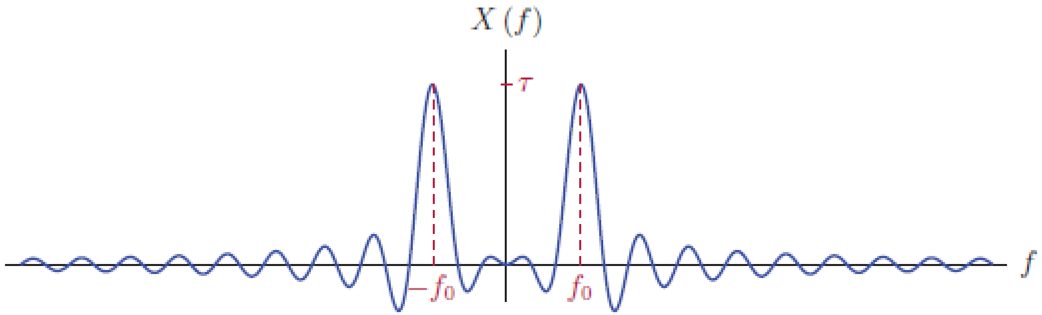
CECC507: Signals and Systems

Lecture Notes 12: Z-Transform for Discrete-Time Signals and Systems: Part B



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Mechatronics

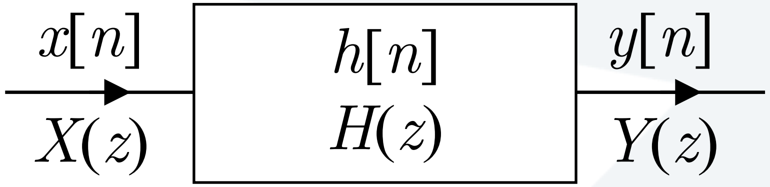
Manara University

Chapter 8

Z-Transform for Discrete-Time Signals and Systems

4. Using the z-Transform with DTLTI Systems

Transfer Function and LTI Systems



Block Diagram Representation

$$y[n] = x[n] * h[n] \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$Y(z) = X(z)H(z)$$

- Since $y[n] = x[n] * h[n]$, the system is characterized in the Laplace domain by $Y(z) = X(z)H(z)$.
- $H(z)$ is the transfer function (or system function) of the system (i.e., the transfer function is the LT of the impulse response).
- A LTI system is completely characterized by its transfer function H.

Relating the transfer function to the difference equation

- Many DTLTI systems of practical interest can be represented using an Nth-order linear difference equation with constant coefficients.
- Consider a system with input x and output y that is characterized by an equation of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

where the a_k and b_k are complex constants and

$$\mathcal{Z}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\} \Rightarrow \sum_{k=0}^{N} \mathcal{Z}\{a_k y[n-k]\} = \sum_{k=0}^{M} \mathcal{Z}\{b_k x[n-k]\}$$

$$\sum_{k=0}^{N} a_k \mathcal{Z}\{y[n-k]\} = \sum_{k=0}^{M} b_k \mathcal{Z}\{x[n-k]\}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The impulse response of the system $h[n] = \mathcal{Z}^{-1}\{H(z)\}$.
- The convolution operation is only applicable to problems involving LTI systems.
- Therefore it follows that the transfer function concept is meaningful only for systems that are both linear and time invariant.
- In determining the transfer function from the difference equation, all initial conditions must be assumed to be zero.

- Example 1: Finding the transfer function from the DE

  A DTLTI system is defined by means of the difference equation:

$$y[n]-0.4y[n-1]+0.89y[n-2]=x[n]-x[n-1]$$

$$Y(z)-0.4z^{-1}Y(z)+0.89z^{-2}Y(z)=X(z)-s^{-1}X(s)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1-z^{-1}}{1-0.4z^{-1}+0.89z^{-2}}=\frac{z(z-1)}{z^{2}-0.4z+0.89}$$

Transfer function and causality

- For a DTLTI system to be causal, its impulse response $h[n]$ needs to be equal to zero for $n<0$.

$$H(z)=\sum_{k=-\infty}^{\infty}h[n]z^{-n}=\sum_{k=0}^{\infty}h[n]z^{-n}$$

- The ROC for the transfer function of a causal system is the outside of a circle in the z-plane. Consequently, the transfer function must also converge at $|z| \to \infty$. Consider a transfer function in the form:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

For the system described by $H(z)$ to be causal we need:

$$\lim_{z \to \infty} H(z) = \lim_{z \to \infty} \frac{b_M}{a_N} z^{M-N} < \infty \Leftrightarrow M - N \le 0 \Rightarrow M \le N$$

- Note: this condition is necessary for a system to be causal, but it is not sufficient. It is also possible for a non-causal system to have a system function with $M \le N$.

Transfer function and stability:

- For a DTLTI system to be stable its impulse response must be absolute integrable.

$$\sum_{k=-\infty}^{\infty} \left|h[n]\right| z^{-n} < \infty$$

- Fourier transform of a signal exists if the signal is absolute integrable.

$$H(\Omega) = H(z)\Big|_{z=e^{j\Omega}}$$

Stability condition:

- For a DTLTI system to be stable, the ROC of its z-domain transfer function must include the unit circle.
- For a causal system to be stable, the transfer function must not have any poles on or outside the unit circle of the z-plane.

- For a anticausal system to be stable, the transfer function must not have any poles on or inside the unit circle of the z-plane.

- For a noncausal system the ROC for the TF, if it exists, is the region between two circles with radii r_1 and r_2, $r_1 < |z| < r_2$. The poles of the TF may be either:

  a. On or inside the circle with radius r_1

  b. On or outside the circle with radius r_2

  and the ROC must include the unit circle.

- Example 19: Impulse response of a stable system

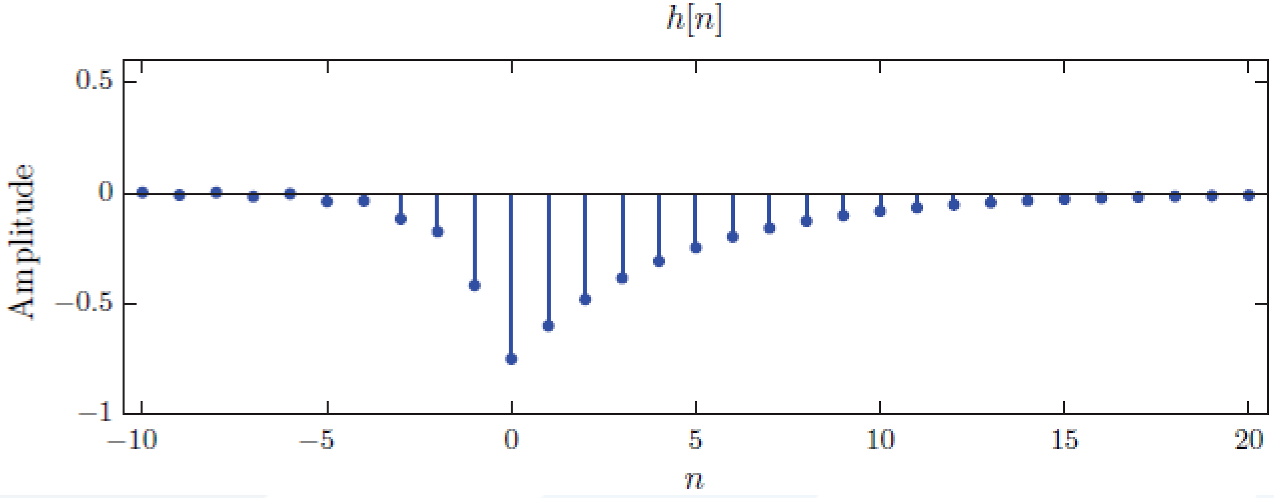
  Determine the impulse response of a stable system characterized by:

$$H(z) = \frac{z(z+1)}{(z-0.8)(z+1.2)(z-2)}$$

The poles of the system are at $p = -1.2,\ 0.8,\ 2$. Since the system is known to be stable, its ROC must include the unit circle. The only possible choice is $0.8 < |z| < 1.2$.

$$H(z) = \frac{z(z+1)}{(z-0.8)(z+1.2)(z-2)} = -\frac{0.75z}{z-0.8} - \frac{0.0312z}{z+1.2} + \frac{0.7813z}{z-2}$$

$$h[n] = -0.75(0.8)^n u[n] + 0.0312(-1.2)^n u[-n-1] - 0.7813(2)^n u[-n-1]$$



5. Simulation Structures for DTLTI Systems

Direct-form implementation

- The general form of the z-domain transfer function for a DTLTI system is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 z^M + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

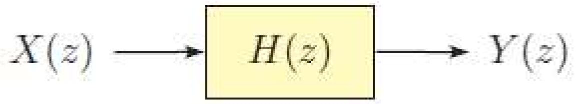
- The method of obtaining a block diagram from an z-domain TF will be derived using a third-order system, but its generalization to higher-order TF is quite straightforward. Consider a DTLTI system described by a TF $H(z)$:

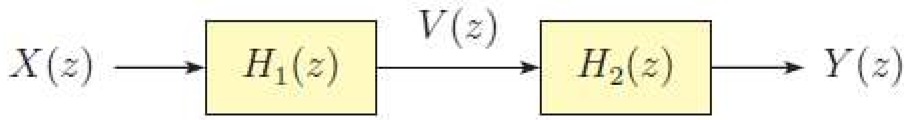
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

Let us use an intermediate function $V(z)$

$$H(z) = H_1(z)H_2(z) = \frac{Y(z)}{V(z)}\frac{V(z)}{X(z)}$$

$$H_1(z) = \frac{V(z)}{X(z)} = b_0 + b_1z^{-1} + b_2z^{-2} + b_3z^{-3}, \quad H_2(z) = \frac{Y(z)}{V(z)} = \frac{1}{1 + a_1z^{-1} + a_2z^{-2} + a_3z^{-3}}$$



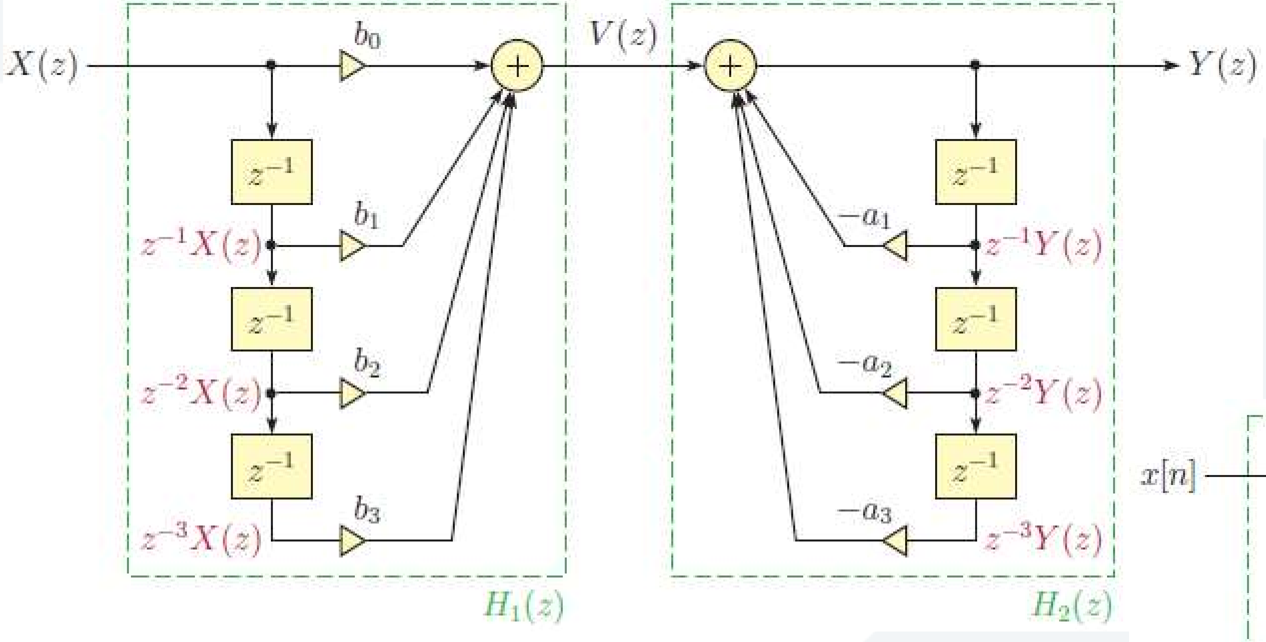


$$V(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) + b_3z^{-3}X(z)$$

$$v[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + b_3x[n-3]$$

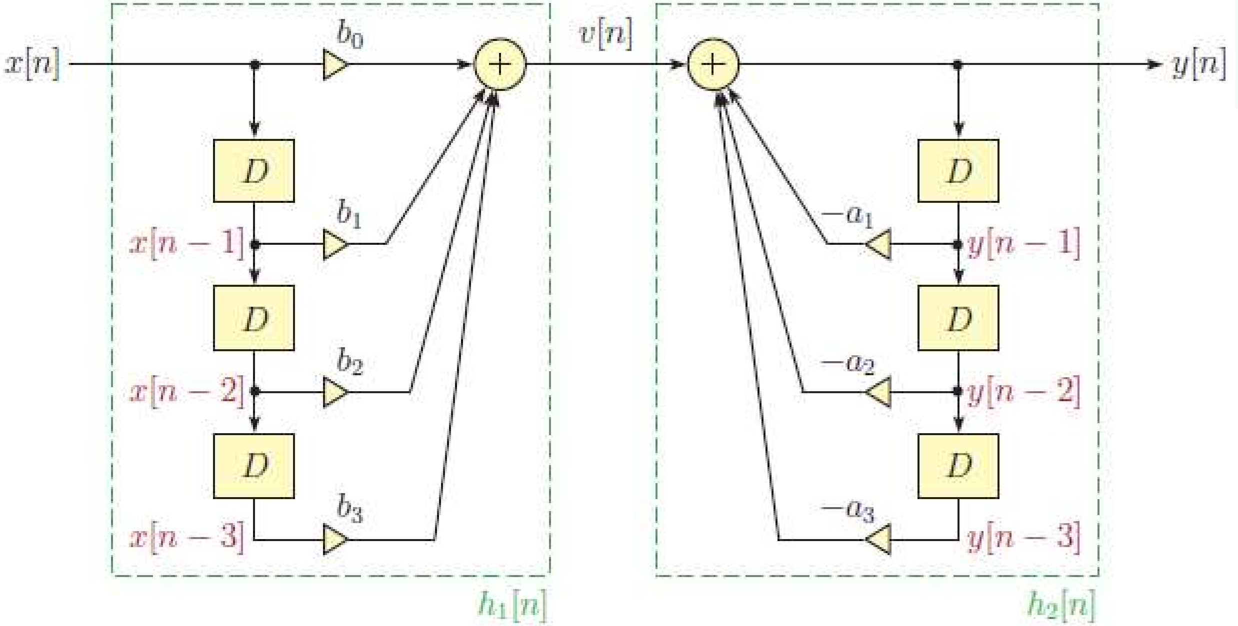
$$Y(z) = V(z) - a_1z^{-1}Y(z) - a_2z^{-2}Y(z) - a_3z^{-3}Y(z)$$

$$y[n] = v[n] - a_1y[n-1] - a_2y[n-2] - a_3y[n-3]$$

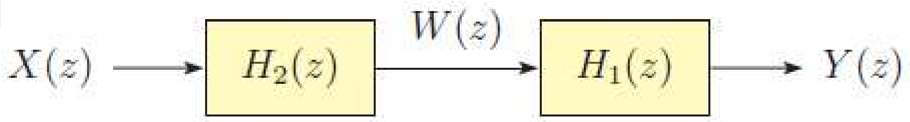


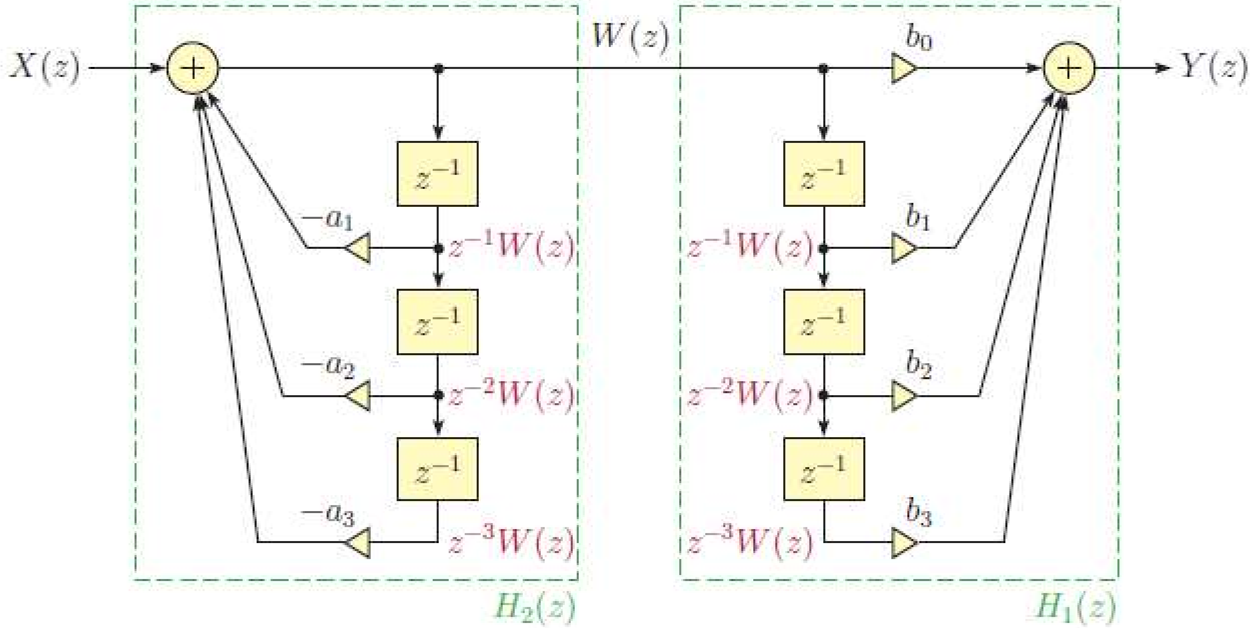
Direct-form I realization of $H(z)$

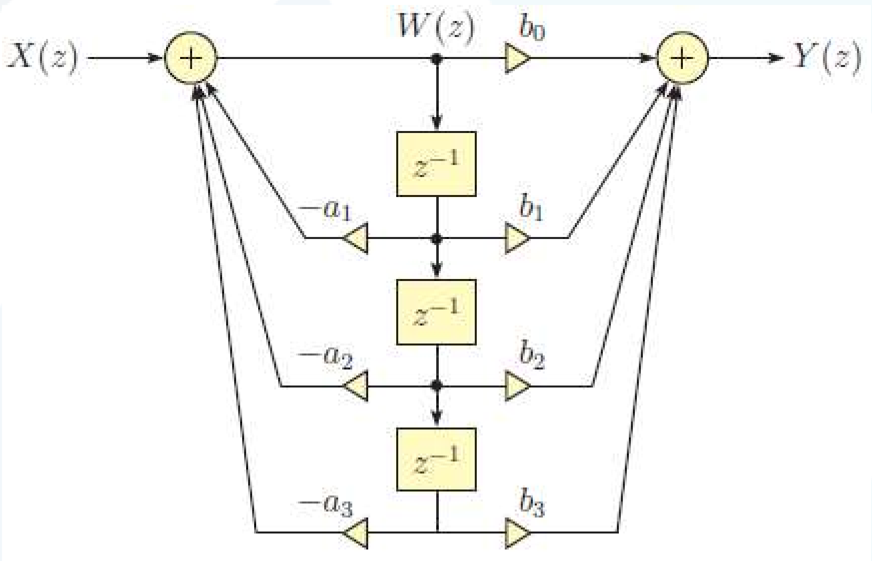
Direct-form I realization of $H(z)$ using time-domain quantities



Since each subsystem, $H_1(z)$ and $H_2(z)$, is linear, it does not matter which one comes first in a cascade connection.





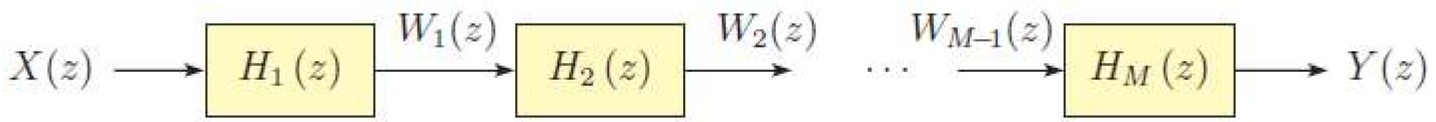


Direct-form II realization of $H(z)$

Cascade and parallel forms

Cascade form

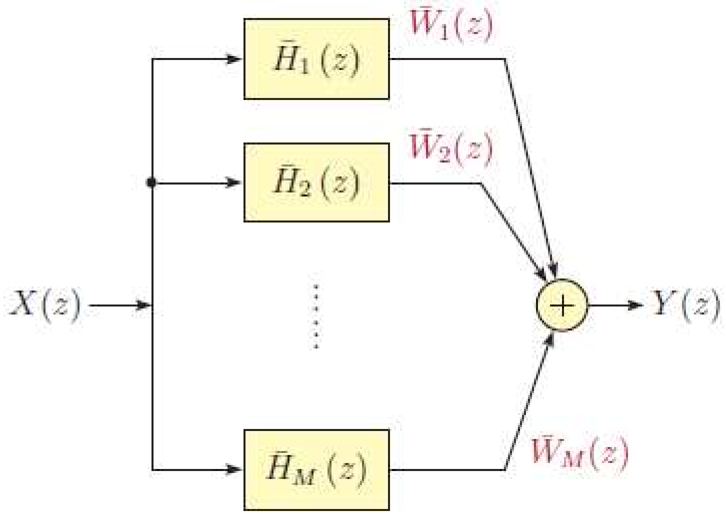
$$H(z) = H_1(z)H_2(z)\cdots H_M(z) = \frac{W_1(z)}{X(z)}\frac{W_2(z)}{W_1(z)}\cdots\frac{Y(z)}{W_{M-1}(z)}$$



Parallel form

$$H(z) = \bar{H}_1(z) + \bar{H}_2(z) + \cdots + \bar{H}_M(z)$$

$$= \frac{\bar{W}_1(z)}{X(z)} + \frac{\bar{W}_2(z)}{X(z)} + \cdots + \frac{\bar{W}_M(z)}{X(z)}$$



6. Unilateral z-Transform

The unilateral z-transform of the signal x is defined as:

$$X_u(z) = Z_u\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n}$$

- If $x[n]$ is a causal signal, then the unilateral transform $X_u(z)$ becomes identical to the bilateral transform $X(z)$.
- The unilateral ZT is related to the bilateral z-transform as follows:

$$Z_u\{x[n]\} = Z\{x[n]u[n]\} = \sum_{n=-\infty}^{\infty} x[n]u[n]z^{-n}$$

- One property of the unilateral z-transform that differs from its counterpart for the bilateral z-transform is the time-shifting property.

$$\mathcal{Z}\{x[n-1]\} = z^{-1}\mathcal{Z}\{x[n]\} = z^{-1}X(z)$$

$$\mathcal{Z}_u\{x[n-1]\} = \sum_{n=0}^{\infty} x[n-1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n} = x[-1] + z^{-1}\sum_{n=0}^{\infty} x[n]z^{-n}$$

$$\mathcal{Z}_u\{x[n-1]\} = x[-1] + z^{-1}X_u(z)$$

$$\mathcal{Z}_u\{x[n-k]\} = \sum_{n=-k}^{-1} x[n]z^{-n-k} + z^{-k}X_u(z), \quad k > 0$$

$$\mathcal{Z}_u\{x[n+k]\} = z^{-k}X_u(z) - \sum_{n=0}^{k-1} x[n]z^{k-n}, \quad k > 0$$

- The unilateral z-transform is useful in the use of z-transform techniques for solving difference equations with specified initial conditions.

- Example 3: Finding the natural response of a system through z-transform

$$y[n]-\tfrac{5}{6}y[n-1]+\tfrac{1}{6}y[n-2]=0$$

Using z-transform techniques, determine the natural response of the system for the initial conditions: $y[-1]=19$, $y[-2]=53$.

$$Z_u\{y[n-1]\}=y[-1]+z^{-1}Y_u(z)=19+z^{-1}Y_u(z)$$

$$Z_u\{y[n-2]\}=y[-1]+y[-2]z^{-1}+z^{-2}Y_u(z)=53+19z^{-1}+z^{-2}Y_u(z)$$

$$Y_u(z)-\tfrac{5}{6}[19+z^{-1}Y_u(z)]+\tfrac{1}{6}[53+19z^{-1}+z^{-2}Y_u(z)]=0$$

$$Y_u(z)=\frac{z(7z-\frac{19}{6})}{z^2-\frac{5}{6}z+\frac{1}{6}}=\frac{z(7z-\frac{19}{6})}{(z-\frac{1}{2})(z-\frac{1}{3})}=\frac{2z}{z-\frac{1}{2}}+\frac{5z}{z-\frac{1}{3}}$$

$$y_h[n]=2(\tfrac{1}{2})^n u[n]+5(\tfrac{1}{3})^n u[n]$$

- Example 4: Finding the forced response of a system through z-transform

Consider a system defined by means of the difference equation:

$$y[n] = 0.9y[n-1] + 0.1x[n]$$

Determine the response of this system for the input signal $x[n] = 20\cos(0.2\pi n)$ if the initial value of the output is $y[-1] = 2.5$.

$$Z\{\cos(\Omega_0 n)u[n]\} = \frac{z[z-\cos(\Omega_0)]}{z^2 - 2\cos(\Omega_0)z + 1} \Rightarrow Z_u\{20\cos(0.2\pi n)\} = \frac{20z[z-\cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$Z_u\{y[n-1]\} = y[-1] + z^{-1}Y_u(z) = 2.5 + z^{-1}Y_u(z)$$

$$Z_u\{y[n]\} = 0.9Z_u\{y[n-1]\} + 0.1Z_u\{x[n]\}$$

$$Y_u(z) = 0.9[2.5 + z^{-1}Y_u(z)] + 0.1\frac{20z[z-\cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$Y_u(z) = 0.9z^{-1}Y_u(z) + 2.25 + \frac{2z[z - \cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$Y_u(z) = \frac{2z^2[z - \cos(0.2\pi)] + 2.25(z - e^{j0.2\pi})(z - e^{-j0.2\pi})}{(z - 0.9)(z - e^{j0.2\pi})(z - e^{-j0.2\pi})}$$

$$Y_u(z) = \frac{2.7129}{z - 0.9} + \frac{0.7685 - j1.4953}{z - e^{j0.2\pi}} + \frac{0.7685 + j1.4953}{z - e^{-j0.2\pi}}$$

The forced response of the system is:

$$y[n] = 2.7129\ (0.9)^n\ u[n] + 1.5371\cos(0.2\pi n)u[n] + 2.9907\sin(0.2\pi n)u[n]$$