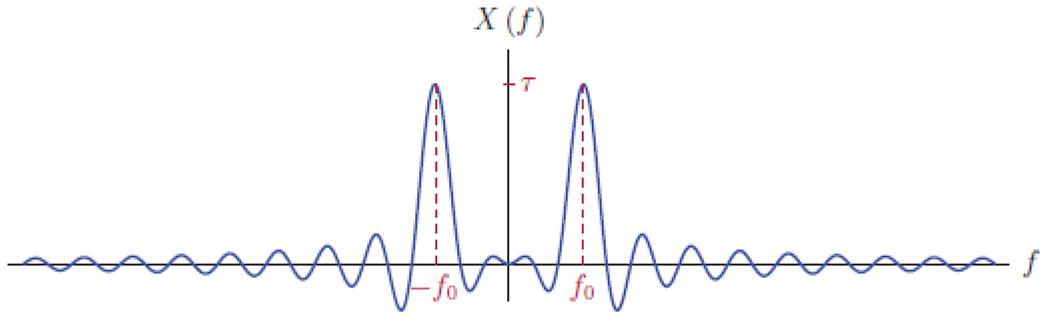
CECC507: Signals and Systems

Lecture Notes 11: Z-Transform for Discrete-Time Signals and Systems: Part A



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Chapter 8

Z-Transform for Discrete-Time Signals and Systems

1. Introduction

- The z-transform (ZT) can be viewed as a generalization of the discrete time Fourier transform.
- The ZT representation exists for some sequences that do not have a discrete Fourier transform representation. So, we can handle some sequences with the ZT that cannot be handled with the DTFT ($x[n] = nu[n]$).

2. Z-Transform

- The z-transform of a discrete-time signal $x[n]$ is defined as:

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z, the independent variable of the transform is a complex number.

- The z-transform defined is sometimes referred to as the bilateral (two sided) z-transform. A simplified variant of the transform termed the unilateral (one-sided) z-transform is introduced as an alternative analysis tool.

Relationship Between ZT and Discrete-Time FT

$$X(r,\Omega) = X(z)\Big|_{z=re^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n](re^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\Omega n} = \mathcal{F}\{r^{-n}x[n]\}$$

$$X(z)\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{F}\{x[n]\}$$

- Example 1: A simple z-transform example

$$x[n] = \{3.\underset{\underset{n=0}{\uparrow}}{7}, 1.3, -1.5, 3.4, 5.2\}$$

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n} = 3.7+1.3z^{-1}-1.5z^{-2}+3.4z^{-3}+5.2z^{-4}$$

The transform converges at all points in the complex z-plane except of $z=0$.

- Example 2: z-transform of a non-causal signal

$$x[n]=\{3.7,\ 1.3,\ -1.\underset{\substack{\uparrow\\ n=0}}{5},\ 3.4,\ 5.2\}$$

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n} = 3.7z^{2}+1.3z^{1}-1.5+3.4z^{-1}+5.2z^{-2}$$

It converges at every point in the z-plane except, the origin and infinity.

- Example 3: z-Transform of the unit-impulse

$$X(z)=Z\{\delta[n]\}=\sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^{0}=1$$

It converges at every point in the z-plane

- Example 4: z-Transform of a time shifted the unit-impulse

$$X(z) = Z\{\delta[n-k]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = z^{-k}$$

1. If $k > 0$ then the transform does not converge at the origin $z = 0$.
2. If $k < 0$ then the transform does not converge at infinity.

Regions of Convergence

- For the z-transform $X(z)$ of $x[n]$ to exist we need that:

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]|\left|r^{-n}e^{-j\Omega n}\right| = \sum_{n=-\infty}^{\infty} |x[n]|r^{-n} < \infty$$

Thus, the ROC depends only on r and not on Ω.

ROC of Finite-Support Signals

The region of convergence (ROC) of the z-transform of a signal $x[n]$ of finite support $[N_0, N_1]$, where $-\infty < N_0 \leq n \leq N_1 < \infty$, is the whole z-plane, excluding the origin $z = 0$ and/or $z = \pm\infty$ depending on N_0 and N_1.

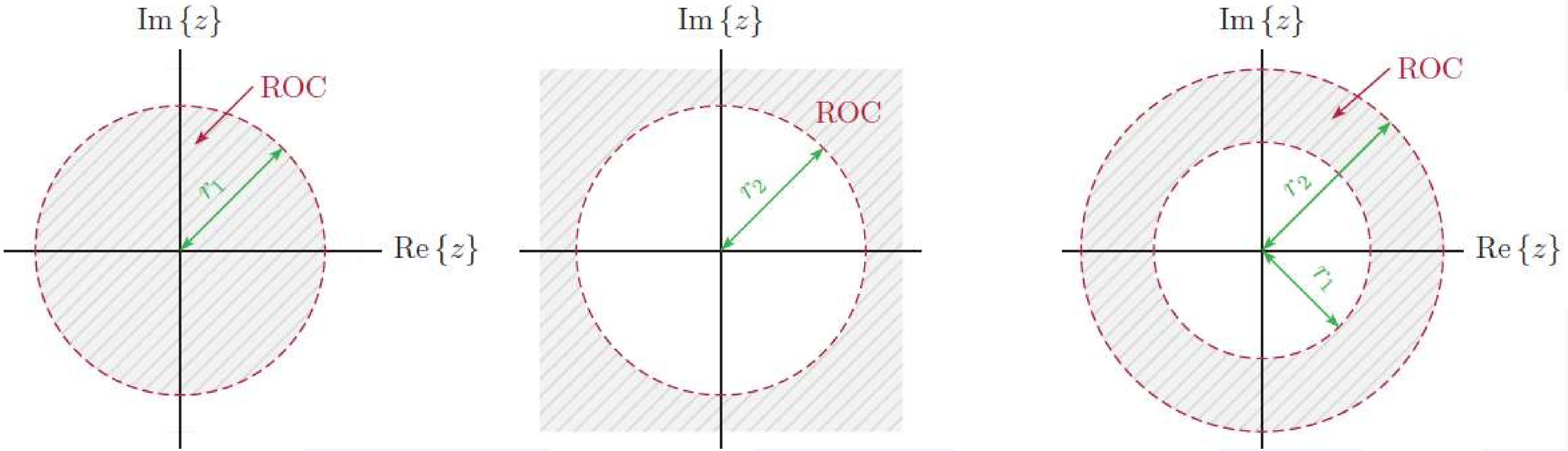
$$X(z) = \sum_{n=N_0}^{N_1} x[n]z^{-n}$$

ROC of Infinite-Support Signals

1. causal signal $x[n]$ has a region of convergence $|z| > r_1$ where r_1 is the largest radius of the poles of $X(z)$, i.e., the ROC is the outside of a circle of radius r_1,

2. anticausal signal $x[n]$ has as region of convergence the inside of the circle defined by the smallest radius r_2 of the poles of $X(z)$, or $|z| < r_2$,

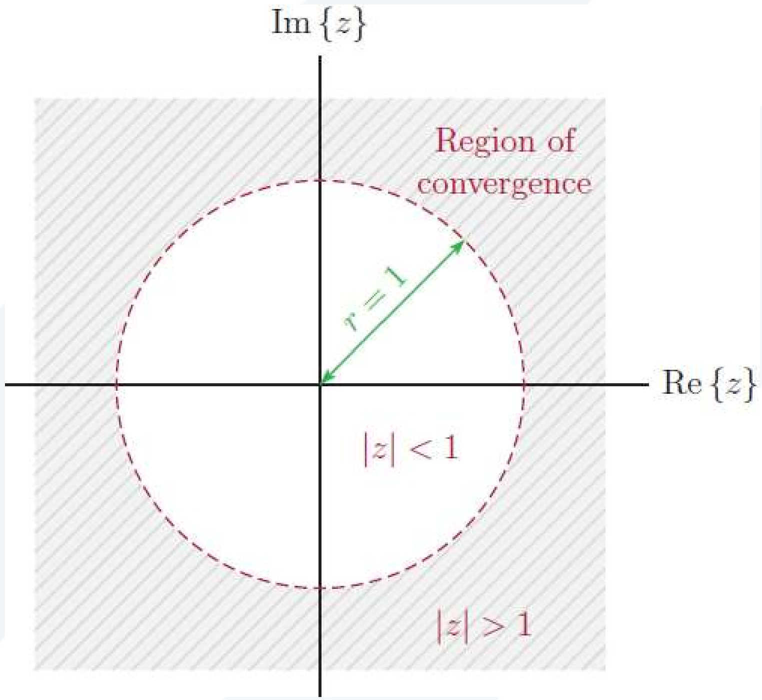
3. noncausal signal $x[n]$ has as region of convergence $r_1 < |z| < r_2$, or the inside of a torus of inside radius r_1 and outside radius r_2 corresponding to the maximum and minimum radii of the poles of $X_c(z)$ and $X_{ac}(z)$, or the z-transforms of the causal and anticausal components of $x[n] = x_c[n] + x_{ac}[n]$.



- Example 5: z-Transform of the unit-step signal

$$X(z) = \mathcal{Z}\{u[n]\} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

converge if: $|z^{-1}| < 1 \Rightarrow |z| > 1$

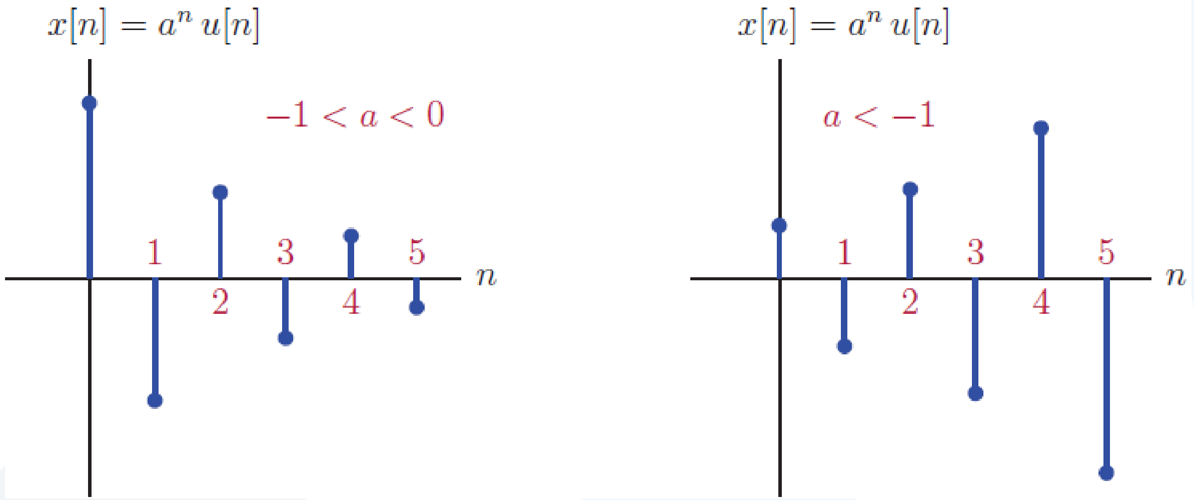


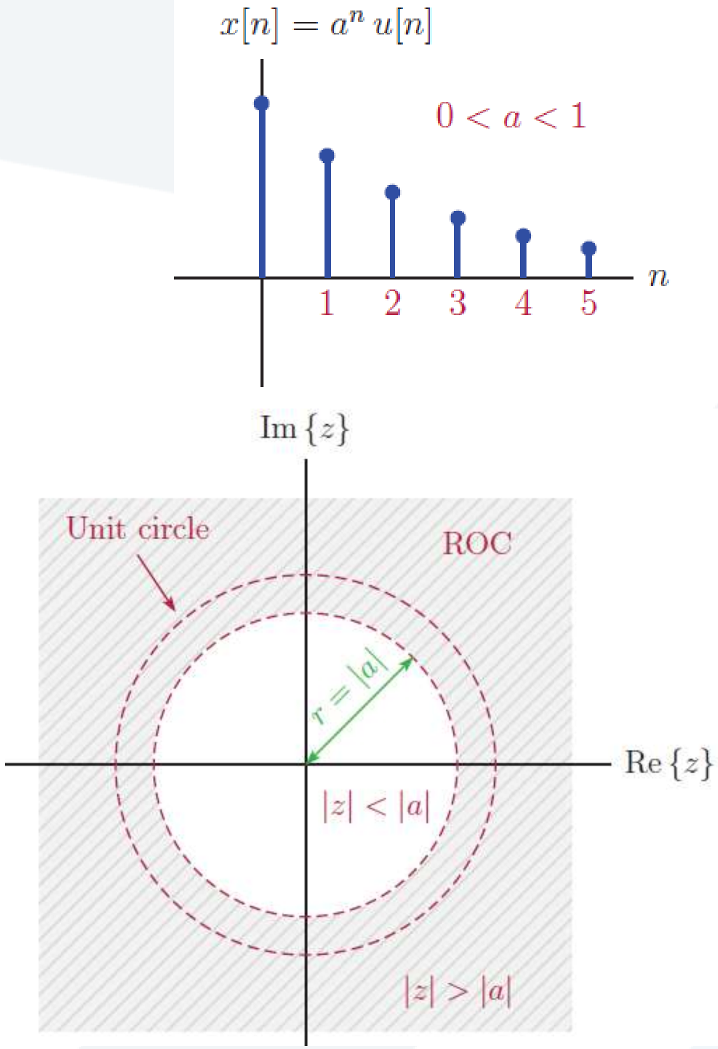
- Example 6: z-Transform of a causal exponential signal

$$x[n] = a^n\, u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

converge if: $|az^{-1}| < 1 \Rightarrow |z| > |a|$





$x[n] = a^n u[n]$
$0 < a < 1$
n
1 2 3 4 5
$x[n] = a^n u[n]$
$a > 1$
n
1 2 3 4 5
Im $\{z\}$
Unit circle
ROC
$r = |a|$
Re $\{z\}$
$|z| < |a|$
$|z| > |a|$
Im $\{z\}$
Unit circle
ROC
$r = |a|$
Re $\{z\}$
$|z| < |a|$
$|z| > |a|$

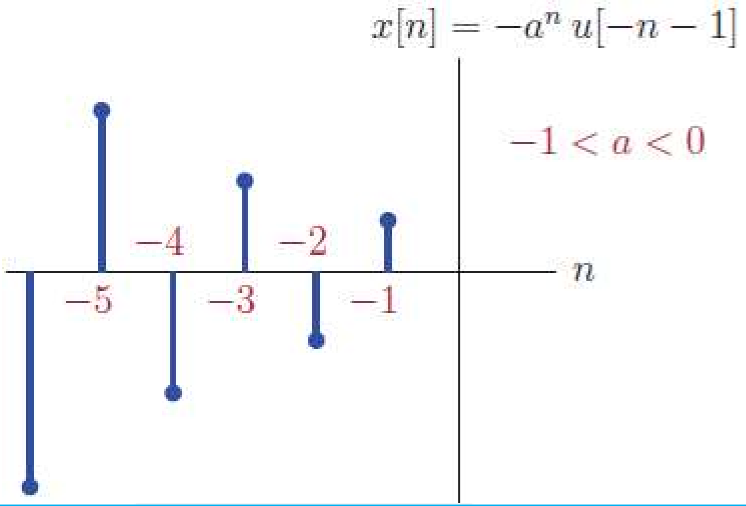
- Example 7: z-Transform of an anti-causal exponential signal

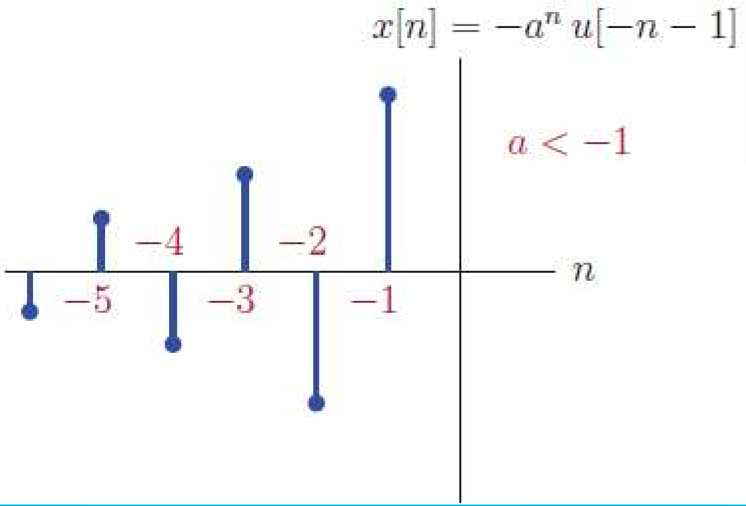
$$x[n] = -a^n\, u[-n-1]$$

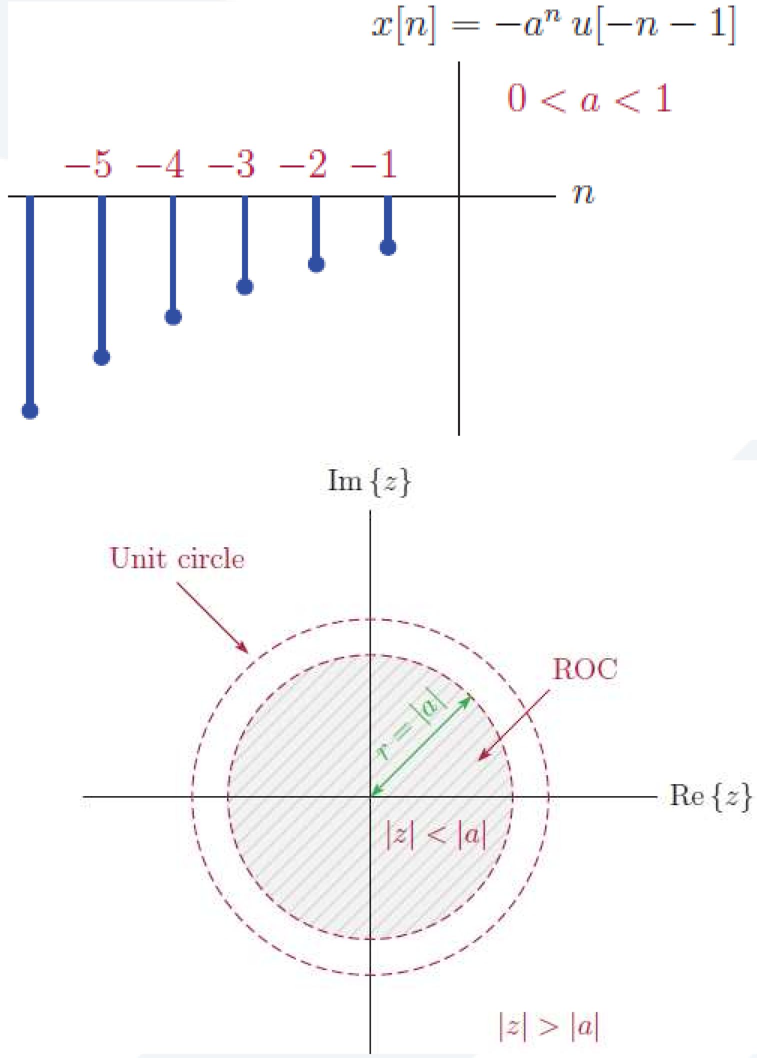
$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} (a^{-1}z)^m$$

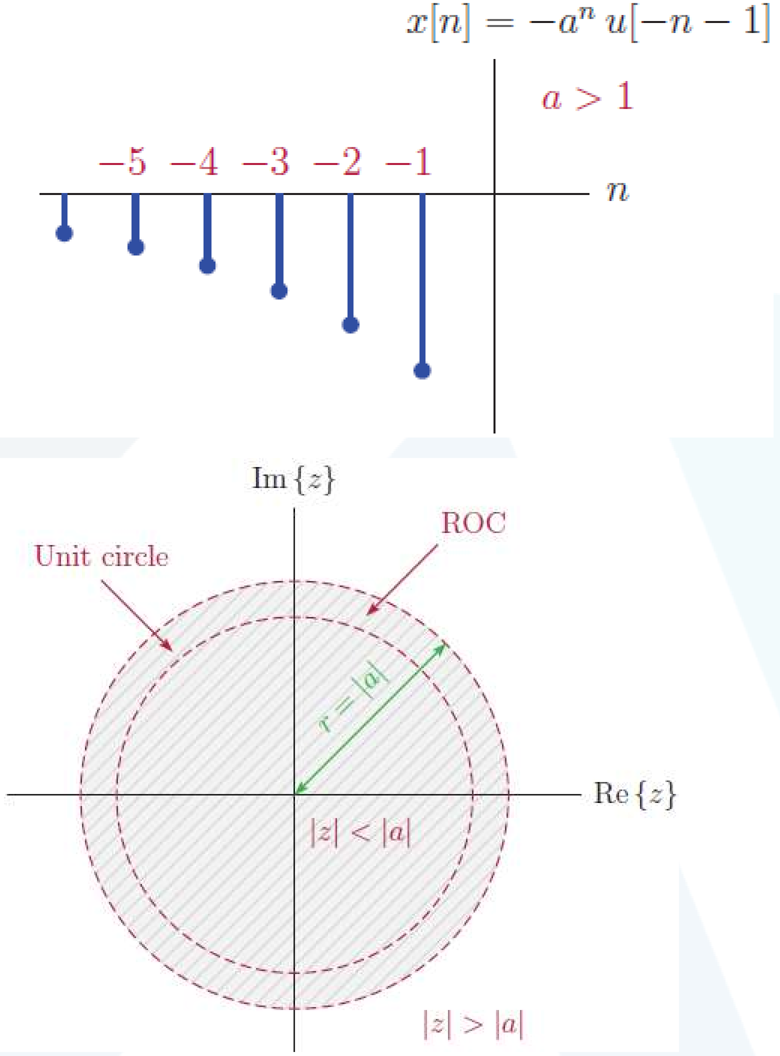
$$= -a^{-1}z \frac{1}{1-a^{-1}z} = \frac{z}{z-a}$$

converge if: $|a^{-1}z| < 1 \Rightarrow |z| < |a|$






$x[n] = -a^n\, u[-n-1]$
$0 < a < 1$
-5 -4 -3 -2 -1
n
Im $\{z\}$
Re $\{z\}$
Unit circle
ROC
$r = |a|$
$|z| < |a|$
$|z| > |a|$


$x[n] = -a^n\, u[-n-1]$
$a > 1$
-5 -4 -3 -2 -1
n
Im $\{z\}$
Re $\{z\}$
Unit circle
ROC
$r = |a|$
$|z| < |a|$
$|z| > |a|$

$$\mathcal{Z}\{a^n u[n]\} = \frac{z}{z-a}, \quad \text{ROC: } |z| > |a|$$

$$\mathcal{Z}\{-a^n u[-n-1]\} = \frac{z}{z-a}, \quad \text{ROC: } |z| < |a|$$

- Note: It is possible for two different signals to have the same transform expression for the z-transform $X(z)$. In order for us to uniquely identify which signal among the two led to a particular transform, the region of convergence must be specified along with the transform.

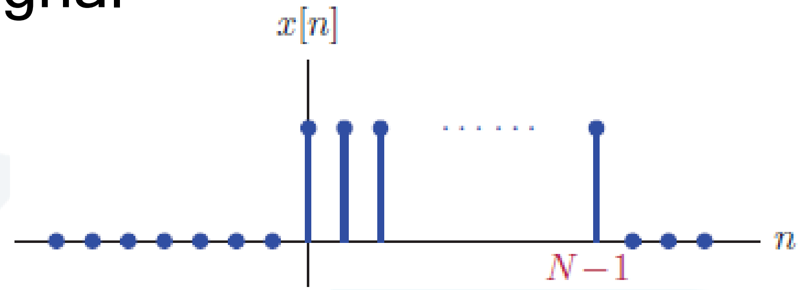
- In the general case, a rational transform $X(z)$ is expressed in the form

$$X(z) = K\frac{B(z)}{A(z)} = K\frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(p-p_1)(p-p_2)\cdots(p-p_N)}$$

The larger of M and N is the order of the transform $X(z)$.

- Example 8: z-Transform of a discrete-time pulse signal

$$X(z) = \sum_{n=0}^{N-1} (1) z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}}, \quad \text{ROC: } |z| > 0$$

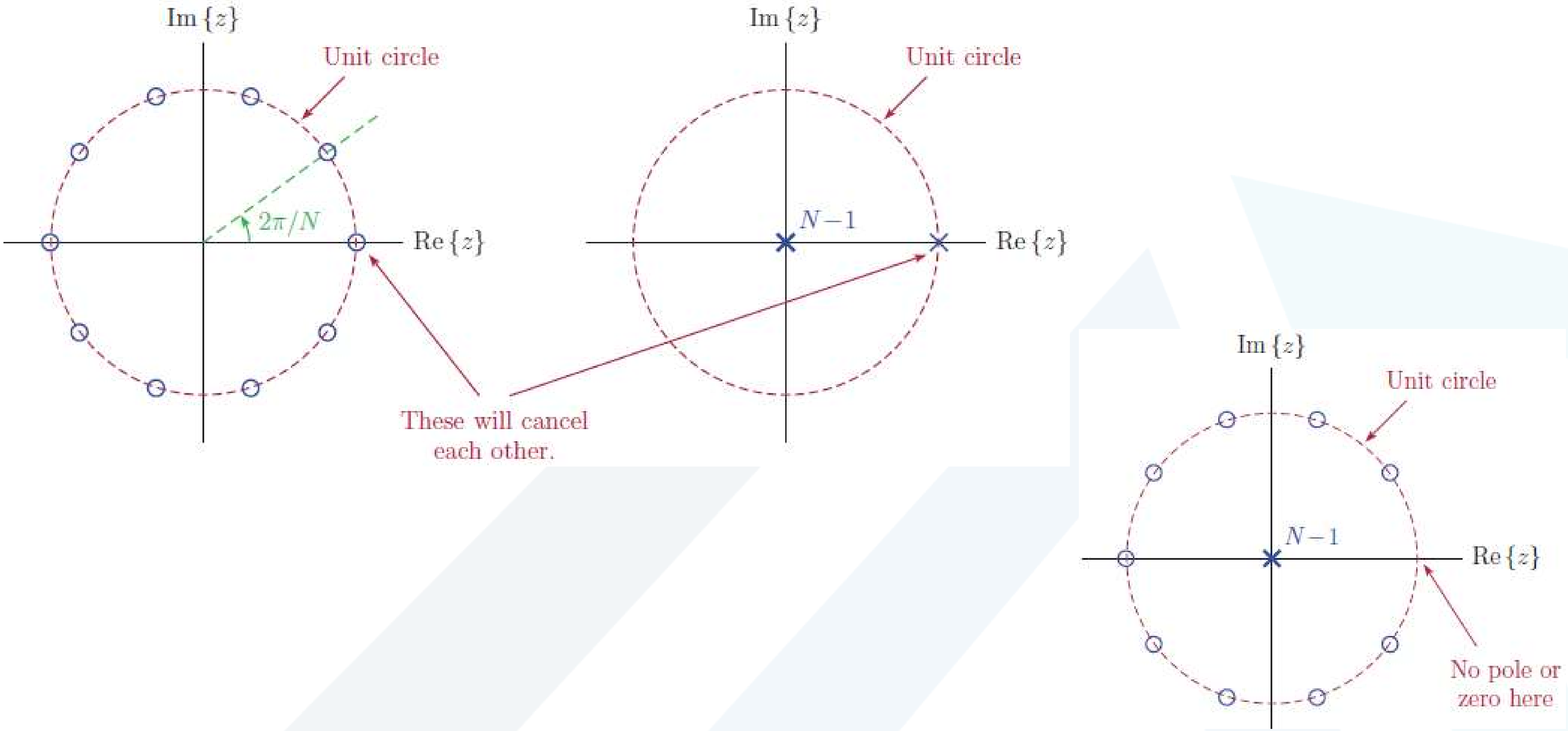


$$X(z) = \frac{z^N - 1}{z^{N-1}(z-1)}$$ It seems as though $X(z)$ might have a pole at $z = 1$

Zeros: $z_k = e^{j2\pi k/N}, \quad k = 1, \cdots, N-1$

Poles: $z = 1$ and $p_k = 0, \quad k = 1, \cdots, N-1$

The factors $(z - 1)$ in numerator and denominator polynomials cancel each other, therefore there is neither a zero nor a pole at $z = 1$.

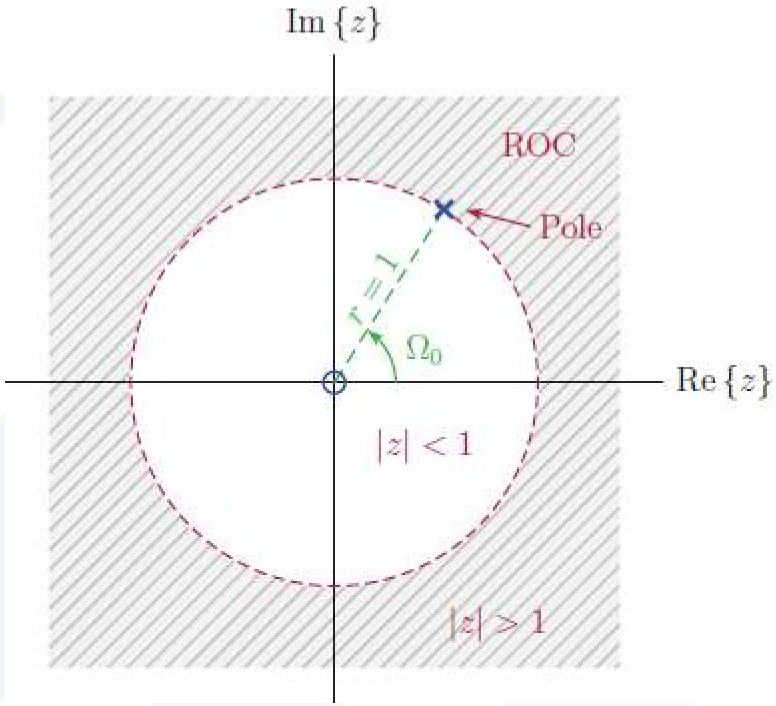

Im {z}
Unit circle
2π/N
Re {z}
These will cancel
each other.
Im {z}
Unit circle
N−1
Re {z}
Im {z}
Unit circle
N−1
Re {z}
No pole or
zero here

- Example 9: z-Transform of complex exponential signal

$$x[n] = e^{j\Omega_0 n}u[n]$$

$$X(z) = \sum_{n=0}^{\infty} e^{j\Omega_0 n} z^{-n} = \sum_{n=0}^{\infty} (e^{j\Omega_0} z^{-1})^n = \frac{1}{1 - e^{j\Omega_0} z^{-1}}$$

$$X(z) = \frac{z}{z - e^{j\Omega_0}} \qquad \text{ROC: } \left|e^{j\Omega_0} z^{-1}\right| < 1 \Rightarrow |z| > 1$$



Properties of z-Transform

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Time shifting	$x[n - k]$	$X(z)z^{-k}$	$R \pm \{0 \text{ or } \infty\}$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1}
Multiply by exp.	$x[n]a^n$	$X(z/a)$	$\|a\|R$
Differentiate in z	$nx[n]$	$-z\, dX(z)/dz$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)\ X_2(z)$	$\supset (R_1 \cap R_2)$
Summation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{z}{z-1} X(z)$	$\supset (R \cap (z > 1))$

- Example 10: z-Transform of a cosine signal

$$x[n] = \cos(\Omega_0 n)u[n]$$

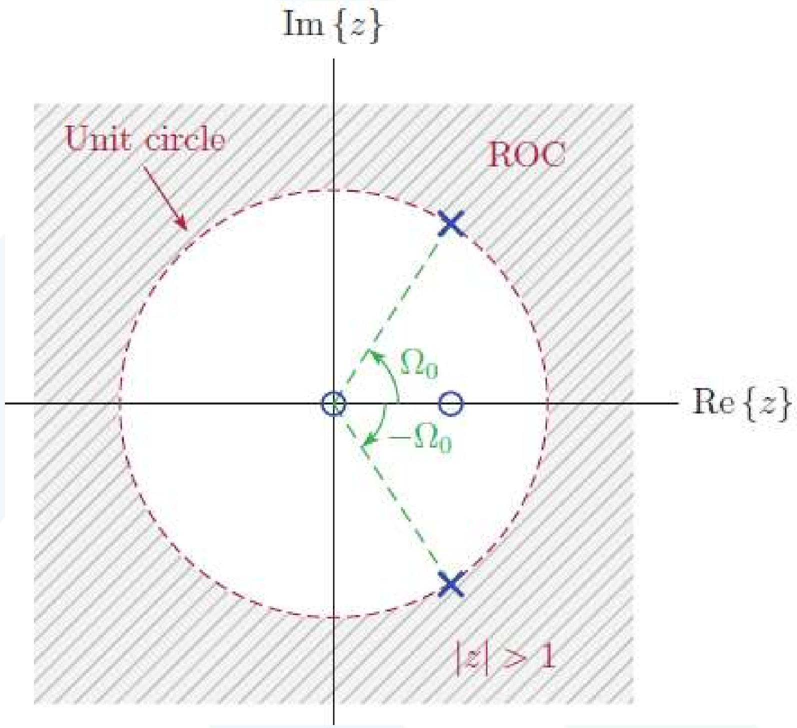
$$\cos(\Omega_0 n)u[n] = \frac{1}{2}e^{j\Omega_0 n}u[n] + \frac{1}{2}e^{-j\Omega_0 n}u[n]$$

$$\mathcal{Z}\{\cos(\Omega_0 n)u[n]\} = \frac{1}{2}\mathcal{Z}\{e^{j\Omega_0 n}u[n]\} + \frac{1}{2}\mathcal{Z}\{e^{-j\Omega_0 n}u[n]\}$$

$$= \frac{1/2}{1-e^{j\Omega_0}z^{-1}} + \frac{1/2}{1-e^{-j\Omega_0}z^{-1}} = \frac{1-\cos(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}}$$

$$= \frac{z[z-\cos(\Omega_0)]}{z^2-2\cos(\Omega_0)z+1}$$

ROC is $|z| > 1$



- Example 11: z-Transform of a sine signal

$$x[n] = \sin(\Omega_0 n)u[n]$$

$$\sin(\Omega_0 n)u[n] = \frac{1}{2j}e^{j\Omega_0 n}u[n] - \frac{1}{2j}e^{-j\Omega_0 n}u[n]$$

$$\mathcal{Z}\{\sin(\Omega_0 t)u[n]\} = \frac{1}{2j}\mathcal{Z}\{e^{j\Omega_0 n}u[n]\} - \frac{1}{2j}\mathcal{Z}\{e^{-j\Omega_0 n}u[n]\}$$

$$= \frac{1/2j}{1 - e^{j\Omega_0}z^{-1}} - \frac{1/2j}{1 - e^{-j\Omega_0}z^{-1}} = \frac{\sin(\Omega_0)z^{-1}}{1 - 2\cos(\Omega_0)z^{-1} + z^{-2}}$$

$$= \frac{\sin(\Omega_0)z}{z^2 - 2\cos(\Omega_0)z + 1}$$

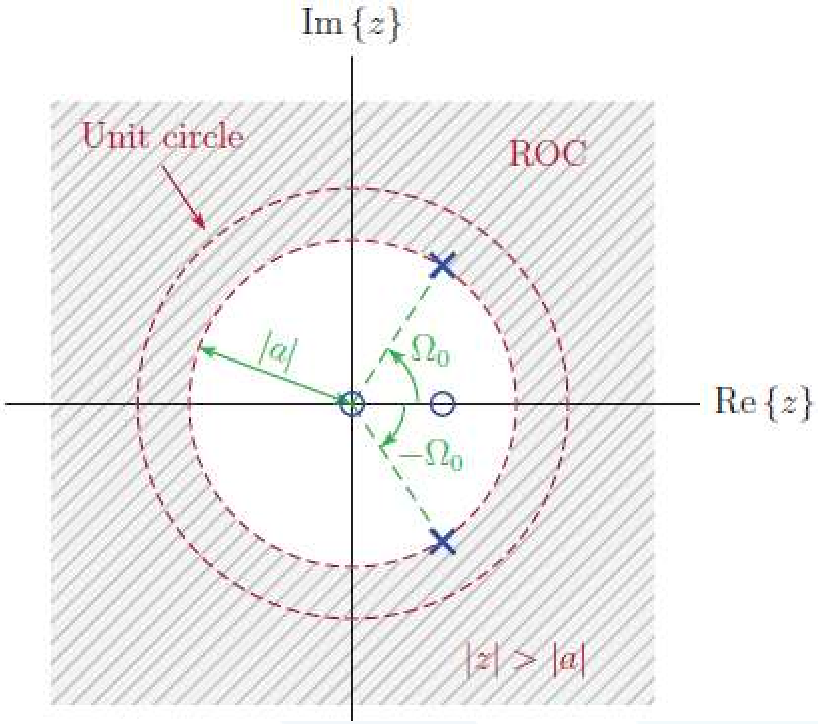
ROC is $|z| > 1$

- Example 12: Multiplication by an exponential signal

$$x[n] = a^n \cos(\Omega_0 n) u[n]$$

$$x[n] = a^n x_1[n], \qquad X_1(z) = \frac{z[1-\cos(\Omega_0)]}{z^2 - 2\cos(\Omega_0) z + 1}$$

$$X_1(z) = X_1(z/a) = \frac{z[z - a\cos(\Omega_0)]}{z^2 - 2a\cos(\Omega_0) z + a^2}$$



The transform $X(z)$ has two poles at:

$$z = ae^{\pm j\Omega_0} \Rightarrow \text{ROC: } |z| > |a|$$

- Example 13: Using the differentiation property

$$x[n] = na^n \, u[n]$$

$$\mathcal{Z}\{a^n u[n]\} = \frac{z}{z-a}, \quad \text{ROC: } |z| > |a|$$

$$X(z) = (-z)\frac{d}{dz}\frac{z}{z-a} = \frac{az}{(z-a)^2}, \quad \text{ROC: } |z| > |a|$$

z-Transform of a unit-ramp signal $x[n] = nu[n]$

Setting $a = 1 \Rightarrow$ $X(z) = \left.\frac{az}{(z-a)^2}\right|_{a=1} = \frac{z}{(z-1)^2}, \quad \text{ROC: } |z| > 1$

- Example 14: Using the convolution property

$$x_1[n] = \{\underset{\substack{\uparrow \\ n=0}}{4}, 3, 2, 1\}, \qquad x_2[n] = \{\underset{\substack{\uparrow \\ n=0}}{3}, 7, 4\}$$

Determine $x[n] = x_1[n] * x_2[n]$ using z-transform techniques.

$$X_1(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}, \qquad X_2(z) = 3 + 7z^{-1} + 4z^{-2}$$

$$X(z) = X_1(z)X_2(z) = 12 + 37z^{-1} + 43z^{-2} + 29z^{-3} + 15z^{-4} + 4z^{-5}$$

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{12}, 37, 43, 29, 15, 4\}$$

- Example 15: Finding the output signal of a DTLTI system using inverse z-transform

$$h[n] = (0.9)^n\, u[n], \qquad x[n] = u[n] - u[n-7]$$

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{z}{z-0.9}, \quad \text{ROC: } |z| > 0.9$$

$$X(z) = \sum_{n=0}^{6} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} = \frac{z^7 - 1}{z^6(z - 1)}, \quad \text{ROC: } |z| > 0$$

$$Y(z) = X(z)H(z)$$

$$= H(z) + z^{-1}H(z) + z^{-2}H(z) + z^{-3}H(z) + z^{-4}H(z) + z^{-5}H(z) + z^{-6}H(z)$$

$$y[n] = h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4] + h[n-5] + h[n-6]$$

$$y[n] = (0.9)^n u[n] + (0.9)^{n-1}u[n-1] + (0.9)^{n-2}u[n-2] + (0.9)^{n-3}u[n-3]$$
$$+ (0.9)^{n-4}u[n-4] + (0.9)^{n-5}u[n-5] + (0.9)^{n-6}h[n-6]$$

Initial and final value Theorems

Initial and final value properties of the z-transform applies to causal signals only.

Initial value: $x[0] = \lim_{z \to \infty} X(z)$

Final value: $\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1)X(z)$

- Example 16: Using the initial value property

$$X(z) = \frac{3z^3 + 2z + 5}{2z^3 - 7z^2 + z - 4}$$

Determine the initial value $x[0]$ of the signal.

$$x[0] = \lim_{z \to \infty} \frac{3z^3 + 2z + 5}{2z^3 - 7z^2 + z - 4} = \frac{3}{2}$$

3. Inverse Z-Transform

- Recall that the inverse z-transform x of X is given by:

$$x[n] = \frac{1}{2\pi j}\oint_{\Gamma} X(z)z^{n-1}dz$$

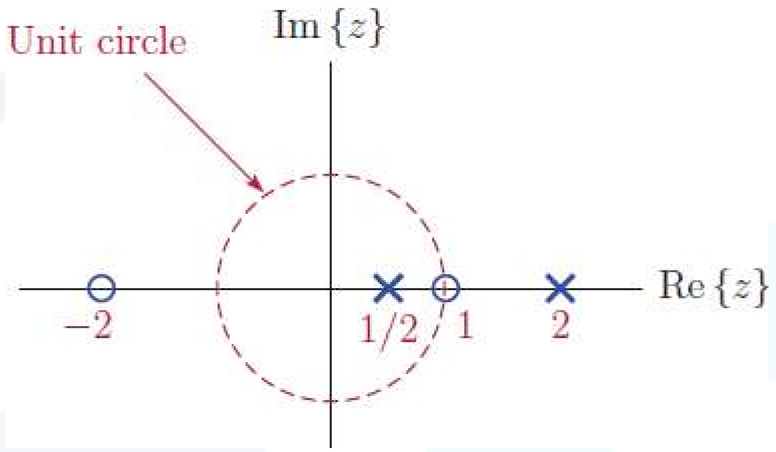
where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X.

- Unfortunately, the above contour integration can often be quite tedious to compute. Consequently, we do not usually compute the inverse z-transform directly using the above equation.

- For rational functions, the inverse z-transform can be more easily computed using partial fraction expansions.

- Example 17: Finding the inverse z-transform using partial fractions

$$X(z) = \frac{(z-1)(z+2)}{(z-1/2)(z-2)}$$



$$\frac{X(z)}{z} = \frac{(z-1)(z+2)}{z(z-1/2)(z-2)} = \frac{-2}{z} + \frac{\frac{5}{3}}{(z-\frac{1}{2})} + \frac{\frac{4}{3}}{(z-2)}$$

$$X(z) = -2 + \frac{\frac{5}{3}z}{(z-\frac{1}{2})} + \frac{\frac{4}{3}z}{(z-2)} = X_1(z) + X_2(z) + X_3(z)$$

$X_1(z)$, is a constant, and its ROC is the entire z-plane. $x_1[n] = \mathcal{Z}^{-1}\{-2\} = -2\delta[n]$

The ROC of $X(z)$ will be determined based on the individual ROCs of the terms $X_2(z)$ and $X_3(z)$. Three possibilities:

Possibility 1: ROC: $|z| < ½$

$X_2(z)$ and $X_3(z)$ must correspond to anti-causal signals. We need:

ROC for $X_2(z)$: $|z| < ½$

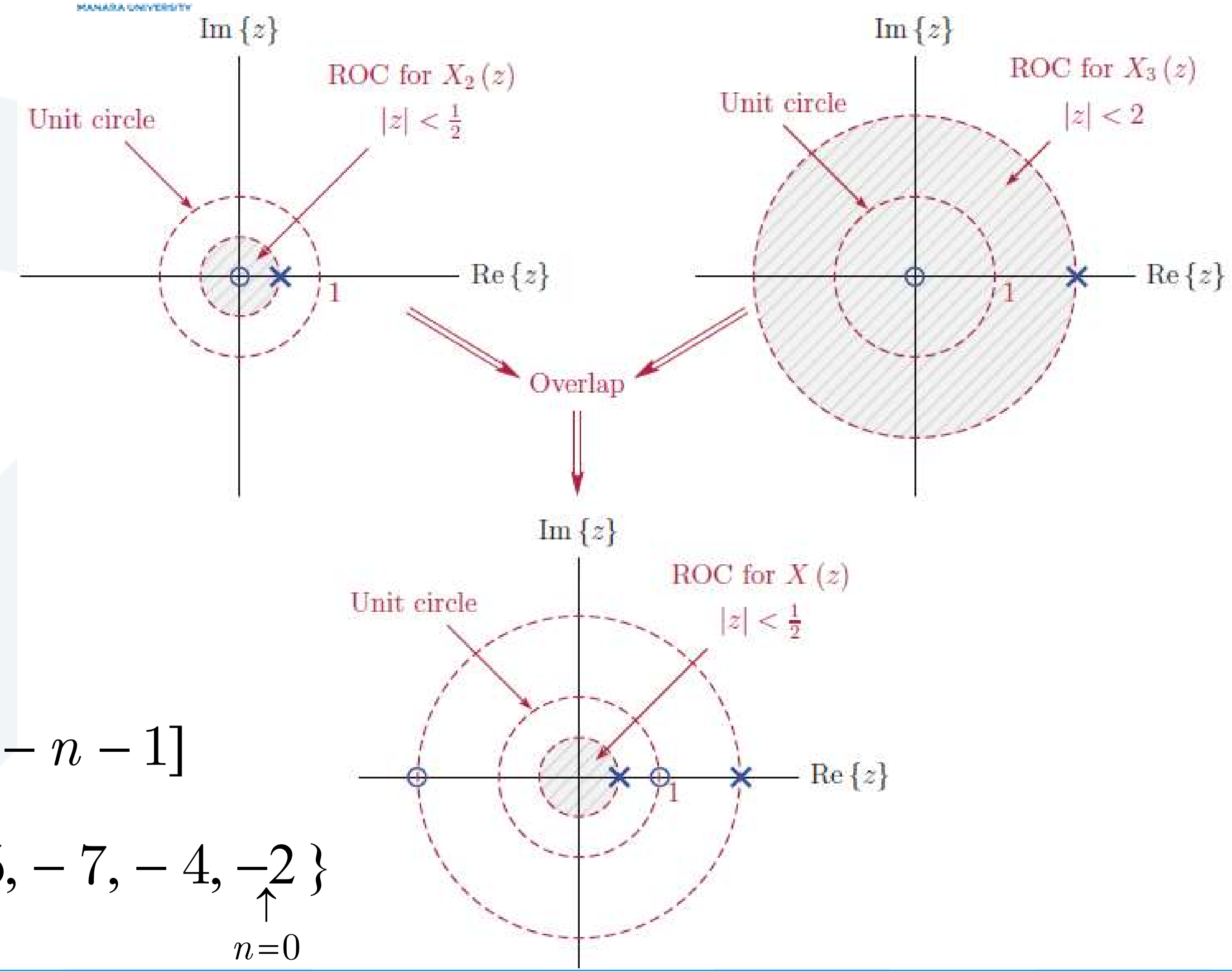
ROC for $X_3(z)$: $|z| < 2$

$$x_2[n] = -\frac{5}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

$$x_3[n] = -\frac{4}{3}\left(2\right)^n u[-n-1]$$

$$x[n] = -2\delta[n] - \left[\frac{5}{3}\left(\frac{1}{2}\right)^n + \frac{4}{3}\left(2\right)^n\right] u[-n-1]$$

$$x[n] = \{\cdots, -53.375, -26.75, -13.5, -7, -4, \underset{\substack{\uparrow \\ n=0}}{-2}\}$$



Possibility 2: ROC: $|z| > 2$

$X_2(z)$ and $X_3(z)$ must correspond to causal signals. We need:

ROC for $X_2(z)$: $|z| > ½$

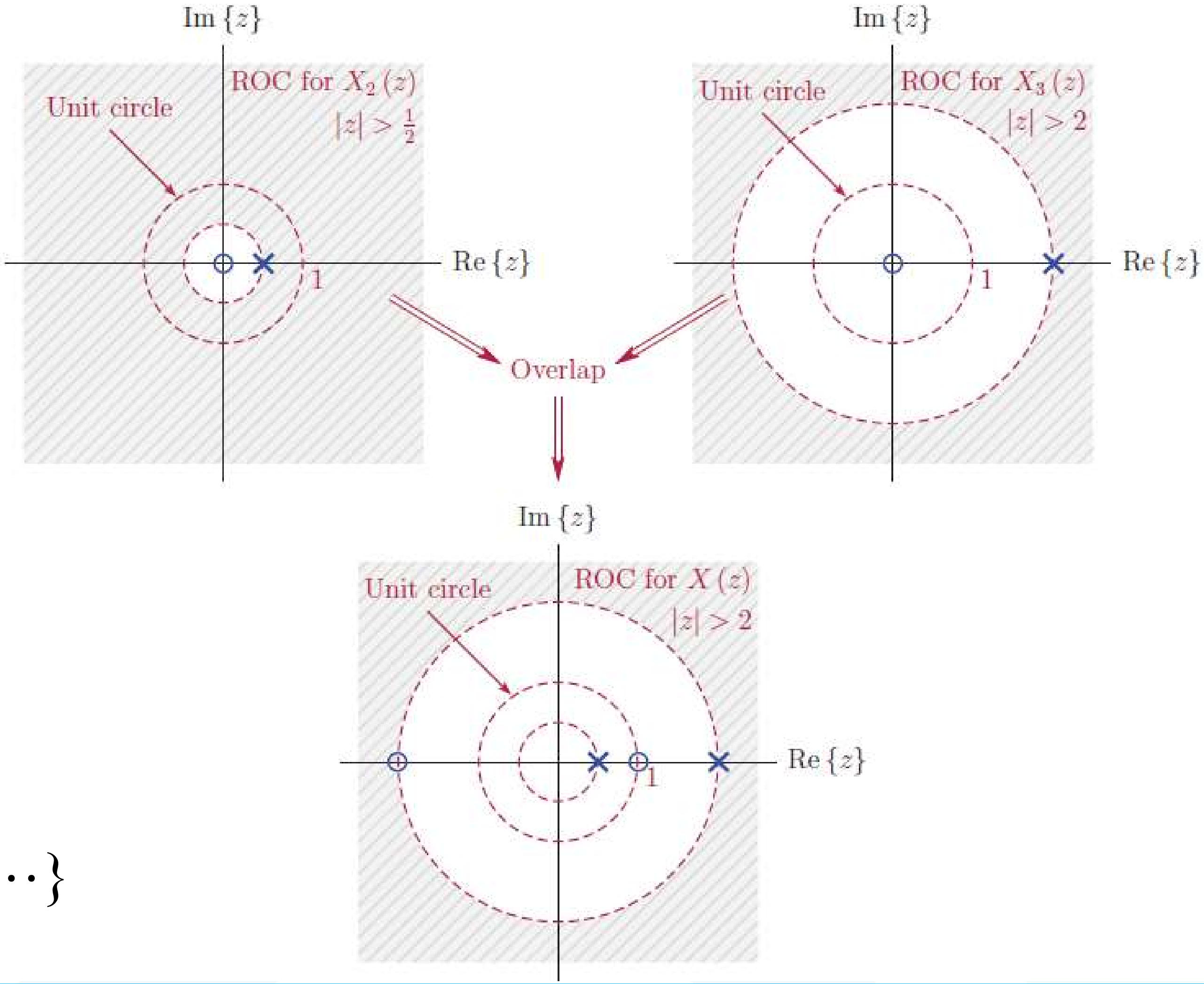
ROC for $X_3(z)$: $|z| > 2$

$$x_2[n] = \frac{5}{3}\left(\frac{1}{2}\right)^n u[n]$$

$$x_3[n] = \frac{4}{3}\left(2\right)^n u[n]$$

$$x[n] = -2\delta[n] + \left[\frac{5}{3}\left(\frac{1}{2}\right)^n + \frac{4}{3}\left(2\right)^n\right]u[n]$$

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 3.5, 5.75, 8.208, 13.385, \cdots\}$$



Possibility 3: ROC: ½ < $|z|$ < 2

$X_2(z)$ and $X_3(z)$ must correspond to noncausal signals. We need:

ROC for $X_2(z)$: $|z| > ½$

ROC for $X_3(z)$: $|z| < 2$

$$x_2[n] = \frac{5}{3}\left(\frac{1}{2}\right)^n u[n]$$

$$x_3[n] = -\frac{4}{3}(2)^n u[-n-1]$$

$$x[n] = -2\delta[n] + \frac{5}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

$$x[n] = \{\cdots, -0.333, -0.667, \underset{\substack{\uparrow \\ n=0}}{-0.333}, 0.833, 0.417, \cdots\}$$