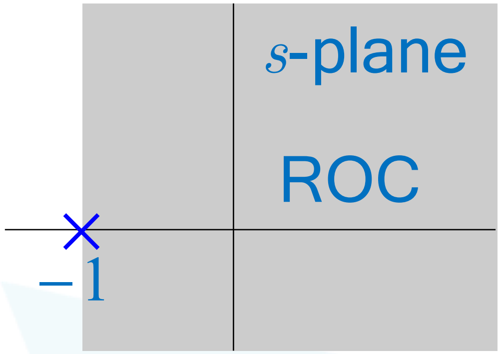
- For a causal signal $x(t)$, the region of convergence of its Laplace transform $X(s)$ is a plane to the right of the poles, $R_c = \{(\sigma, \omega): \sigma > \max\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a anticausal signal $x(t)$, the ROC of its Laplace transform $X(s)$ is a plane to the left of the poles, $R_{ac} = \{(\sigma, \omega): \sigma < \min\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a noncausal signal $x(t)$, the region of convergence of its Laplace transform $X(s)$ is the intersection of the ROC corresponding to the causal component, R_c, and R_{ac} corresponding to the anticausal component, $R_c \cap R_{ac}$
- Example 3: Find the Laplace transform of $x_1(t)$

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

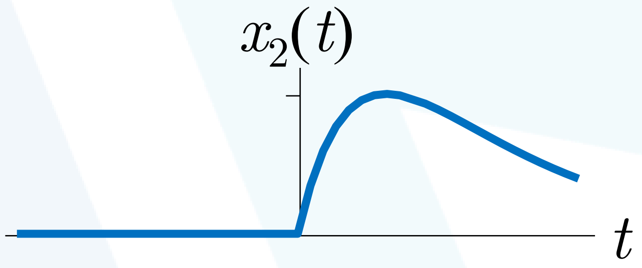


$$X_1(s) = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt = \int_0^{\infty} e^{-t} e^{-st} dt = \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^{\infty} = \frac{1}{s+1},$$

$$\text{Re}\{s\} > -1$$



- Example 4: Find the Laplace transform of $x_2(t)$

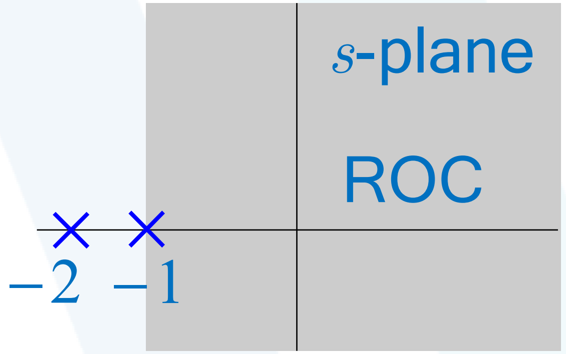


$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

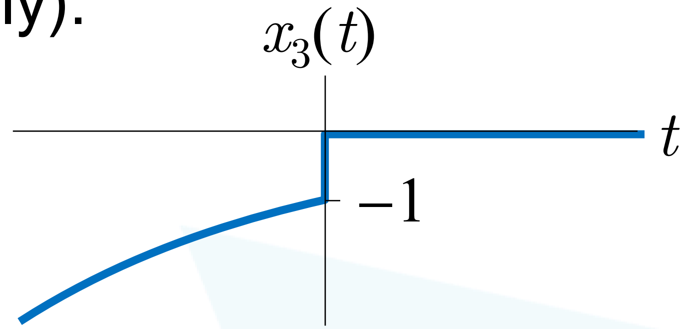
$$X_2(s) = \int_0^{\infty} (e^{-t} - e^{-2t}) e^{-st} dt$$

$$= \int_0^{\infty} e^{-t} e^{-st} dt - \int_0^{\infty} e^{-2t} e^{-st} dt = \frac{1}{(s+1)(s+2)},$$

$$\text{Re}\{s\} > -1$$



- Note: Left-sided signals have left-sided LT(bilateral only).

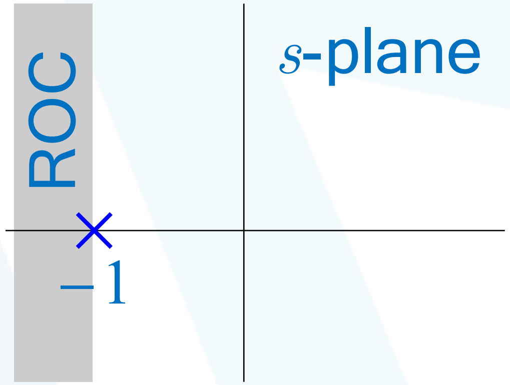


- Example 5: LT of an anti-causal exponential signal

$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X_3(s) = \int_{-\infty}^{\infty} x_3(t) e^{-st} dt = \int_{-\infty}^{0} -e^{-t} e^{-st} dt = \left. \frac{-e^{-(s+1)t}}{-(s+1)} \right|_{-\infty}^{0} = \frac{1}{s+1},$$

$$\mathrm{Re}\{s\} < -1$$



Left and Right Sided ROCs

- It is possible for two different signals to have the same transform expression for $X(s)$.

$$L\{e^{-t}u(t)\} = \frac{1}{s+1}, \quad \text{ROC: } \mathrm{Re}\{s\} > -1$$

$$L\{-e^{-t}u(-t)\} = \frac{1}{s+1}, \quad \text{ROC: } \mathrm{Re}\{s\} < -1$$

In order for us to uniquely identify which signal among the two led to a particular transform, the ROC must be specified along with the transform.

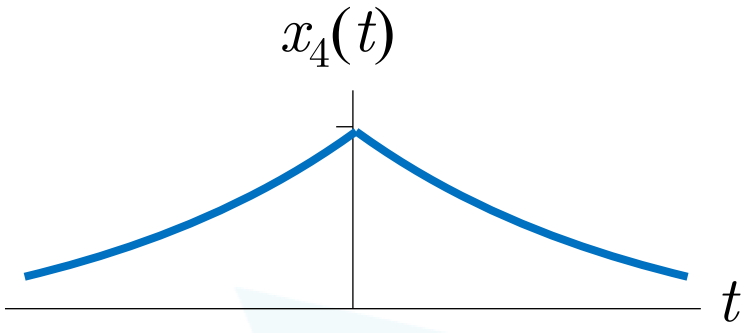
- Example 6: Find the Laplace transform of $x_4(t)$

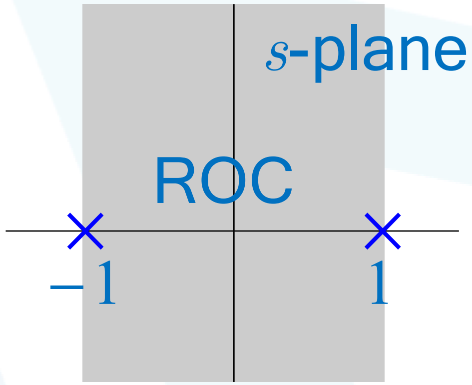
$$x_4(t) = e^{-|t|}$$

$$X_4(s) = \int_{-\infty}^{\infty} e^{-|t|} e^{-st} dt = \int_{-\infty}^{0} e^{(1-s)t} dt + \int_{0}^{\infty} e^{(1+s)t} dt$$

$$= \left. \frac{e^{(1-s)t}}{(1-s)} \right|_{-\infty}^{0} + \left. \frac{-e^{(1+s)t}}{-(1+s)} \right|_{0}^{\infty} = \underbrace{\frac{1}{1-s}}_{\text{Re}<1} + \underbrace{\frac{1}{1+s}}_{\text{Re}>-1} = \frac{2}{1-s^2},$$

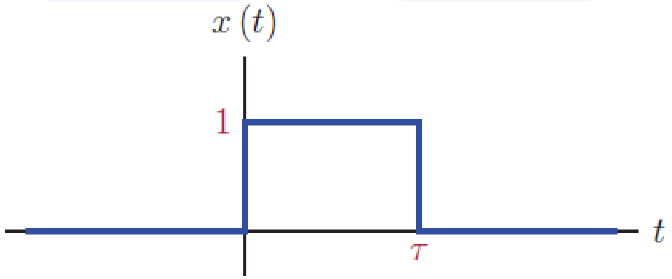
$$-1 < \text{Re}\{s\} < 1$$





- Example 7: Laplace transform of a pulse signal

$$x(t) = \Pi\left(\frac{t - \tau/2}{\tau}\right)$$



$$X(s)=\int_0^{\tau}(1)e^{-st}dt=\left.\frac{e^{-st}}{-s}\right|_0^{\tau}=\frac{1-e^{-s\tau}}{s}$$

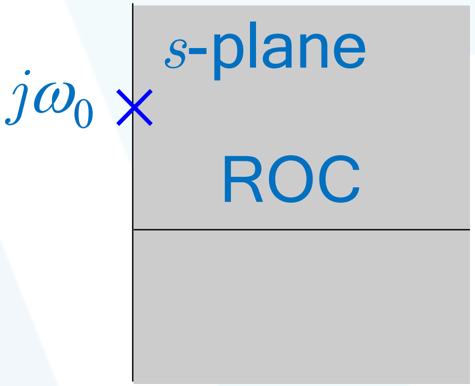
$$X(s)\big|_{s=0}=\left.\frac{\tau e^{-s\tau}}{1}\right|_{s=0}=\tau \Rightarrow X(s)\text{ converge at }s=0$$

- Example 8: Laplace transform of complex exponential signal

$$x(t)=e^{j\omega_0 t}u(t)$$

$$X(s)=\int_{-\infty}^{\infty}e^{j\omega_0 t}u(t)e^{-st}dt=\int_0^{\infty}e^{(j\omega_0-s)t}dt=\left.\frac{e^{(j\omega_0 t-st)}}{j\omega_0-s}\right|_0^{\infty}=\frac{1}{s-j\omega_0},$$

$$\mathrm{Re}\{s\}>0$$



Properties of Laplace Transform

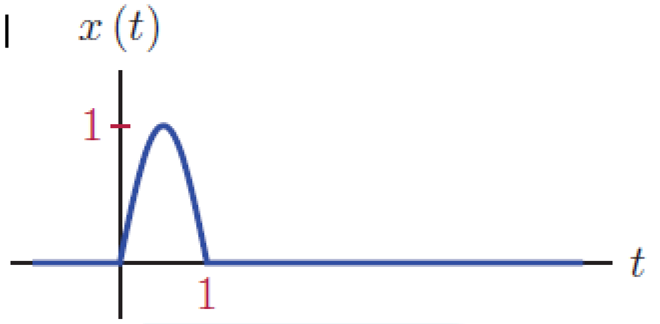
Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t - T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s + \alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at)$	$\frac{1}{\lvert a\rvert} X(\frac{s}{a})$	aR
Differentiate in t	$dx(t)/dt$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\mathrm{Re}(s) > 0))$
Convolve in t	$x_1 * x_2(t)$	$X_1(s)\ X_2(s)$	$\supset (R_1 \cap R_2)$

Laplace Transform Pairs

1	$\delta(t)$	1	All s
2	$u(t)$	$1/s$	$\text{Re}\{s\} > 0$
3	$-u(-t)$	$1/s$	$\text{Re}\{s\} < 0$
4	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$	$\text{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\dfrac{n!}{s^{n+1}}$	$\text{Re}\{s\} < 0$
6	$e^{-at} u(t)$	$\dfrac{1}{s+a}$	$\text{Re}\{s\} > -a$
7	$-e^{-at} u(-t)$	$\dfrac{1}{s+a}$	$\text{Re}\{s\} < -a$

8	$t^n e^{-at} u(t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > -a$
9	$-t^n e^{-at} u(-t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} < -a$
10	$[\cos \omega_0 t] u(t)$	$\dfrac{s}{s^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > 0$
11	$[\sin \omega_0 t] u(t)$	$\dfrac{\omega_0}{s^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > 0$
12	$[e^{-at} \cos \omega_0 t] u(t)$	$\dfrac{s+a}{(s+a)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$
13	$[e^{-at} \sin \omega_0 t] u(t)$	$\dfrac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\mathrm{Re}\{s\} > -a$

- Example 9: Laplace transform of a truncated sine functio



$$x(t) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

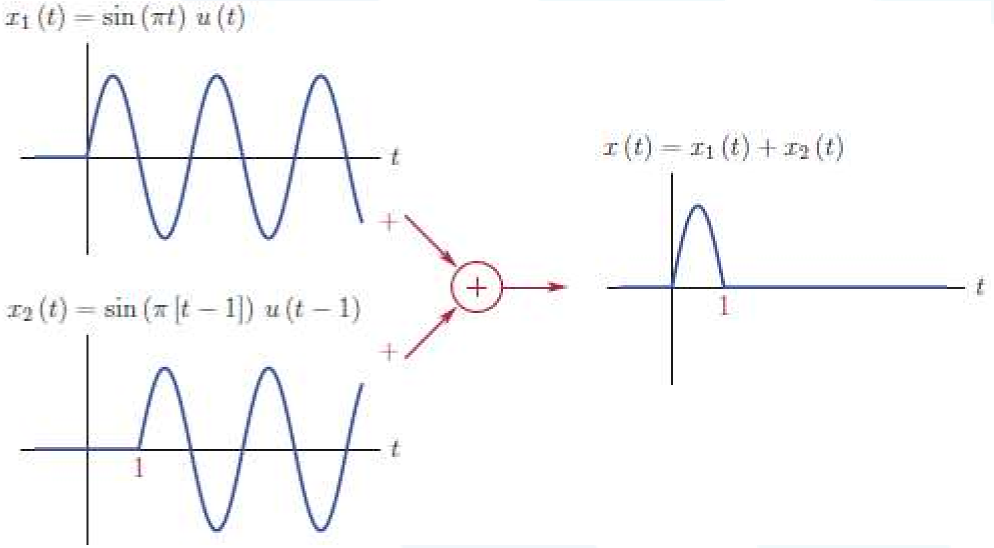
$$X(s) = \int_0^1 \sin(\pi t) e^{-st} dt = \frac{1}{2j} \int_0^1 (e^{j\pi t} - e^{-j\pi t}) e^{-st} dt = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}$$

Another method

$$x(t) = \sin(\pi t) u(t) + \sin(\pi[t - 1]) u(t - 1)$$

$$X(s) = \frac{\pi}{s^2 + \pi^2} + \frac{\pi}{s^2 + \pi^2} e^{-s} = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}$$

ROC: entire s-plane except points where $\text{Re}\{s\} \to -\infty$



- **Example 10:** Using the convolution property of the Laplace transform

$$x_1(t) = e^{-t}u(t), \;\; x_2(t) = \delta(t) - e^{-2t}u(t)$$

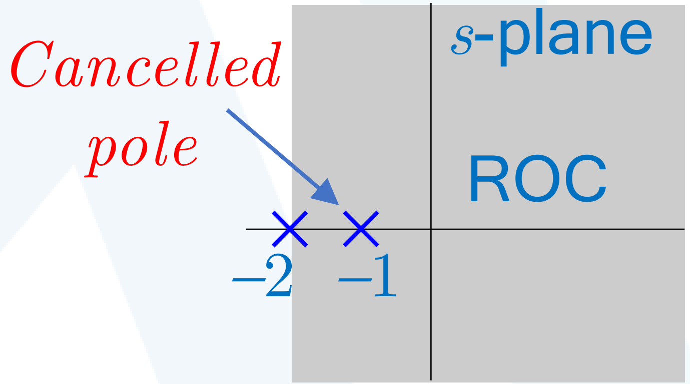
Determine $x(t) = x_1(t) * x_2(t)$ using Laplace transform techniques.

$$X_1(s) = \frac{1}{s+1}, \quad \text{ROC: Re}\{s\} > -1$$

$$X_2(s) = 1 - \frac{1}{s+2} = \frac{s+1}{s+2}, \quad \text{ROC: Re}\{s\} > -2$$

$$X(s) = X_1(s)X_2(s) = \frac{1}{s+2}, \quad \text{ROC: Re}\{s\} > -2$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}u(t)$$



Initial Value Theorem

For a function x with Laplace transform X, if x is causal and contains no impulses or higher order singularities at the origin, then:

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

- When X is known but x is not, the initial value theorem eliminates the need to explicitly find x in order to evaluate $x(0^+)$.

- Example 11: Calculate the initial value of the function $x(t)$, whose LT is:

$$X(s) = \frac{2(s+1)}{(s+1)^2 + 5^2}$$

$$x(0^+) = \lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{2s(s+1)}{(s+1)^2 + 5^2} = 2$$

Verification: $x(t) = 2e^{-t}\cos(5t)\,u(t)$

Final Value Theorem

For a function x with Laplace transform X, if x is causal and $x(t)$ has a finite limit as $t \to \infty$, then:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

- When X is known but x is not, the final value theorem eliminates the need to explicitly find x in order to evaluate limit $_{t\to\infty}\, x(t)$.

- Example 12: Calculate the final value of the function $x(t)$, whose Laplace transform is:

$$X(s) = \frac{s+2}{s(s+1)}$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s(s+2)}{s(s+1)} = \lim_{s\to 0} \frac{s+2}{s+1} = 2$$

Verification: $x(t) = (2 - e^{-t})\,u(t)$

Inverse Laplace Transform

The inverse LT x of X is given by $L^{-1}\{X(s)\} = x(t) = \dfrac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds,$

where Re(s) $= \sigma$ is in the ROC of X.

- We do not usually compute the inverse Laplace transform directly using the above equation.
- For rational functions, the inverse Laplace transform can be more easily computed using partial fraction expansions (PFE).
- Example 13: Calculate the inverse LT of the function $H(s) = 1/(s + a)$

$h(t) = e^{-at}u(t)$ with ROC: Re$\{s\} > -a$

$h(t) = -e^{-at}u(-t)$ with ROC: Re$\{s\} < -a$

- **Example 14:** Using PFE with complex poles

The Laplace transform of a signal $x(t)$ is $X(s) = \dfrac{s+1}{s(s^2+9)}$ with the ROC specified as Re $\{s\} > 0$. Determine $x(t)$.

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+j3} + \frac{k_3}{s-j3}$$

$$k_1 = \frac{1}{9}, \quad k_2 = -\frac{1}{18} + j\frac{1}{6}, \quad k_3 = \frac{1}{18} - j\frac{1}{6}$$

Based on the specified ROC, the signal $x(t)$ is causal

$$x(t) = \frac{1}{9}u(t) - \frac{1}{18}[e^{-j3t} + e^{j3t}]u(t) + j\frac{1}{6}[e^{-j3t} - e^{j3t}]u(t)$$

$$x(t) = \frac{1}{9}u(t) - \frac{1}{9}\cos(3t)u(t) + \frac{1}{3}\sin(3t)u(t)$$

- Example 15: Multiple-order poles

A causal signal $x(t)$ has the Laplace transform $X(s) = \dfrac{s(s+1)}{(s+1)^3(s+2)}$

$$X(s) = \frac{s(s+1)}{(s+1)^3(s+2)} = \frac{-3}{s+1} + \frac{3}{(s+1)^2} - \frac{2}{(s+1)^3} + \frac{3}{s+2}$$

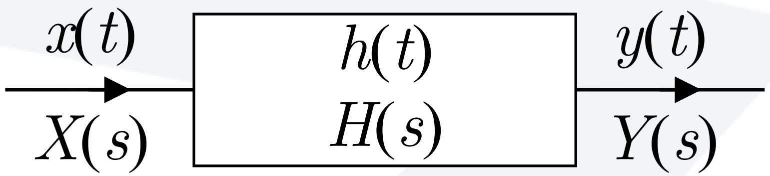
$$L\{e^{-t}u(t)\} = \frac{1}{s+1}, \qquad L\{te^{-t}u(t)\} = -\frac{d}{ds}\left[\frac{1}{s+1}\right] = \frac{1}{(s+1)^2}$$

$$L\{t^2e^{-t}u(t)\} = -\frac{d}{ds}\left[\frac{1}{(s+1)^2}\right] = \frac{2}{(s+1)^3}$$

$$x(t) = -3e^{-t}u(t) + 3te^{-t}u(t) - t^2e^{-3t}u(t) + 3e^{-2t}u(t)$$

3. Using the Laplace Transform with CTLTI Systems

Transfer Function and LTI Systems



- Since $y(t) = x(t) * h(t)$, the system is characterized in the Laplace domain by $Y(s) = X(s)H(s)$.
- $H(s)$ is the transfer function (or system function) of the system.
- A LTI system is completely characterized by its transfer function H.

Relating the transfer function to the differential equation

- Many LTI systems of practical interest can be represented using an Nth-order linear differential equation with constant coefficients.

- Consider a system with input x and output y that is characterized by an equation of the form:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

where the a_k and b_k are complex constants and

$$\mathcal{L}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{L}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\} \Rightarrow \sum_{k=0}^{N} \mathcal{L}\left\{a_k \frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} \mathcal{L}\left\{b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{L}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{L}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

- The transfer function is always rational.
- The impulse response of the system $h(t) = \mathcal{L}^{-1}\{H(s)\}$.
- The convolution operation is only applicable to problems involving LTI systems.
- Therefore it follows that the transfer function concept is meaningful only for systems that are both linear and time invariant.
- In determining the transfer function from the differential equation, all initial conditions must be assumed to be zero.
- Example 16: Finding the transfer function from the DE

  A CTLTI system is defined by means of the differential equation:

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 17\frac{dy(t)}{dt} + 13y(t) = \frac{d^2x(t)}{dt^2} + x(t)$$

$$s^3Y(s) + 5s^2Y(s) + 17sY(s) + 13Y(s) = s^2X(s) + X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 1}{s^3 + 5s^2 + 17s + 13}$$

Transfer function and causality

- **Theorem**: For a LTI system with a rational transfer function H, causality of the system is equivalent to the ROC of H being the right sided to the right of the rightmost pole or, if H has no poles, the entire complex plane.
- For a CTLTI system to be causal, its impulse response $h(t)$ needs to be equal to zero for $t < 0$.

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{0}^{\infty} h(t)e^{-st}dt$$

- Consider a transfer function in the form:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

For the system described by $H(s)$ to be causal we need:

$$\lim_{s\to\infty} H(s) = \lim_{s\to\infty} \frac{b_M}{a_N} s^{M-N} < \infty \Leftrightarrow M - N \le 0 \Rightarrow M \le N$$

Causality condition:

- In the transfer function of a causal CTLTI system the order of the numerator must not be greater than the order of the denominator.

Transfer function and stability:

- For a CTLTI system to be stable its impulse response must be absolute integrable.

$$\int_{-\infty}^{\infty} \left|h(t)\right| dt < \infty$$

Stability condition:

- For a CTLTI system to be stable, the ROC of its s-domain transfer function must include the imaginary axis.
- For a causal system to be stable, the transfer function must not have any poles on the imaginary axis or in the right half s-plane.
- For a anticausal system to be stable, the transfer function must not have any poles on the imaginary axis or in the right half s-plane.
- For a noncausal system the ROC for the transfer function, if it exists, is the region expressed in the form $\sigma_1 < \text{Re}\{s\} < \sigma_2$. For stability we need $\sigma_1 < 0$ and $\sigma_2 > 0$. The poles of the transfer function may be either:
    a. On or to the left of the vertical line $\sigma = \sigma_1$
    b. On or to the right of the vertical line $\sigma = \sigma_2$

- Example 17: Impulse response of a stable system

A stable system is characterized by the transfer function:

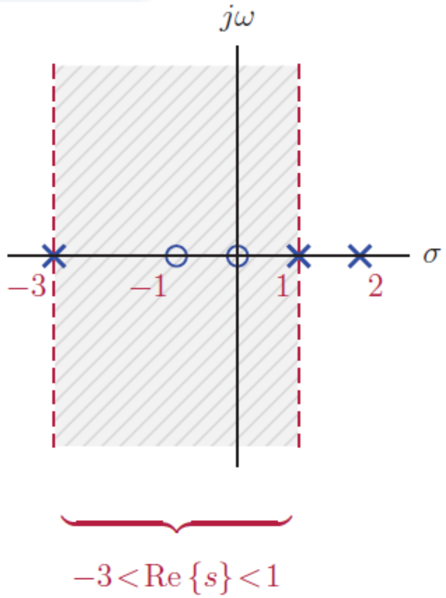
$$H(s) = \frac{15s(s+1)}{(s+3)(s-1)(s-2)}$$

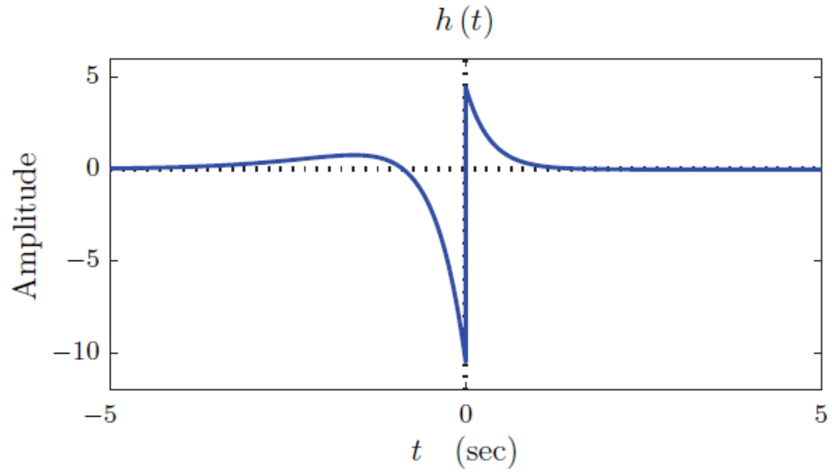
Determine the ROC of the TF. Afterwards find the impulse response of the system.

The 3 poles are at $s = -3, 1, 2$. Since the system is known to be stable, its ROC must include the j-ω axis. The only possible choice is $-3 < \text{Re}\{s\} < 1$.

$$H(s) = \frac{4.5}{s+3} - \frac{7.5}{s-1} + \frac{18}{s-2}$$

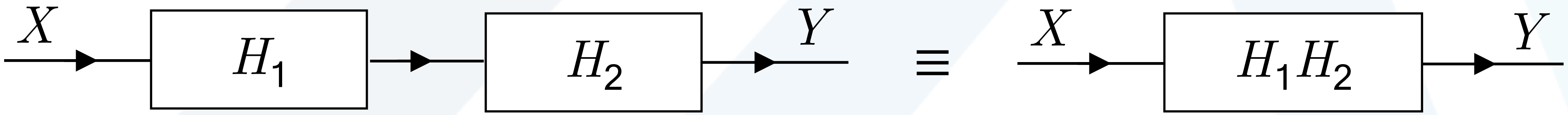
$$h(t) = 4.5e^{-3t}u(t) + 7.5e^{t}u(-t) - 18e^{2t}u(-t)$$





Interconnection of LTI Systems

- The series interconnection of the LTI systems with transfer functions H_1 and H_2 is the LTI system with transfer function $H_1 H_2$.



- The parallel interconnection of the LTI systems with transfer functions H_1 and H_2 is the LTI system with transfer function $H_1 + H_2$.