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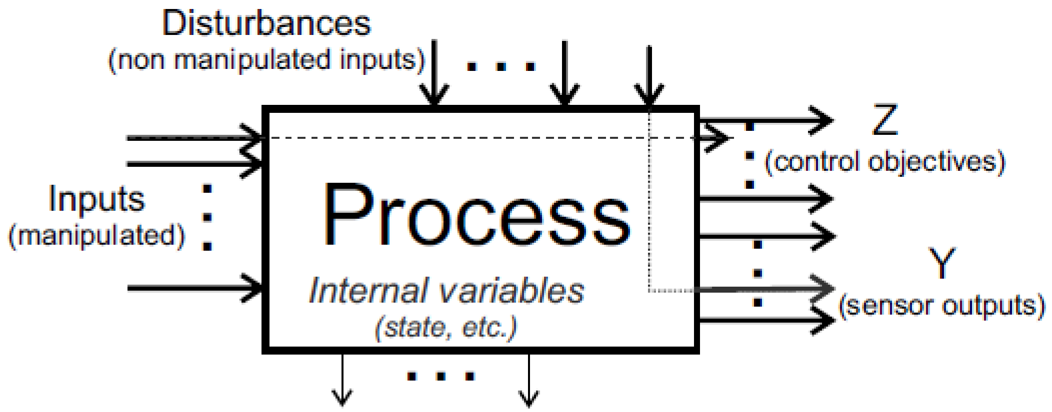
نظم متعددة المتغيرات
Multivariable Systems



الروبوت والأنظمة الذكية

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Controllability

and

Observability

Controllability and Observability

In fact, the conditions of controllability and observability may govern the existence of a complete solution to the control system design problem. The solution to this problem may not exist if the system considered is not controllable. Although most physical systems are controllable and observable, corresponding mathematical models may not possess the property of controllability and observability. Then it is necessary to know the conditions under which a system is controllable and observable. This section deals with controllability and the next section discusses observability.

Controllability and Observability

A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $\boldsymbol{X}(\boldsymbol{t_0})$ to any other state in a finite interval of time.

A system is said to be observable at time t_0 if, with the system in state $\boldsymbol{X}(\boldsymbol{t_0})$, it is possible to determine this state from the observation of the output over a finite time interval.

The concepts of controllability and observability were introduced by Kalman. They play an important role in the design of control systems in state space.

Controllability and Observability

In fact, the conditions of controllability and observability may govern the existence of a complete solution to the control system design problem. The solution to this problem may not exist if the system considered is not controllable. Although most physical systems are controllable and observable, corresponding mathematical models may not possess the property of controllability and observability. Then it is necessary to know the conditions under which a system is controllable and observable.

Then we derive alternative forms of the condition for complete state controllability followed by discussions of complete output controllability. Finally, we present the concept of stabilizability.

Complete State Controllability of Continuous-Time Systems

Consider the continuous-time system.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where
$\mathbf{x}$ = state vector (n-vector)
u = control signal (scalar)
$\mathbf{A}$ = $n \times n$ matrix
$\mathbf{B}$ = $n \times 1$ matrix

The system described by Equation is said to be state controllable at $t = t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $t_0 \le t \le t_1$ If every state is controllable, then the system is said to be completely state controllable.

Complete State Controllability of Continuous-Time Systems

We shall now derive the condition for complete state controllability. Without loss of generality, we can assume that the final state is the origin of the state space and that the initial time is zero, or $t_0 = 0$.

The solution of Equation is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau$$

Applying the definition of complete state controllability just given, we have

$$\mathbf{x}(t_1) = \mathbf{0} = e^{\mathbf{A}t_1}\mathbf{x}(0) + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}u(\tau)\,d\tau$$

$$\mathbf{x}(0) = -\int_0^{t_1} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)\,d\tau$$

Complete State Controllability of Continuous-Time Systems

Referring to Equation

$$e^{\mathbf{A}t} = \alpha_0(t)\mathbf{I} + \alpha_1(t)\mathbf{A} + \alpha_2(t)\mathbf{A}^2 + \cdots + \alpha_{m-1}(t)\mathbf{A}^{m-1}$$

$\boldsymbol{e^{-At}}$ can be written

$$e^{-\mathbf{A}\tau} = \sum_{k=0}^{n-1} \alpha_k(\tau)\mathbf{A}^k$$

Substituting gives

$$\mathbf{x}(0) = -\sum_{k=0}^{n-1} \mathbf{A}^k\mathbf{B} \int_0^{t_1} \alpha_k(\tau)u(\tau)\,d\tau$$

Let us put

$$\int_0^{t_1} \alpha_k(\tau)u(\tau)\,d\tau = \beta_k$$

Complete State Controllability of Continuous-Time Systems

Then Equation becomes

$$\mathbf{x}(0) = -\sum_{k=0}^{n-1} \mathbf{A}^k \mathbf{B} \beta_k$$

$$= -\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right] \begin{bmatrix} \beta_0 \\ \hline \beta_1 \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \beta_{n-1} \end{bmatrix}$$

If the system is completely state controllable, then, given any initial state $\boldsymbol{X}(\mathbf{0})$, Equation must be satisfied. This requires that the rank of the **nxn** matrix

$$\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right]$$

be ***n***.

Complete State Controllability of Continuous-Time Systems

From this analysis, we can state the condition for complete state controllability as follows: The system given by Equation ($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$) is completely state controllable if and only if the vectors $\boldsymbol{B}, \boldsymbol{AB}, \dots, \boldsymbol{A}^{n-1}\boldsymbol{B}$ are linearly independent, or the **nxn** matrix

$$\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right]$$

is of rank ***n***.

The result just obtained can be extended to the case where the control vector ***u*** is **r**-dimensional. If the system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where **u** is an **r**-vector, then it can be proved that the condition for complete state controllability is that the **nxnr** matrix

$$\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right]$$

be of rank ***n***, or contain ***n*** linearly independent column vectors.

Complete State Controllability of Continuous-Time Systems

The matrix

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$$

is commonly called the *controllability matrix.*

EXAMPLE

Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Since

$$\left[\mathbf{B} \mid \mathbf{AB}\right] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{singular}$$

the system is not completely state controllable.

EXAMPLE

Consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

For this case,

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \text{nonsingular}$$

The system is therefore completely state controllable.

Alternative Form of the Condition for Complete State Controllability

Consider the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

If the eigenvectors of **A** are distinct, then it is possible to find a transformation matrix **P** such that

Alternative Form of the Condition for Complete State Controllability

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & \lambda_n \end{bmatrix}$$

Note that if the eigenvalues of **A** are distinct, then the eigenvectors of **A** are distinct; however, the converse is not true. For example, an **nxn** real symmetric matrix having multiple eigenvalues has ***n*** distinct eigenvectors. Note also that each column of the **P** matrix is an eigenvector of **A** associated with $\boldsymbol{\lambda}_i (i = 1,2, \dots, n)$.

Alternative Form of the Condition for Complete State Controllability

Let us define $\mathbf{x} = \mathbf{P}\mathbf{z}$ Substituting we obtain

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u}$$

By defining

$$\mathbf{P}^{-1}\mathbf{B} = \mathbf{F} = (f_{ij})$$

we can rewrite Equation as

$$\begin{aligned}
\dot{z}_1 &= \lambda_1 z_1 + f_{11}u_1 + f_{12}u_2 + \cdots + f_{1r}u_r \\
\dot{z}_2 &= \lambda_2 z_2 + f_{21}u_1 + f_{22}u_2 + \cdots + f_{2r}u_r \\
&\cdot \\
&\cdot \\
&\cdot \\
\dot{z}_n &= \lambda_n z_n + f_{n1}u_1 + f_{n2}u_2 + \cdots + f_{nr}u_r
\end{aligned}$$

Alternative Form of the Condition for Complete State Controllability

If the elements of any one row of the **nxr** matrix **F** are all zero, then the corresponding state variable cannot be controlled by any of the $\boldsymbol{u_i}$. Hence, the condition of complete state controllability is that if the eigenvectors of **A** are distinct, then the system is completely state controllable if and only if no row of $\boldsymbol{P^{-1}B}$ has all zero elements. It is important to note that, to apply this condition for complete state controllability, we must put the matrix $\boldsymbol{P^{-1}AP}$ in Equation ($\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{Bu}$) in diagonal form.

Alternative Form of the Condition for Complete State Controllability

If the **A** matrix in Equation ($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$) does not possess distinct eigenvectors, then diagonalization is impossible. In such a case, we may transform **A** into a Jordan canonical form. If, for example, **A** has eigenvalues $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_4, \boldsymbol{\lambda}_4, \boldsymbol{\lambda}_6, \ldots\ldots, \boldsymbol{\lambda}_n$, and has **n-3** distinct eigenvectors, then the Jordan canonical form of **A** is

$$\mathbf{J} = \left[\begin{array}{ccc|cc|cccc} \lambda_1 & 1 & 0 & & & & & & 0 \\ 0 & \lambda_1 & 1 & & & & & & \\ 0 & 0 & \lambda_1 & & & & & & \\ \hline & & & \lambda_4 & 1 & & & & \\ & & & 0 & \lambda_4 & & & & \\ \hline & & & & & \lambda_6 & & & \\ & & & & & & \cdot & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ 0 & & & & & & & & \lambda_n \end{array}\right]$$

Alternative Form of the Condition for Complete State Controllability

The square submatrices on the main diagonal are called *Jordan blocks.* Suppose that we can find a transformation matrix **S** such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$$

If we define a new state vector **z** by

$$\mathbf{x} = \mathbf{S}\mathbf{z}$$

then substitution yields

$$\begin{aligned}\dot{\mathbf{z}} &= \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{z} + \mathbf{S}^{-1}\mathbf{B}\mathbf{u} \\ &= \mathbf{J}\mathbf{z} + \mathbf{S}^{-1}\mathbf{B}\mathbf{u}\end{aligned}$$

The condition for complete state controllability of the system of Equation ($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$) may then be stated as follows: The system is completely state controllable if and only if (1)

Alternative Form of the Condition for Complete State Controllability

(1) no two Jordan blocks in **J** of last Equation are associated with the same eigenvalues,

(2) the elements of any row of $\boldsymbol{S}^{-1}\boldsymbol{B}$ that correspond to the last row of each Jordan block are not all zero, and

(3) the elements of each row of $\boldsymbol{S}^{-1}\boldsymbol{B}$ that correspond to distinct eigenvalues are not all zero.

EXAMPLE

The following systems are completely state controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc:cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hdashline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 3 & 0 \\ 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

EXAMPLE

The following systems are not completely state controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} -2 & 1 & 0 & & 0 \\ 0 & -2 & 1 & & \\ 0 & 0 & -2 & & \\ \hline & & & -5 & 1 \\ 0 & & & 0 & -5 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} u$$

Condition for Complete State Controllability in the *s* Plane

The condition for complete state controllability can be stated in terms of transfer functions or transfer matrices. It can be proved that a necessary and sufficient condition for complete state controllability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs, the system cannot be controlled in the direction of the canceled mode.

EXAMPLE

Consider the following transfer function:

$$\frac{X(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

Clearly, cancellation of the factor (**s+2.5**) occurs in the numerator and denominator of this transfer function. (Thus one degree of freedom is lost.) Because of this cancellation, this system is not completely state controllable.

EXAMPLE

The same conclusion can be obtained by writing this transfer function in the form of a state equation. **A** state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Since

$$[\mathbf{B} \;\vdots\; \mathbf{AB}] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

the rank of the matrix $[\mathbf{B} \;\vdots\; \mathbf{AB}]$ is 1.

Therefore, we arrive at the same conclusion: The system is not completely state controllable.

Output Controllability

In the practical design of a control system, we may want to control the output rather than the state of the system. Complete state controllability is neither necessary nor sufficient for controlling the output of the system. For this reason, it is desirable to define separately complete output controllability. Consider the system described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{y}$ = output vector (m-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

$\mathbf{C}$ = $m \times n$ matrix

$\mathbf{D}$ = $m \times r$ matrix

Output Controllability

The system described by last Equations is said to be completely output controllable if it is possible to construct an unconstrained control vector **u**(t) that will transfer any given initial output $\boldsymbol{y}(\boldsymbol{t_0})$ to any final output $\boldsymbol{y}(\boldsymbol{t_1})$ in a finite time interval $\boldsymbol{t_0} \leq \boldsymbol{t} \leq \boldsymbol{t_1}$.

It can be proved that the condition for complete output controllability is as follows:

The system described by last Equations is completely output controllable if and only if the **mx(n+1)r** matrix

$$\left[\mathbf{CB} \;\vdots\; \mathbf{CAB} \;\vdots\; \mathbf{CA}^2\mathbf{B} \;\vdots\; \cdots \;\vdots\; \mathbf{CA}^{n-1}\mathbf{B} \;\vdots\; \mathbf{D} \right]$$

is of rank $\boldsymbol{m}$. Note that the presence of the **Du** term always helps to establish output controllability.

Uncontrollable System

An uncontrollable system has a subsystem that is physically disconnected from the input.

Stabilizability

For a partially controllable system, if the uncontrollable modes are stable and the unstable modes are controllable, the system is said to be stabilizable. For example, the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

is not state controllable. The stable mode that corresponds to the eigenvalue of –1 is not controllable. The unstable mode that corresponds to the eigenvalue of 1 is controllable. Such a system can be made stable by the use of a suitable feedback. Thus this system is stabilizable.

OBSERVABILITY

In this section we discuss the observability of linear systems. Consider the unforced system described by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{y}$ = output vector (m-vector)

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{C}$ = $m \times n$ matrix

The system is said to be completely observable if every state $\boldsymbol{X}(\boldsymbol{t_0})$ can be determined from the observation of $\boldsymbol{y}(\boldsymbol{t})$ over a finite time interval, $\boldsymbol{t_0} \leq \boldsymbol{t} \leq \boldsymbol{t_1}$

OBSERVABILITY

The system is, therefore, completely observable if every transition of the state eventually affects every element of the output vector. The concept of observability is useful in solving the problem of reconstructing unmeasurable state variables from measurable variables in the minimum possible length of time. In this section we treat only linear, time-invariant systems. Therefore, without loss of generality, we can assume that $\boldsymbol{t_0 = 0}$.

The concept of observability is very important because, in practice, the difficulty encountered with state feedback control is that some of the state variables are not accessible for direct measurement, with the result that it becomes necessary to estimate the unmeasurable state variables in order to construct the control signals.

OBSERVABILITY

That such estimates of state variables are possible if and only if the system is completely observable.

In discussing observability conditions, we consider the unforced system. The reason for this is as follows: If the system is described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

then

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

and $\mathbf{y}(t)$ is

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}$$

OBSERVABILITY

Since the matrices **A**, **B**, **C**, and **D** are known and **u**(*t*) is also known, the last two terms on the right-hand side of this last equation are known quantities. Therefore, they may be subtracted from the observed value of **y**(*t*). Hence, for investigating a necessary and sufficient condition for complete observability, it suffices to consider the system described by Equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Complete Observability of Continuous-Time Systems

Consider the system described by Equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

The output vector **y**(t) is $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{A}^k$$

where ***n*** is the degree of the characteristic polynomial.

Hence, we obtain

Complete Observability of Continuous-Time Systems

$$\mathbf{y}(t) = \sum_{k=0}^{n-1} \alpha_k(t)\mathbf{C}\mathbf{A}^k\mathbf{x}(0)$$

or

$$\mathbf{y}(t) = \alpha_0(t)\mathbf{C}\mathbf{x}(0) + \alpha_1(t)\mathbf{C}\mathbf{A}\mathbf{x}(0) + \cdots + \alpha_{n-1}(t)\mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(0)$$

If the system is completely observable, then, given the output **y**(t) over a time interval $\mathbf{0} \leq \boldsymbol{t} \leq \boldsymbol{t_1}$, **x**(0) is uniquely determined from last Equation. It can be shown that this requires the rank of the **nmxn** matrix

$$\begin{bmatrix} \mathbf{C} \\ \hline \mathbf{CA} \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \mathbf{CA}^{n-1} \end{bmatrix}$$

to be ***n***.

Complete Observability of Continuous-Time Systems

From this analysis, we can state the condition for complete observability as follows:

The system described by Equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

is completely observable if and only if the **nxnm** matrix

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

is of rank ***n*** or has ***n*** linearly independent column vectors. This matrix is called the *observability matrix.*

EXAMPLE

Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is this system controllable(state &output) and observable?

EXAMPLE

Since the rank of the matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

is **2**, the system is completely state controllable.

For output controllability, let us find the rank of the matrix $[\mathbf{CB} \mid \mathbf{CAB}]$. Since

$$[\mathbf{CB} \mid \mathbf{CAB}] = [0 \quad 1]$$

the rank of this matrix is **1**.

Hence, the system is completely output controllable.

EXAMPLE

To test the observability condition, examine the rank of

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*]$$

Since

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

the ra[illegible]ence, the system is completely observable. $[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*]$

Conditions for Complete Observability in the s Plane

The conditions for complete observability can also be stated in terms of transfer functions or transfer matrices.

The necessary and sufficient conditions for complete observability is that no cancellation occur in the transfer function or transfer matrix. If cancellation occurs, the canceled mode cannot be observed in the output.

EXAMPLE

Show that the following system is not completely observable:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

Where

$$y = \mathbf{C}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [4 \quad 5 \quad 1]$$

Note that the control function ***u*** does not affect the complete observability of the system. To examine complete observability, we may simply set ***u=0***. For this system, we have

EXAMPLE

For this system, we have

$$\left[\mathbf{C}^* \,\vdots\, \mathbf{A}^*\mathbf{C}^* \,\vdots\, (\mathbf{A}^*)^2\mathbf{C}^*\right] = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

Note that

$$\begin{vmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

Hence, the rank of the matrix $\left[\mathbf{C}^* \,\vdots\, \mathbf{A}^*\mathbf{C}^* \,\vdots\, (\mathbf{A}^*)^2\mathbf{C}^*\right]$ is less than **3**. Therefore, the system is not completely observable.

EXAMPLE

In fact, in this system, cancellation occurs in the transfer function of the system. The transfer function between $\boldsymbol{X_1(s)}$ and $\boldsymbol{U(s)}$ is

$$\frac{X_1(s)}{U(s)} = \frac{1}{(s+1)(s+2)(s+3)}$$

and the transfer function between $\boldsymbol{Y(s)}$ and $\boldsymbol{X_1(s)}$ is

$$\frac{Y(s)}{X_1(s)} = (s+1)(s+4)$$

Therefore, the transfer function between the output $\boldsymbol{Y(s)}$ and the input $\boldsymbol{U(s)}$ is

$$\frac{Y(s)}{U(s)} = \frac{(s+1)(s+4)}{(s+1)(s+2)(s+3)}$$

EXAMPLE

Clearly, the two factors ***(s+1)*** cancel each other. This means that there are nonzero initial states $\boldsymbol{X}(\mathbf{0})$, which cannot be determined from the measurement of $\boldsymbol{y}(\boldsymbol{t})$.

Comments

The transfer function has no cancellation if and only if the system is completely state controllable and completely observable. This means that the canceled transfer function does not carry along all the information characterizing the dynamic system.

Alternative Form of the Condition for Complete Observability

Consider the system described by Equations

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Suppose that the transformation matrix **P** transforms **A** into a diagonal matrix, or $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$

where **D** is a diagonal matrix. Let us define $\mathbf{x} = \mathbf{P}\mathbf{z}$

Then, can be written

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{z} = \mathbf{D}\mathbf{z}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\mathbf{z}$$

Hence,

$$\mathbf{y}(t) = \mathbf{C}\mathbf{P}e^{\mathbf{D}t}\mathbf{z}(0)$$

Alternative Form of the Condition for Complete Observability

$$\mathbf{y}(t) = \mathbf{CP}\begin{bmatrix} e^{\lambda_1 t} & & & & & 0 \\ & e^{\lambda_2 t} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & e^{\lambda_n t} \end{bmatrix}\mathbf{z}(0) = \mathbf{CP}\begin{bmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ \cdot \\ \cdot \\ \cdot \\ e^{\lambda_n t} z_n(0) \end{bmatrix}$$

The system is completely observable if none of the columns of the **mxn** matrix **CP** consists of all zero elements. This is because, if the ith column of **CP** consists of all zero elements, then the state variable $z_i(\mathbf{0})$ will not appear in the output equation and therefore cannot be determined from observation of $\boldsymbol{y}(\boldsymbol{t})$.

Alternative Form of the Condition for Complete Observability

Thus, $\boldsymbol{X}(\mathbf{0})$, which is related to $\boldsymbol{Z}(\mathbf{0})$ by the nonsingular matrix **P**, cannot be determined.

If the matrix **A** cannot be transformed into a diagonal matrix, then by use of a suitable transformation matrix **S**, we can transform **A** into a Jordan canonical form, or

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$$

where **J** is in the Jordan canonical form.

Let us define $\mathbf{x} = \mathbf{S}\mathbf{z}$

Then, can be written

$$\dot{\mathbf{z}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{z} = \mathbf{J}\mathbf{z}$$
$$\mathbf{y} = \mathbf{C}\mathbf{S}\mathbf{z}$$

Alternative Form of the Condition for Complete Observability

Hence,

$$\mathbf{y}(t) = \mathbf{CS}e^{\mathbf{J}t}\mathbf{z}(0)$$

The system is completely observable if

(1) no two Jordan blocks in **J** are associated with the same eigenvalues,

(2) no columns of **CS** that correspond to the first row of each Jordan block consist of zero elements, and

(3) no columns of **CS** that correspond to distinct eigenvalues consist of zero elements.

To clarify condition (2), in next Example we have encircled by dashed lines the columns of **CS** that correspond to the first row of each Jordan block.

EXAMPLE

Are the following systems completely observable?

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & & 0 \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1 \\ 0 & & & 0 & -3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

EXAMPLE

The following systems are completely observable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad y = \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{ccc|cc} 2 & 1 & 0 & & 0 \\ 0 & 2 & 1 & & \\ 0 & 0 & 2 & & \\ \hline & & & -3 & 1 \\ 0 & & & 0 & -3 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Principle of Duality

We shall now discuss the relationship between controllability and observability. We shall introduce the principle of duality, due to Kalman, to clarify apparent analogies between controllability and observability.

Consider the system $\boldsymbol{S_1}$ described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where
$\mathbf{x}$ = state vector (n-vector)
$\mathbf{u}$ = control vector (r-vector)
$\mathbf{y}$ = output vector (m-vector)
$\mathbf{A} = n \times n$ matrix
$\mathbf{B} = n \times r$ matrix
$\mathbf{C} = m \times n$ matrix

Principle of Duality

and the dual system $\boldsymbol{S_2}$ defined by

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*\mathbf{v}$$

$$\mathbf{n} = \mathbf{B}^*\mathbf{z}$$

where $\mathbf{z}$ = state vector (n-vector)

$\mathbf{v}$ = control vector (m-vector)

$\mathbf{n}$ = output vector (r-vector)

$\mathbf{A}^*$ = conjugate transpose of $\mathbf{A}$

$\mathbf{B}^*$ = conjugate transpose of $\mathbf{B}$

$\mathbf{C}^*$ = conjugate transpose of $\mathbf{C}$

The principle of duality states that the system $\boldsymbol{S_1}$ is completely state controllable (observable) if and only if system $\boldsymbol{S_2}$ is completely observable (state controllable).

Principle of Duality

To verify this principle, let us write down the necessary and sufficient conditions for complete state controllability and complete observability for systems $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$.

For system $\boldsymbol{S_1}$:

1. A necessary and sufficient condition for complete state controllability is that the rank of the **nxnr** matrix

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$$

be ***n***.

2. A necessary and sufficient condition for complete observability is that the rank of the **nxnm** matrix

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

be ***n***.

Principle of Duality

For system S_2 :

1. A necessary and sufficient condition for complete state controllability is that the rank of the **nxnm** matrix

$$\left[\mathbf{C}^* \;\vdots\; \mathbf{A}^*\mathbf{C}^* \;\vdots\; \cdots \;\vdots\; (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

be n.

2. A necessary and sufficient condition for complete observability is that the rank of the **nxnr** matrix

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$$

be n.

Principle of Duality

By comparing these conditions, the truth of this principle is apparent. By use of this principle, the observability of a given system can be checked by testing the state controllability of its dual.

Detectability

For a partially observable system, if the unobservable modes are stable and the observable modes are unstable, the system is said to be detectable. Note that the concept of detectability is dual to the concept of stabilizability.

ex

Consider a completely state controllable system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

Define the controllability matrix as **M**:

$$\mathbf{M} = \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$$

Show that

$$\mathbf{M}^{-1}\mathbf{AM} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

Where $a_1, a_2, \ldots, a_n$ are the coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

ex

Solution

Let us consider the case where **n=3**.We shall show that

$$\mathbf{AM} = \mathbf{M}\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

The left-hand side is

$$\mathbf{AM} = \mathbf{A}\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right] = \left[\mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B} \;\vdots\; \mathbf{A}^3\mathbf{B}\right]$$

The right-hand side is

$$\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right]\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = \left[\mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B} \;\vdots\; -a_3\mathbf{B} - a_2\mathbf{AB} - a_1\mathbf{A}^2\mathbf{B}\right]$$

ex

The Cayley–Hamilton theorem states that matrix **A** satisfies its own characteristic equation or, in the case of **n=3**,

$$\mathbf{A}^3 + a_1\mathbf{A}^2 + a_2\mathbf{A} + a_3\mathbf{I} = \mathbf{0}$$

Using last Equation, the third column of the right-hand side of above Equation becomes

$$-a_3\mathbf{B} - a_2\mathbf{AB} - a_1\mathbf{A}^2\mathbf{B} = (-a_3\mathbf{I} - a_2\mathbf{A} - a_1\mathbf{A}^2)\mathbf{B} = \mathbf{A}^3\mathbf{B}$$

Thus, becomes

$$[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = [\mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \mathbf{A}^3\mathbf{B}]$$

Hence, the left-hand side and the right-hand side of Equation are the same. We have thus shown that Equation is true. Consequently,

ex

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

The preceding derivation can be easily extended to the general case of any positive integer **n**.

exx

Consider a completely state controllable system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

Define $\mathbf{M} = [\mathbf{B} \mid \mathbf{A}\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}]$

And

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where the a_i 's are coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

exx

Define also $\mathbf{T} = \mathbf{MW}$

Show that

$$\mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \qquad \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

exx

Solution. Let us consider the case where **n=3**.We shall show that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = (\mathbf{M}\mathbf{W})^{-1}\mathbf{A}(\mathbf{M}\mathbf{W}) = \mathbf{W}^{-1}(\mathbf{M}^{-1}\mathbf{A}\mathbf{M})\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

we have

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{M} = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}$$

Hence, Equation can be rewritten as

$$\mathbf{W}^{-1}\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix}\mathbf{W} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

exx

Therefore, we need to show that

$$\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \mathbf{W} = \mathbf{W} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

The left-hand side of Equation is

$$\begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The right-hand side of Equation is

$$\begin{bmatrix} a_2 & a_1 & 1 \\ a_1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} = \begin{bmatrix} -a_3 & 0 & 0 \\ 0 & a_1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

exx

Thus, we have shown that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Next, we shall show that

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note that Equation can be written as

$$\mathbf{B} = \mathbf{T}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{M}\mathbf{W}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Noting that

exx

Noting that

$$\mathbf{T}\begin{bmatrix}0\\0\\1\end{bmatrix} = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]\begin{bmatrix}a_2 & a_1 & 1\\a_1 & 1 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]\begin{bmatrix}1\\0\\0\end{bmatrix} = \mathbf{B}$$

we have

$$\mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

The derivation shown here can be easily extended to the general case of any positive integer ***n***.