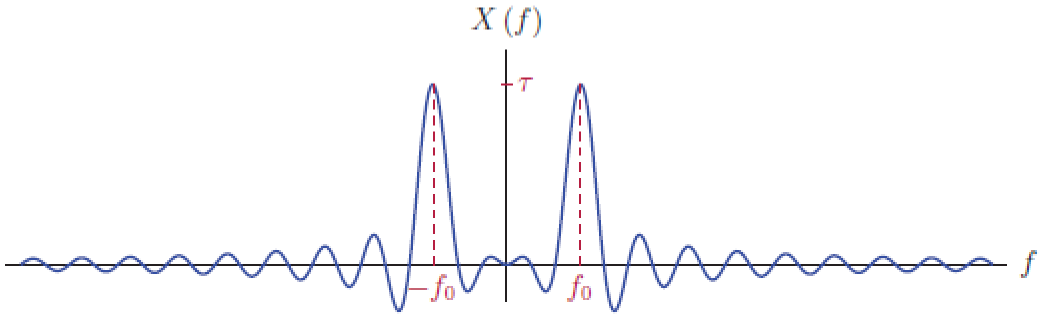
CECC507: Signals and Systems

Lecture Notes 10: Laplace Transform for Continuous-Time Signals and Systems: Part B



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Chapter 7

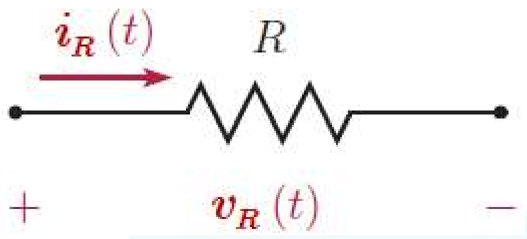
Laplace Transform for Continuous-Time Signals and Systems

Application: Circuit Analysis

Electronic Circuits

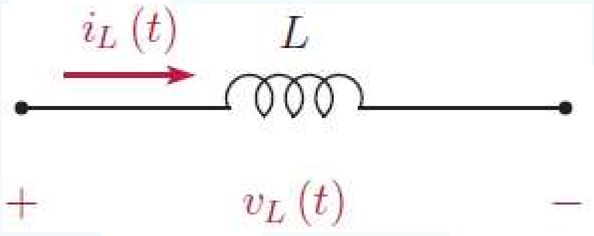
- A resistor $v_R(t) = Ri_R(t)$ or $i_R(t) = \frac{1}{R} v_R(t)$

  $V_R(s) = RI_R(s)$ or $I_R(s) = \frac{1}{R} V_R(s)$



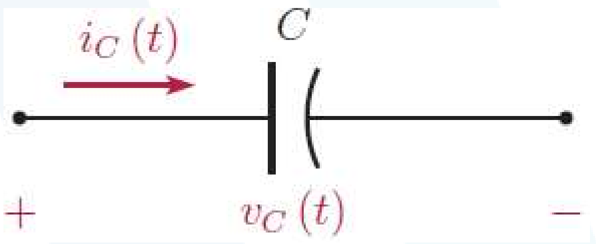
- An inductor $v_L(t) = L\frac{d}{dt} i_L(t)$ or $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau$

  $V_L(s) = sLI_L(s)$ or $I_L(s) = \frac{1}{sL} V_L(s)$



- A capacitor $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$ or $i_C(t) = C\frac{d}{dt} v_C(t)$

  $V_C(s) = \frac{1}{sC} I_C(s)$ or $I_C(s) = sCV_C(s)$

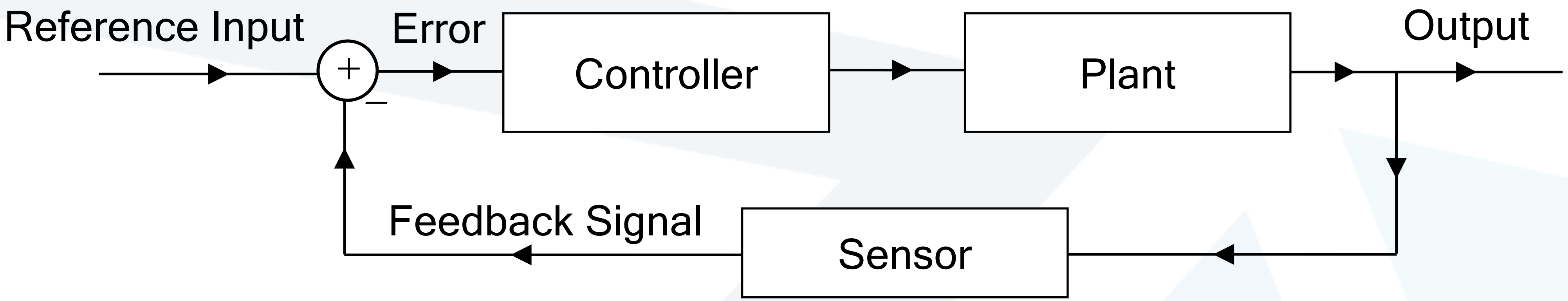


Application: Design and Analysis of Control Systems

Control Systems

- The desired values of the quantities being controlled are collectively viewed as the input of the control system.
- The actual values of the quantities being controlled are collectively viewed as the output of the control system.
- A control system whose behavior is not influenced by the actual values of the quantities being controlled is called an open loop (or non-feedback) system.
- A control system whose behavior is influenced by the actual values of the quantities being controlled is called a closed loop (or feedback) system.
- An example of a simple control system would be a thermostat system, which controls the temperature in a room or building.

Feedback Control Systems



- input: desired value of the quantity to be controlled.
- output: actual value of the quantity to be controlled.
- error: difference between the desired and actual values.
- plant: system to be controlled.
- controller: device that monitors the error and changes the input of the plant. with the goal of forcing the error to zero.

- sensor: device used to measure the actual output.

A control system includes two very important components:

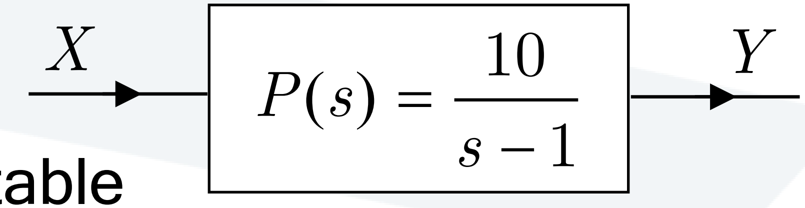
- Transducer: Since it is possible that the output signal $y(t)$ and the reference signal $x(t)$ might not be of the same type, a transducer is used to change $y(t)$ so it is compatible with the reference input $x(t)$.
- Actuator: A device that makes possible the execution of the control action on the plant, so that the output of the plant follows the reference input.

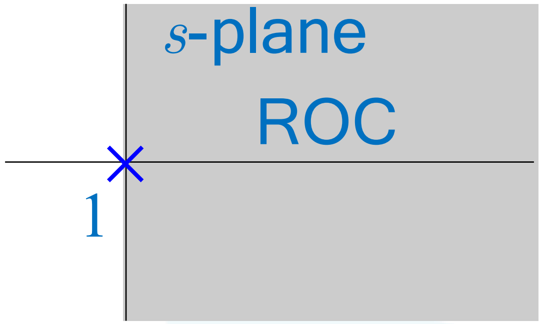
Stability Analysis of Feedback Systems

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the Laplace domain than in the time domain.

Example 1: Stabilization Example: Unstable Plant

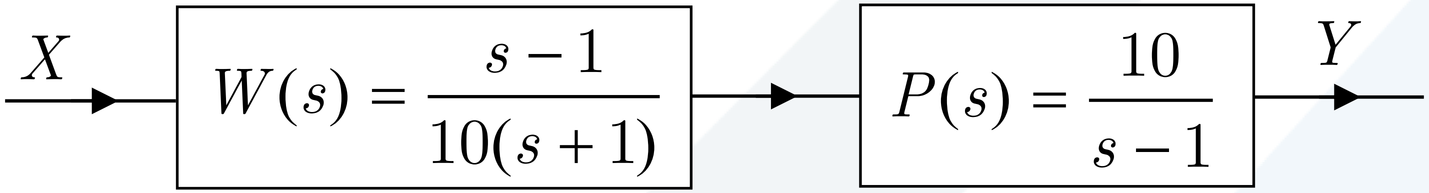
- Causal LTI plant
- System is not BIBO stable





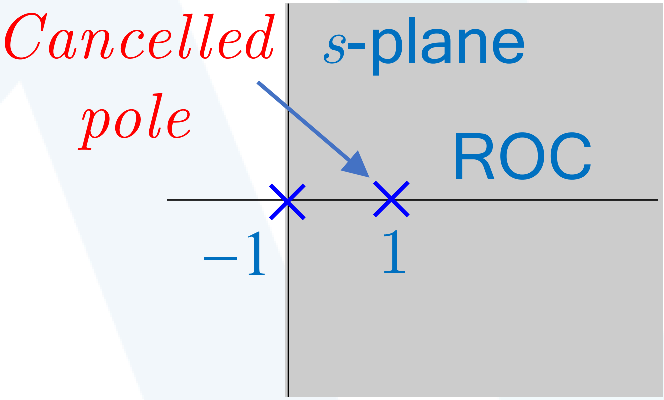
Example 2: Stabilization Example: Using Pole-Zero Cancellation

- System formed by series interconnection of plant and causal LTI compensator:



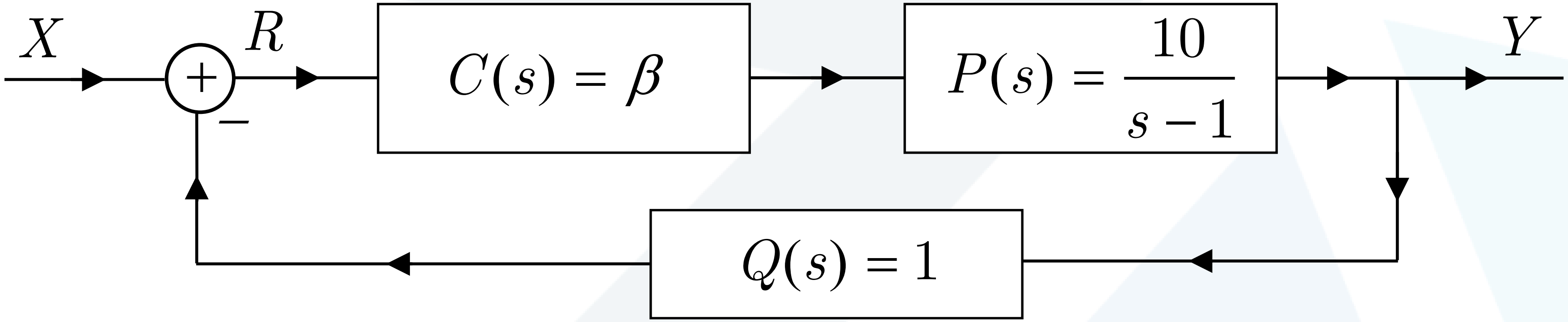
- Transfer function H of overall system (BIBO stable):

$$H(s) = W(s)P(s) = \frac{s-1}{10(s+1)} \frac{10}{s-1} = \frac{1}{(s+1)}$$



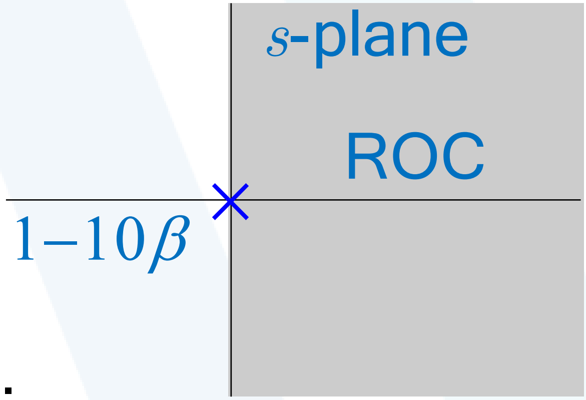
Example 3: Stabilization Example: Using Feedback

- Feedback system (with causal LTI compensator and sensor):



- Transfer function H of overall system:

$$H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)} = \frac{10\beta}{s - (1 - 10\beta)}$$



- Feedback system is BIBO stable if and only if $1-10\beta < 0$.

4. Bode Plots

- Bode plots of the frequency response are used in the analysis and design of feedback control systems. A Bode plot consists of the dB magnitude $20\log_{10}|H(\omega)|$ and the phase $\measuredangle H(\omega)$, each graphed as a function of $\log_{10}(\omega)$.

$$H(s) = K_1 \frac{(1 - s/z_1)(1 - s/z_2)\cdots(1 - s/z_M)}{(1 - s/p_1)(1 - s/p_2)\cdots(1 - s/p_N)}$$

- Let us write $H(s)$ as a cascade combination of $M + N$ subsystems:

$$H(s) = K_1 H_1(s) H_2(s) \cdots H_M(s) H_{M+1}(s) H_{M+2}(s) \cdots H_{M+N}(s)$$

with

$$H_i(s) = 1 - s/z_i, \quad i = 1, \cdots, M$$

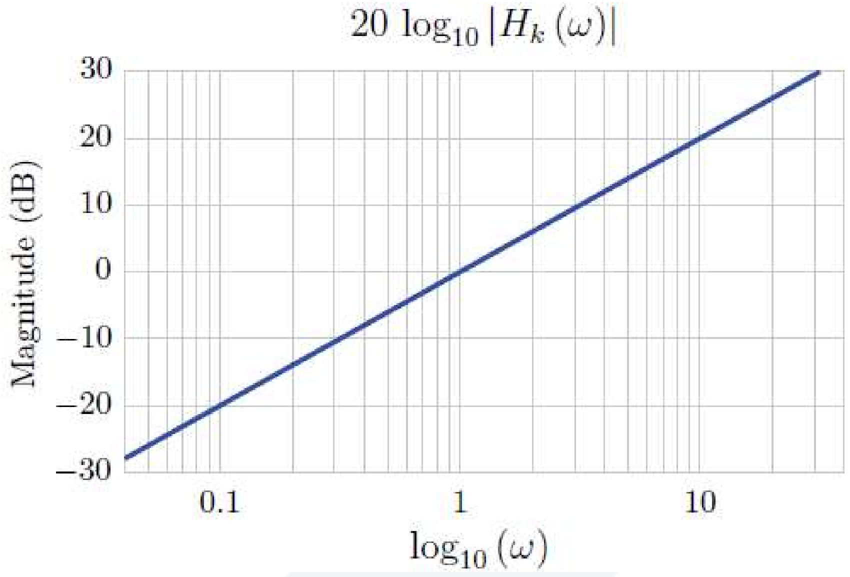
$$H_{M+i}(s) = \frac{1}{1 - s/p_i}, \quad i = 1, \cdots, N$$

- Zero at the origin

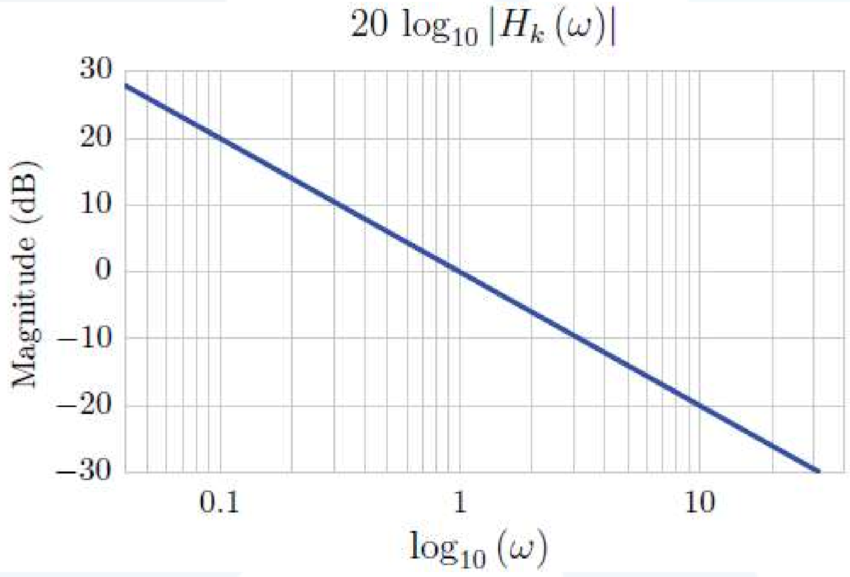
$$H_k(s) = s \Rightarrow 20\log_{10}\left|H_k(\omega)\right| = 20\log_{10}(\omega), \quad \measuredangle H_k(\omega) = 90^\circ$$

- Pole at the origin

$$H_k(s) = 1/s \Rightarrow 20\log_{10}\left|H_k(\omega)\right| = -20\log_{10}(\omega), \quad \measuredangle H_k(\omega) = -90^\circ$$



dB magnitude for $H_k(s) = s$,



dB magnitude for $H_k(s) = 1/s$

- Single real zero

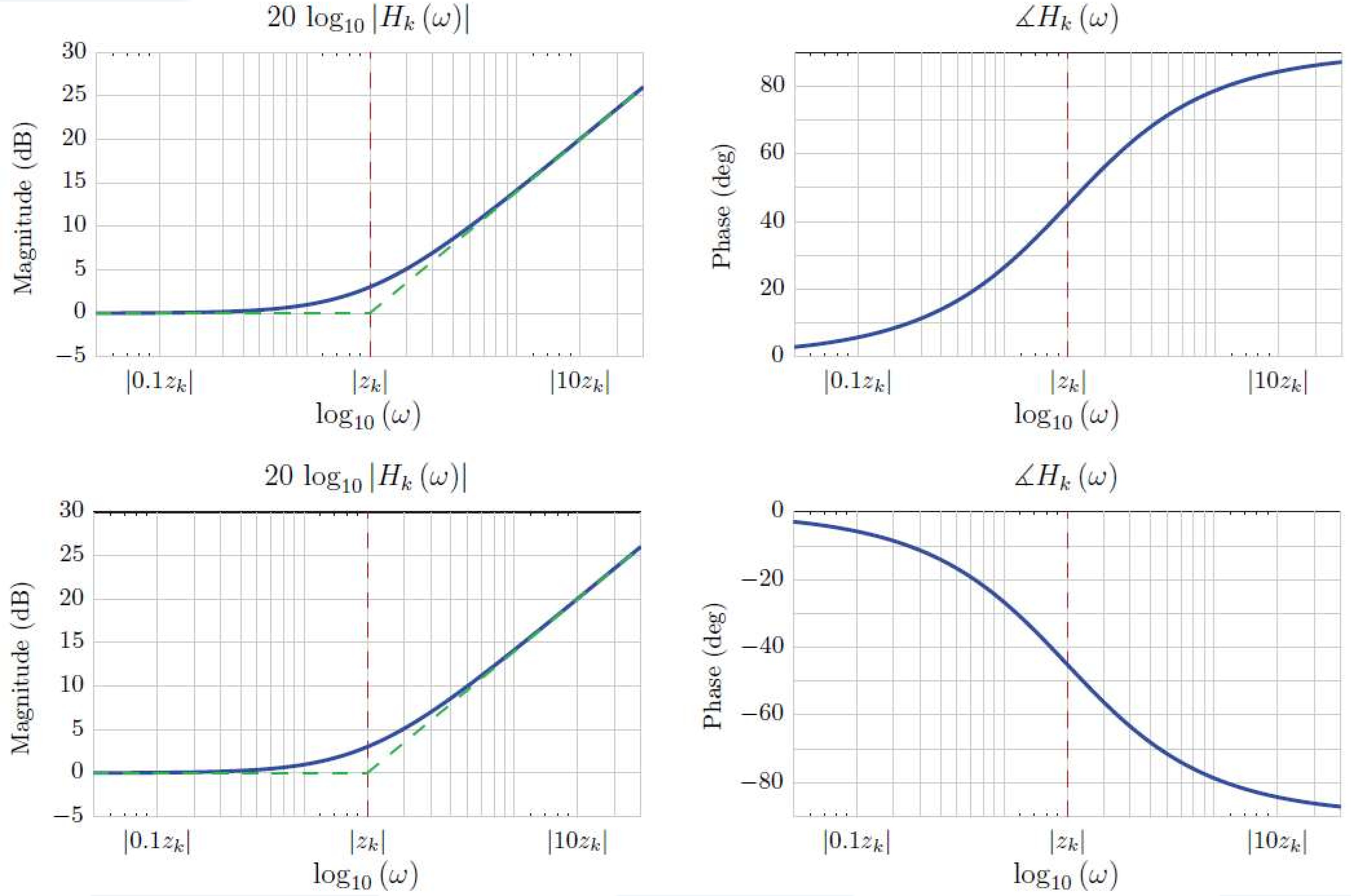
$$H_k(\omega) = H_k(s)\big|_{s=j\omega} = 1 - j\omega/z_k$$

$$20\log_{10}\left|H_k(\omega)\right| = 20\log_{10}\sqrt{1+\left(\omega/z_k\right)^2} = 10\log_{10}\left[1+\left(\omega/z_k\right)^2\right],$$

$$\measuredangle H_k(\omega) = \measuredangle\left(1 - j\omega/z_k\right) = \tan^{-1}\left(-\omega/z_k\right) = -\tan^{-1}\left(\omega/z_k\right)$$

Magnitude: For $\omega \ll |z_k|$ the magnitude is asymptotic to 0 dB. For $\omega \gg |z_k|$ it becomes asymptotic to a straight line with a slope of 20 dB per decade. At $\omega = |z_k|$ it is approximately equal to 3 dB.

Phase: For $\omega \ll |z_k|$ the phase is asymptotic to 0°. For $\omega \gg |z_k|$ the phase is 90° for $z_k < 0$ and –90° for $z_k > 0$. At $\omega = |z_k|$ the phase is 45° for $z_k < 0$ and –45° for $z_k > 0$.



20 log10 |Hk (ω)|
∡Hk (ω)
Magnitude (dB)
Phase (deg)
30
25
20
15
10
5
0
−5
80
60
40
20
0
−20
−40
−60
−80
|0.1zk|
|zk|
|10zk|
log10 (ω)

$z_k < 0$

$z_k > 0$

- Single real pole

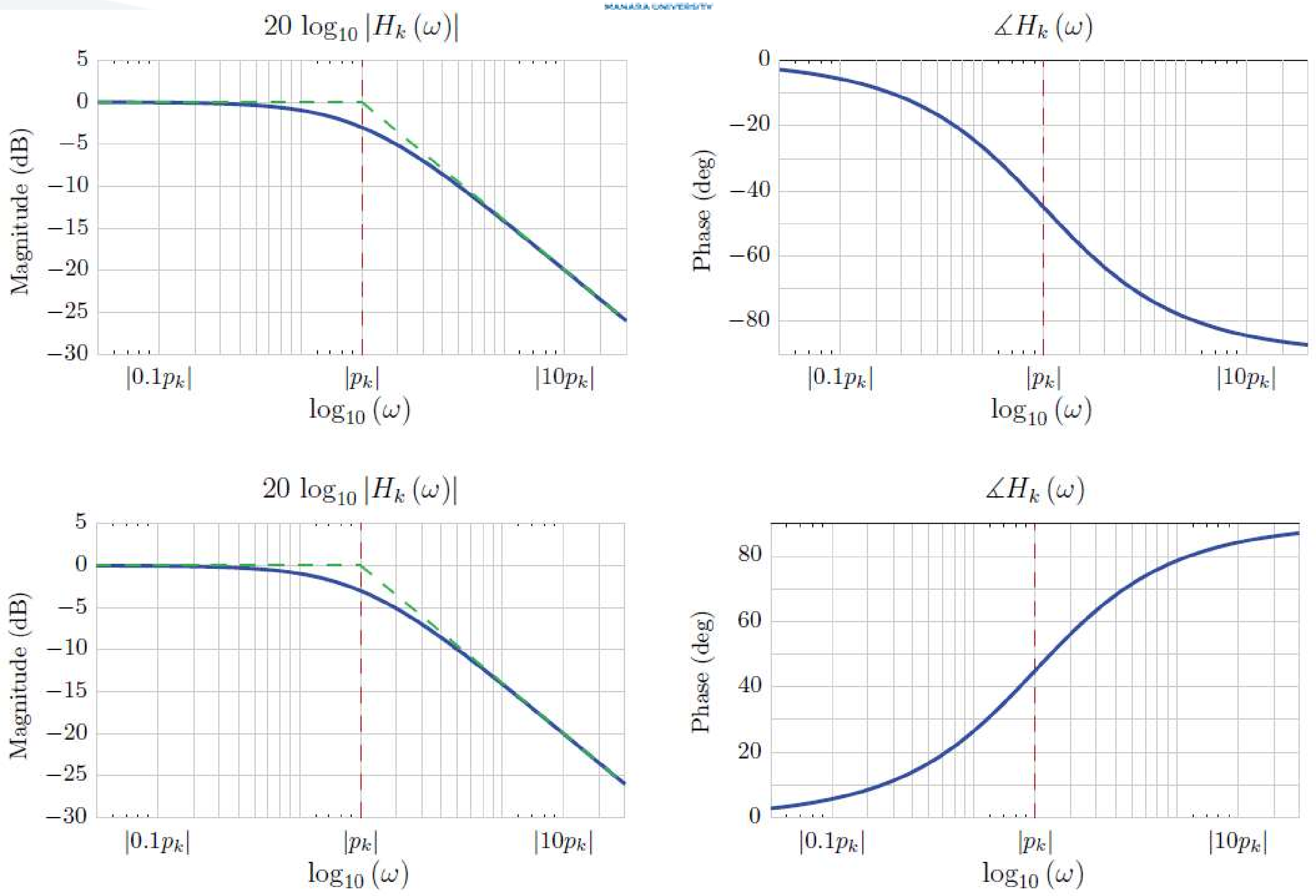
$$H_k(\omega) = H_k(s)\big|_{s=j\omega} = 1/(1 - j\omega/p_k)$$

$$20\log_{10}\left|H_k(\omega)\right| = 20\log_{10}\frac{1}{\sqrt{1+(\omega/p_k)^2}} = -10\log_{10}\left[1+(\omega/p_k)^2\right],$$

$$\measuredangle H_k(\omega) = \measuredangle 1/(1 - j\omega/p_k) = -\tan^{-1}(-\omega/p_k) = \tan^{-1}(\omega/p_k)$$

Magnitude: For $\omega << |p_k|$ the magnitude is asymptotic to 0 dB. For $\omega >> |p_k|$ it becomes asymptotic to a straight line with a slope of –20 dB per decade. At $\omega = |p_k|$ it is approximately equal to –3 dB.

Phase: For $\omega << |p_k|$ the phase is asymptotic to 0°. For $\omega >> |p_k|$ the phase is –90° for $p_k < 0$ and 90° for $p_k > 0$. At $\omega = |p_k|$ the phase is –45° for $z_k < 0$ and 45° for $z_k > 0$.



$20\ \log_{10}|H_k(\omega)|$
Magnitude (dB)
5
0
−5
−10
−15
−20
−25
−30
$|0.1p_k|$
$|p_k|$
$|10p_k|$
$\log_{10}(\omega)$
$\measuredangle H_k(\omega)$
Phase (deg)
0
−20
−40
−60
−80
80
60
40
20

$p_k < 0$

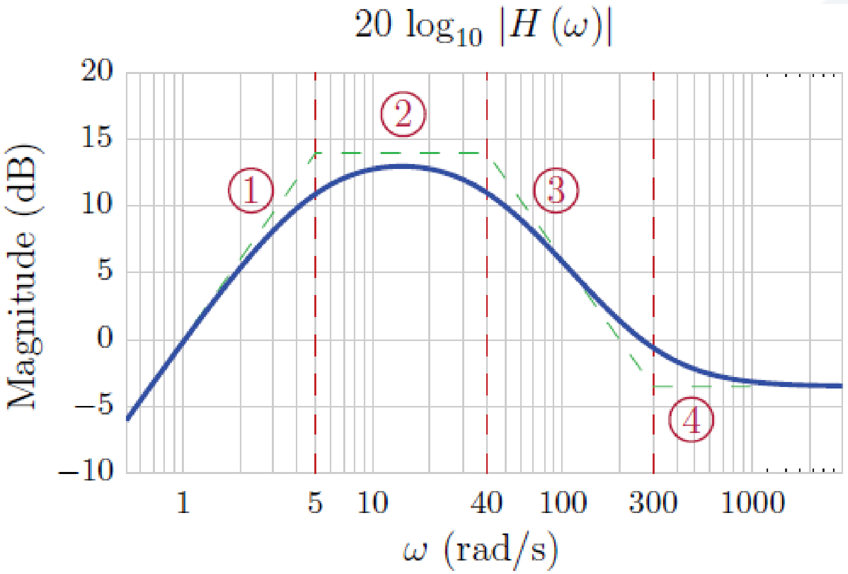
$p_k > 0$

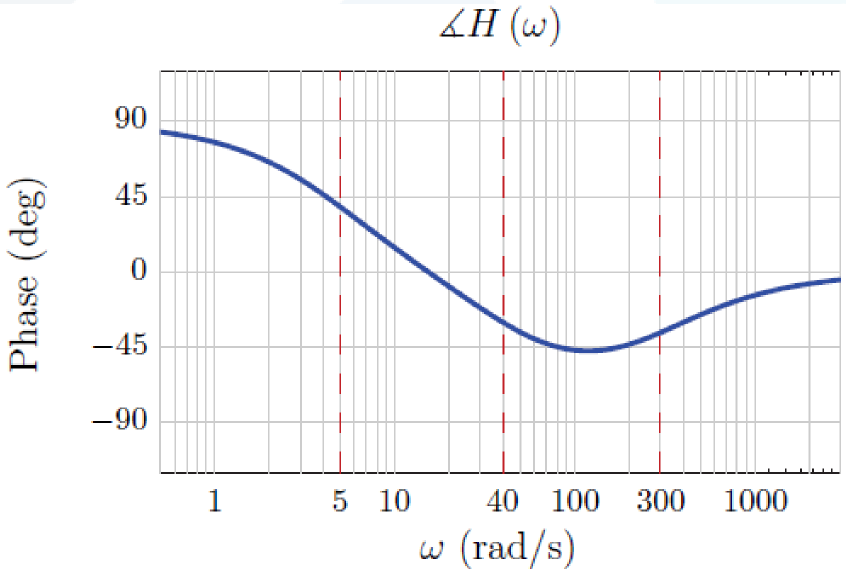
- Example 4: Constructing a Bode plot

$$H(s) = \frac{s(1 + s/300)}{(1 + s/5)(1 + s/40)}$$

$$H(s) = H_1(s)H_2(s)H_3(s)H_4(s)$$

$$H_1(s) = s, \quad H_2(s) = (1 + s/300), \quad H_3(s) = 1/(1 + s/5), \quad H_4(s) = 1/(1 + s/40)$$





- Conjugate pair of poles

Consider a causal and stable second-order system with a pair of complex conjugate poles, that is, $p_2 = p_1^*$

$$H(s) = \frac{1}{(1 - s/p_1)(1 - s/p_1^*)} = \frac{|p_1|^2}{(s - p_1)(s - p_1^*)}$$

Let us put $H(s)$ into the standard form $H(s) = \dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

$$\omega_0^2 = |p_1|^2, \qquad \zeta = -\frac{\mathrm{Re}\{p_1\}}{|p_1|}$$

Since the system is causal and stable, Re $\{p_1\} < 0$. Consequently, when the poles of the system form a complex conjugate pair, we have $0 < \zeta < 1$.

- Analysis of the second-order system

$$H(s) = \frac{1}{(1 - s/p_1)(1 - s/p_2)} = \frac{p_1 p_2}{(s - p_1)(s - p_2)}$$

$$\omega_0^2 = p_1 p_2, \quad 2\zeta\omega_0 = -\text{Re}\{p_1\} - \text{Re}\{p_2\}$$

$$\omega_0 = \sqrt{p_1 p_2}, \quad \zeta = \frac{-\text{Re}\{p_1\} - \text{Re}\{p_2\}}{2\sqrt{p_1 p_2}}$$

The parameter ω_0 is called the natural undamped frequency of the system. The parameter ζ is called the damping ratio.

$$p_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

$\zeta > 1$: The poles p_1 and p_2 are real-valued and distinct. The system is said to be overdamped.

$\zeta = 1$: The 2 poles are $p_1 = p_2 = -\zeta\omega_0$. The system is said to be critically damped.

$\zeta < 1$: The two poles are a complex conjugate pair:

$$p_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2} = -\zeta\omega_0 \pm j\omega_d$$

In this case the system is said to be underdamped.

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2} = \frac{1}{1-\left(\omega/\omega_0\right)^2 + j2\zeta\left(\omega/\omega_0\right)}$$

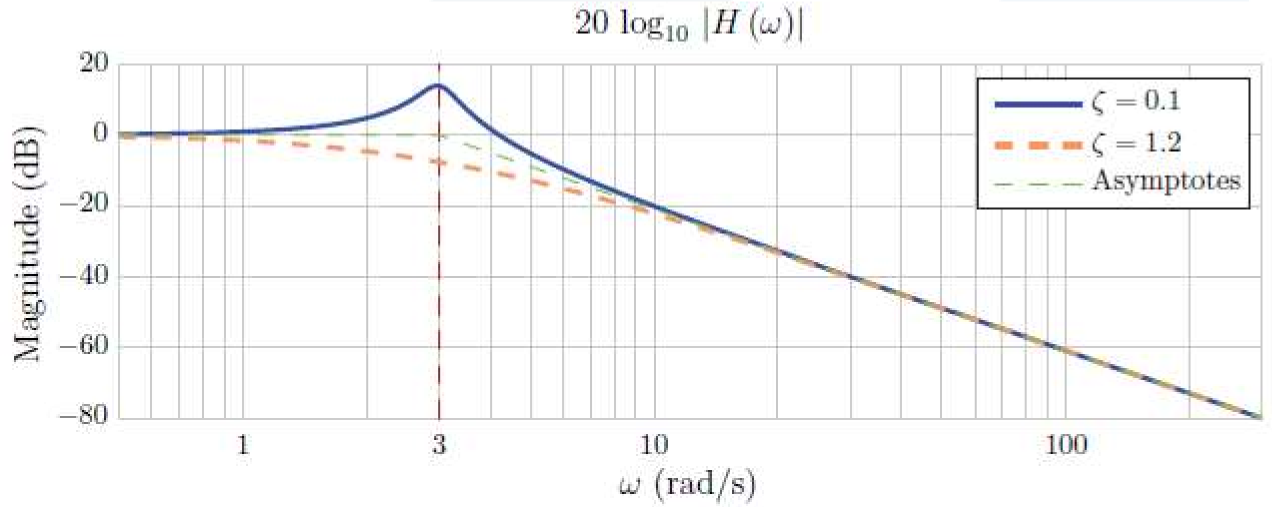
$$20\log_{10}\left|H(\omega)\right| = -10\log_{10}\left\{\left[1-\left(\omega/\omega_0\right)^2\right]^2 + \left[2\zeta\left(\omega/\omega_0\right)\right]^2\right\}$$

$$\measuredangle H(\omega) = -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_0)}{1-(\omega/\omega_0)^2}\right]$$

Define the quality factor as $Q = 1/2\zeta$

Magnitude: For $\omega << \omega_0$ the magnitude is asymptotic to 0 dB. For $\omega >> \omega_0$ it becomes asymptotic to a straight line with a slope of –40 dB per decade. At $\omega = \omega_0$ the actual magnitude is $20 \log_{10} Q = -20 \log_{10} (2\zeta)$.

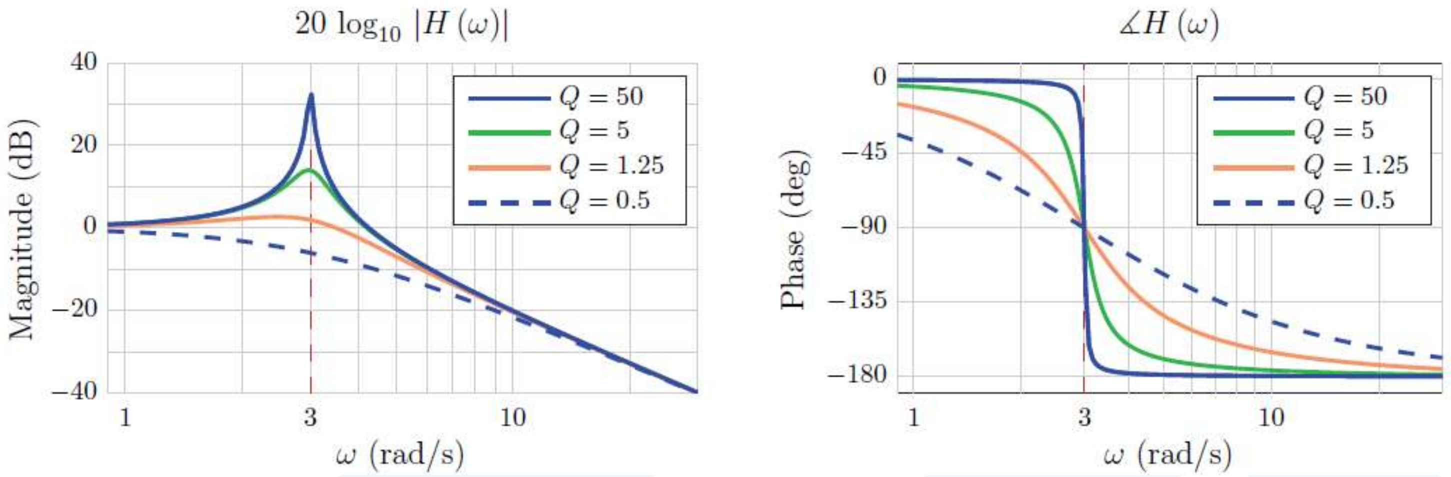
Phase: For $\omega << \omega_0$ the phase is asymptotic to 0°. For $\omega >> \omega_0$ the phase is –180°. At $\omega = \omega_0$ the phase is −90°.



Overdamped: $\zeta > 1 \Rightarrow Q < 0.5$

Critically damped: $\zeta = 1 \Rightarrow Q = 0.5$

Underdamped: $\zeta < 1 \Rightarrow Q > 0.5$



- The response of the second-order system to unit-impulse: $H(s) = \dfrac{k_1}{s - p_1} + \dfrac{k_2}{s - p_2}$

$$k_1 = \frac{p_1 p_2}{p_1 - p_2} = \frac{\omega_0}{2\sqrt{\zeta^2 - 1}}, \quad k_2 = \frac{p_1 p_2}{p_2 - p_1} = -\frac{\omega_0}{2\sqrt{\zeta^2 - 1}}$$

$$h(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} = \frac{\omega_0}{2\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t} \left[e^{\omega_0 \sqrt{\zeta^2 - 1}\, t} - e^{-\omega_0 \sqrt{\zeta^2 - 1}\, t} \right]$$

If $\zeta < 1$, $h(t) = \dfrac{\omega_0}{\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t} \sin\left(\omega_0 \sqrt{\zeta^2 - 1}\, t\right) u(t)$

If $\zeta = 1$, $H(s) = \dfrac{\omega_0}{(s + \omega_0)^2} \Rightarrow h(t) = \omega_0^2 t e^{-\omega_0 t} u(t)$

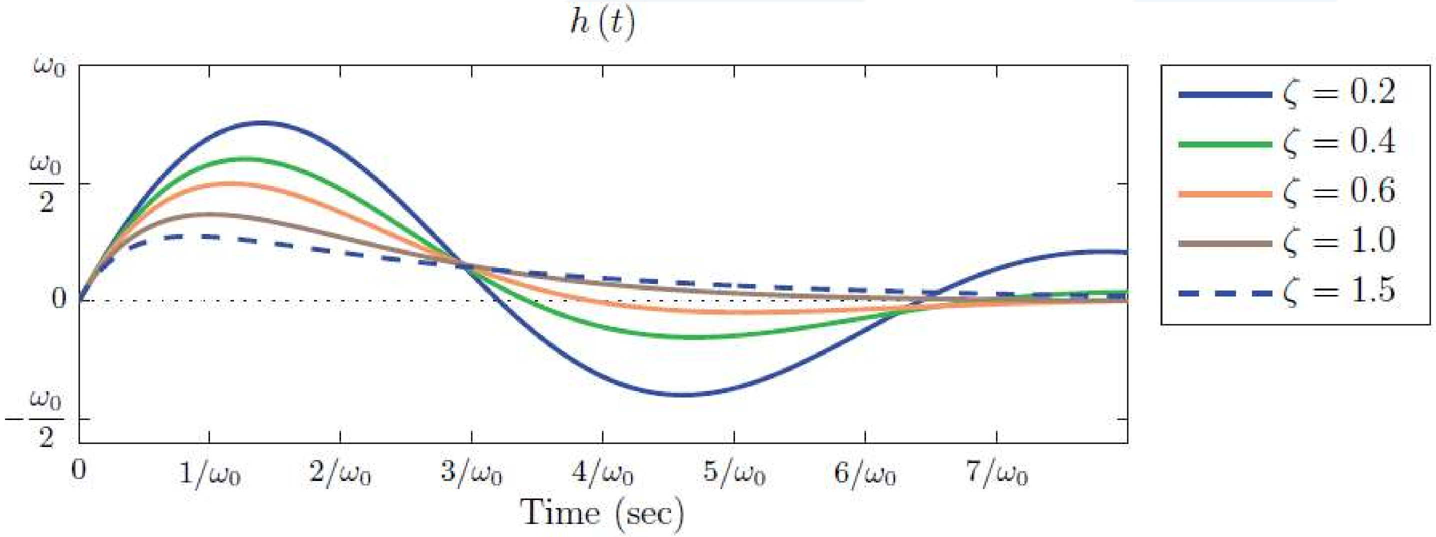
- The response of the second-order system to unit-step:

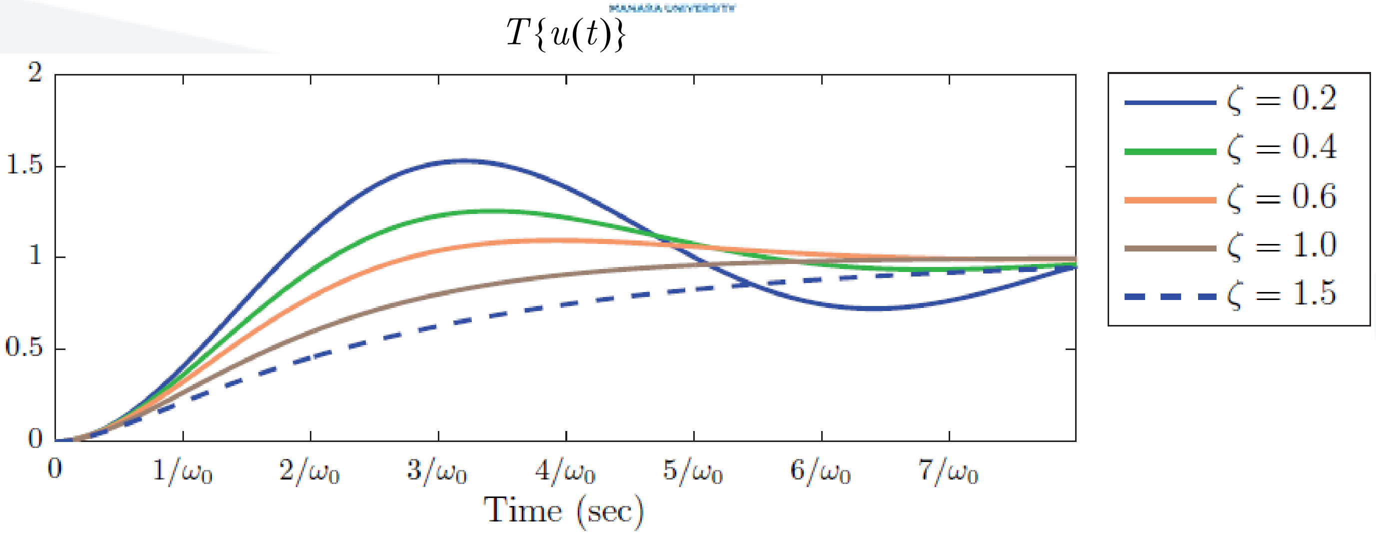
$$T\{u(t)\} = h(t) * u(t) = \int_0^t h(\tau) d\tau, \quad t \geq 0$$

If $\zeta \neq 1$, $T\{u(t)\} = 1 + \dfrac{1}{p_1 - p_2} [p_2 e^{p_1 t} - p_1 e^{p_2 t}] u(t)$

$$T\{u(t)\} = 1 + \frac{e^{-\omega_0 \zeta t}}{2\omega_0\sqrt{1-\zeta^2}}\left[\left(-\omega_0\zeta - \omega_0\sqrt{1-\zeta^2}\right)e^{\omega_0\sqrt{1-\zeta^2}\,t} + \left(\omega_0\zeta - \omega_0\sqrt{1-\zeta^2}\right)e^{-\omega_0\sqrt{1-\zeta^2}\,t}\right]u(t)$$

If $\zeta = 1$, $T\{u(t)\} = [1 - e^{-\omega_0 t} - \omega_0 t e^{-\omega_0 t}]u(t)$





5. Simulation Structures for CTLTI Systems

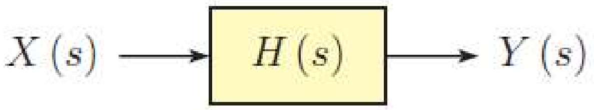
Direct-form implementation

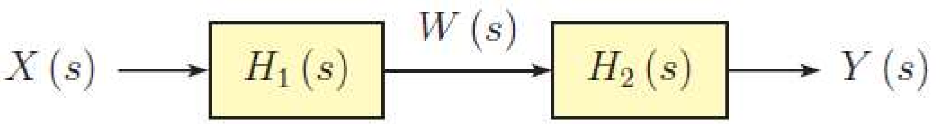
- The method of obtaining a block diagram from an s-domain TF will be derived using a third-order system, but its generalization to higher-order TF is quite straightforward. Consider a CTLTI system described by a TF $H(s)$:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Let us use an intermediate function $W(s)$

$$H(s) = \frac{Y(s)}{W(s)} \frac{W(s)}{X(s)} = \frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}}$$

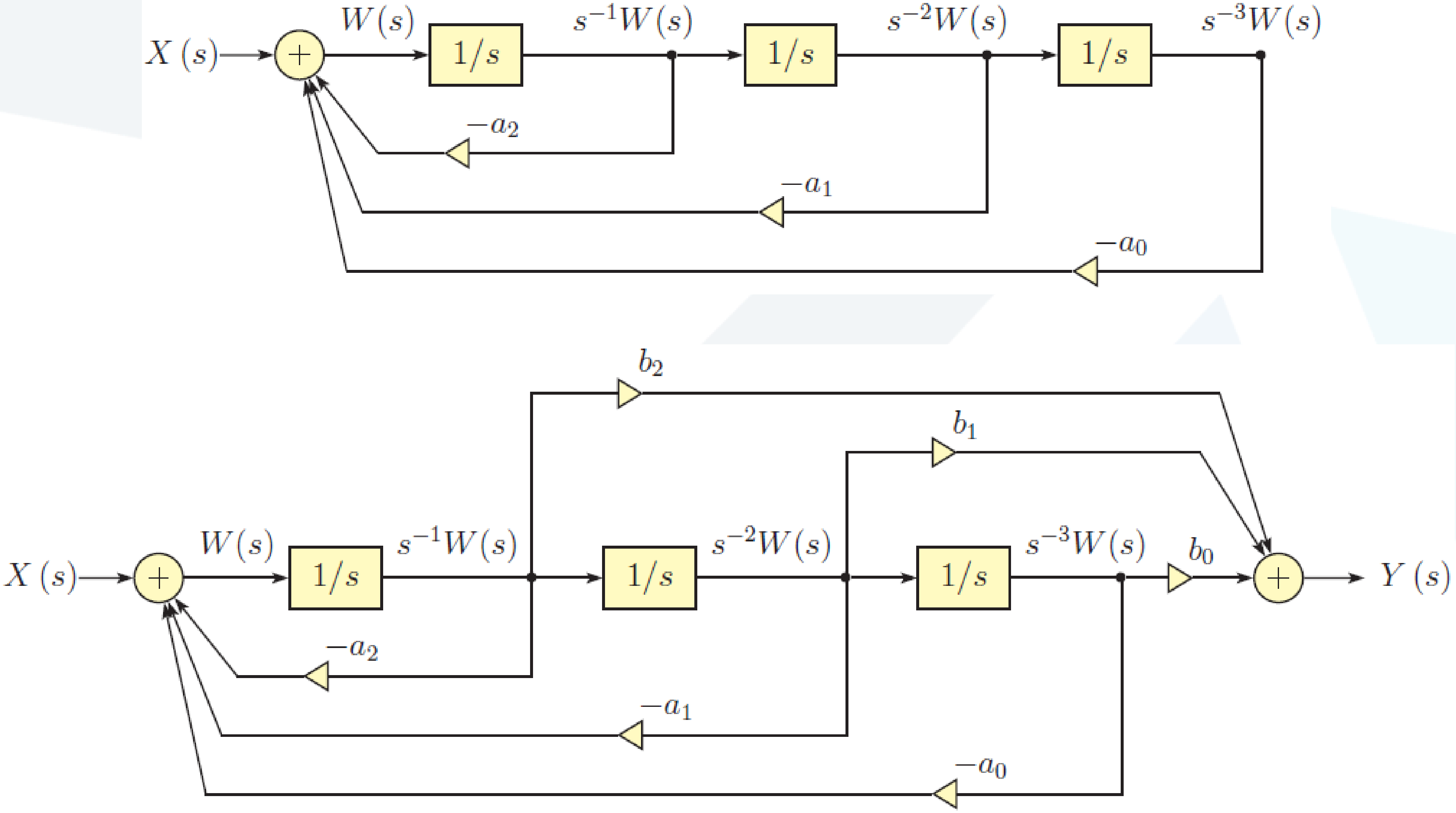




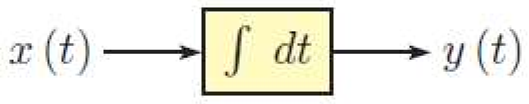
$$H_1(s) = \frac{W(s)}{X(s)} = \frac{1}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}}, \quad H_2(s) = \frac{Y(s)}{W(s)} = b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}$$

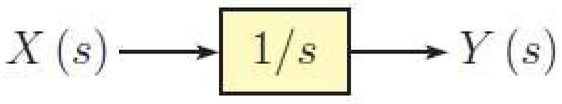
$$W(s) = X(s) - a_2 s^{-1} W(s) - a_1 s^{-2} W(s) - a_0 s^{-3} W(s)$$

$$Y(s) = b_2 s^{-1} W(s) + b_1 s^{-2} W(s) + b_0 s^{-3} W(s)$$



Completed block diagram for simulating the transfer function $H(s)$

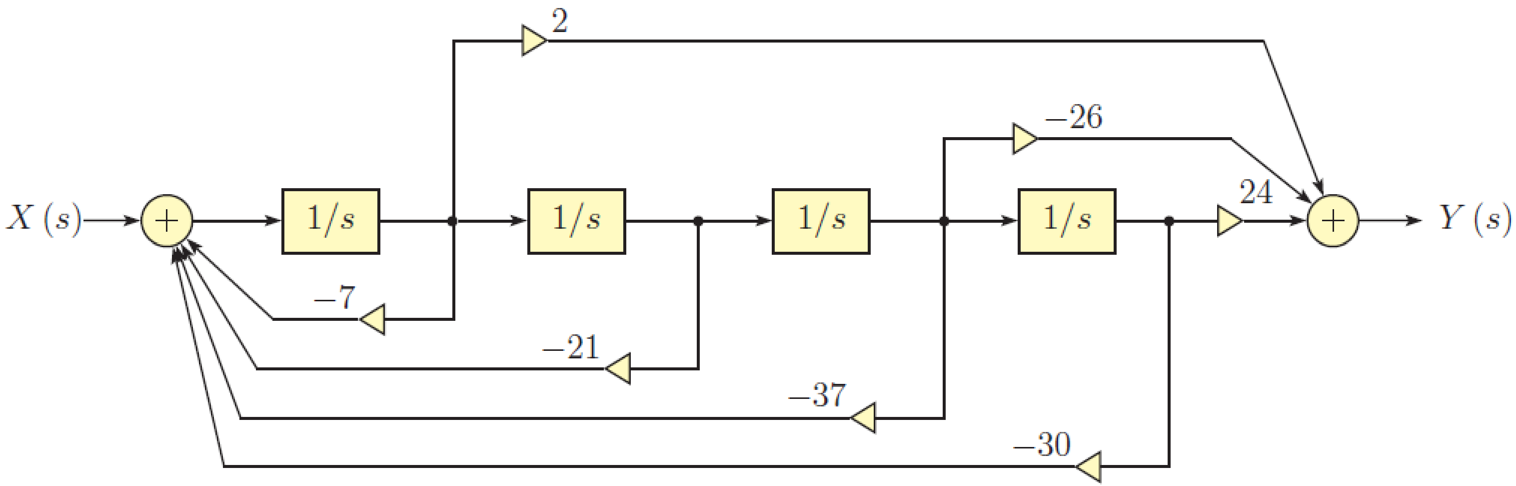




- Example 5: Obtaining a block diagram from transfer function

A CTLTI system is described through the transfer function:

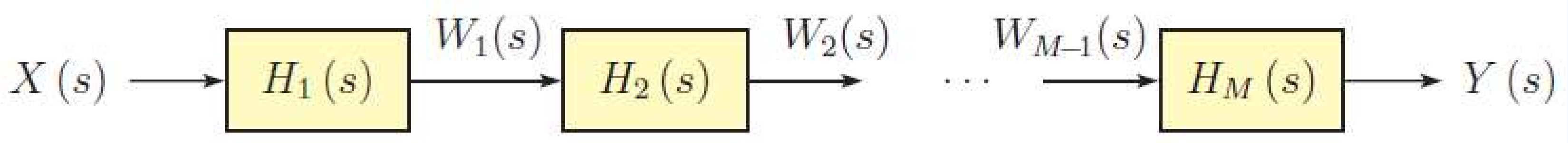
$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s^3 - 26s + 24}{s^4 + 7s^3 + 21s^2 + 37s + 30}$$



Cascade and parallel forms

Cascade form

$$H(s) = H_1(s)H_2(s)\cdots H_M(s) = \frac{W_1(s)}{X(s)}\frac{W_2(s)}{W_1(s)}\cdots\frac{Y(s)}{W_{M-1}(s)}$$



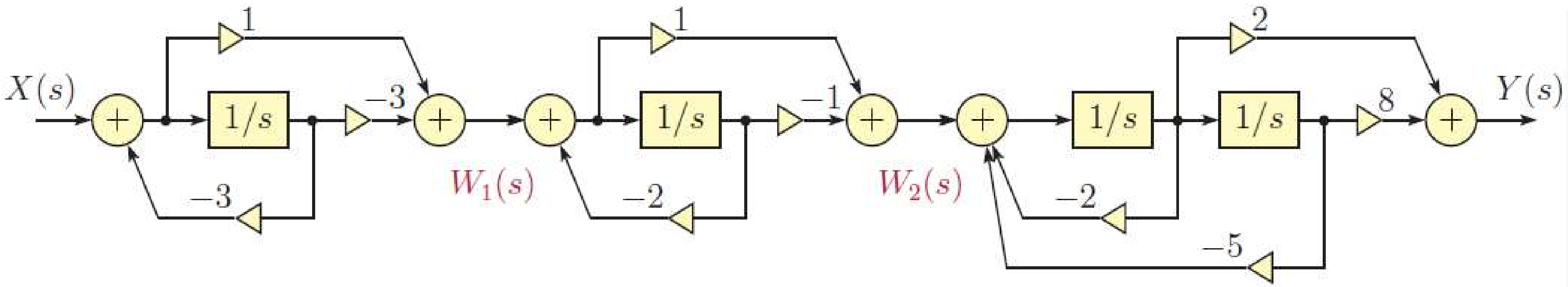
- Example 6: Obtaining a block diagram from transfer function

  Develop a cascade form block diagram for simulating the system used in example 2.

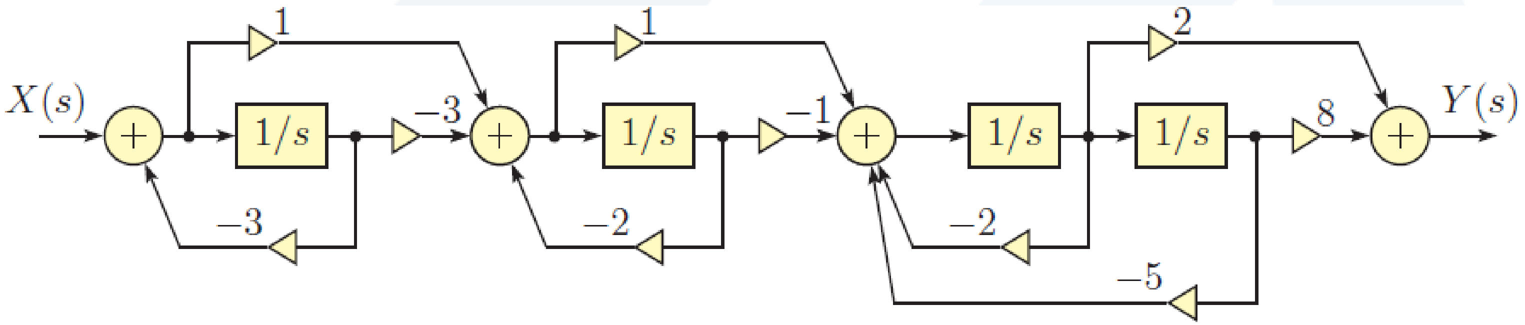
$$H(s) = \frac{Y(s)}{X(s)} = \frac{2(s+4)(s-3)(s-1)}{(s+1-j2)(s+1+j2)(s+3)(s+2)}$$

$$H(s) = H_1(s)H_2(s)H_3(s)$$

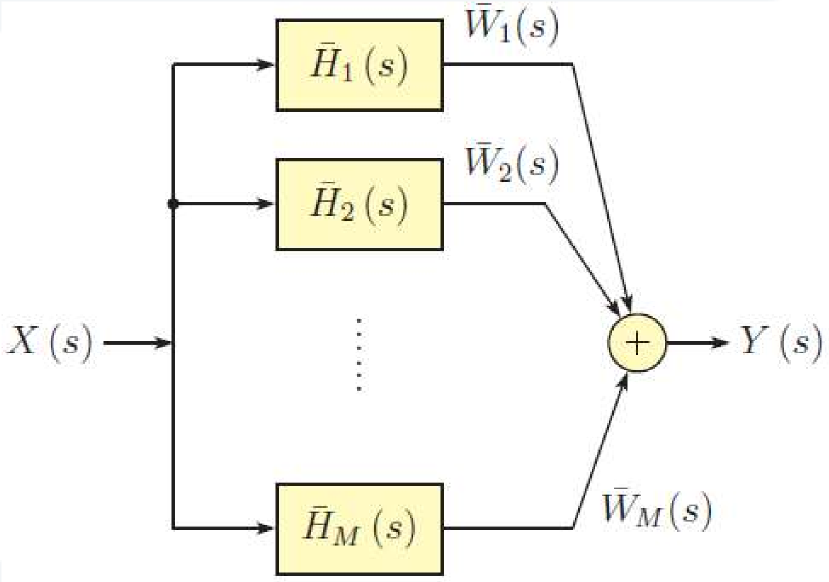
$$H_1(s) = \frac{2(s+4)}{(s+1-j2)(s+1+j2)} = \frac{2s+8}{s^2+2s+5}, \quad H_2(s) = \frac{s-3}{s+3}, \quad H_3(s) = \frac{s-1}{s+2}$$



Further simplified cascade form block diagram



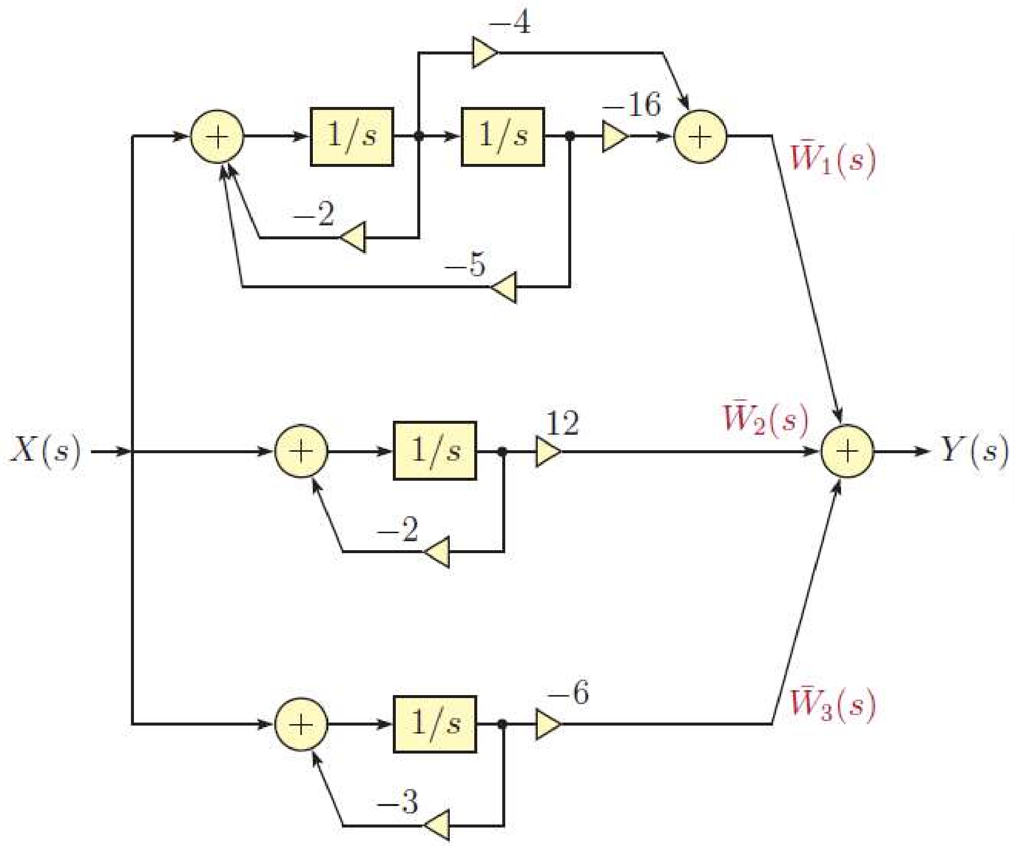
Parallel form $$H(s) = \bar{H}_1(s) + \bar{H}_2(s) + \cdots + \bar{H}_M(s) = \frac{\bar{W}_1(s)}{X(s)} + \frac{\bar{W}_2(s)}{X(s)} + \cdots + \frac{\bar{W}_M(s)}{X(s)}$$



- Example 7: Obtaining a block diagram from transfer function

  Develop a parallel form BD for simulating the system used in example 2.

$$H(s) = \frac{2s+8}{s^2+2s+5} + \frac{12}{s+2} + \frac{-6}{s+3}$$



6. Unilateral Laplace Transform

The unilateral Laplace transform of the function x is defined as:

$$\mathcal{L}_u\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

- The unilateral LT is related to the bilateral Laplace transform as follows:

$$\mathcal{L}_u\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt = \mathcal{L}\{x(t)u(t)\}$$

- With the unilateral LT, the same inverse transform equation is used as in the bilateral case.
- The unilateral LT is only invertible for causal functions.
- For a noncausal function x, we can only recover $x(t)$ for $t \geq 0$.

Unilateral Versus Bilateral Laplace Transform

In the unilateral case:

- The time-domain convolution property has the additional requirement that the functions being convolved must be causal.
- The time/Laplace-domain scaling property has the additional constraint that the scaling factor must be positive.
- The time-domain differentiation property has an extra term in the expression of $\mathcal{L}_u(dx(t)/dt)$, namely $-x(0^-)$.
- The time-domain integration property has a different lower limit in the time-domain integral (0^- instead of $-\infty$);
- The time-domain shifting property does not hold (except in special cases).

Properties of the Unilateral Laplace Transform

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Multiply by t	$t\,x(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at),\ a > 0$	$\frac{1}{a}X(\frac{s}{a})$	aR
Differentiate in t	$dx(t)/dt$	$sX(s) - x(0^-)$	$\supset R$
Integrate in t	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\text{Re}(s) > 0))$
Convolve in t	$x_1 * x_2(t)$	$X_1(s)\ X_2(s)$	$\supset (R_1 \cap R_2)$

Unilateral Laplace Transform Pairs

Pair	$x(t);\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$

Pair	$x(t);\ t \geq 0$	$X(s)$
6	$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$