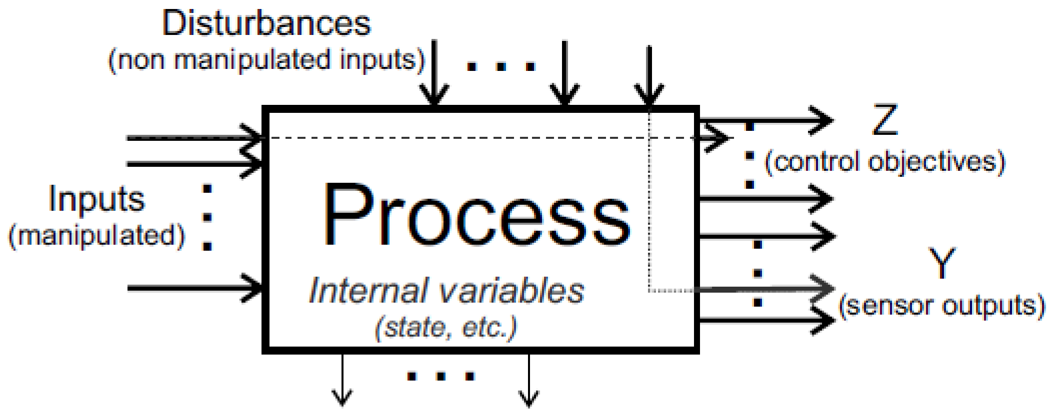
# نظم متعددة المتغيرات
# Multivariable Systems




## الروبوت والأنظمة الذكية

**مدرس المقرر**
د.بلال شيحا

## Principle of Duality

By comparing these conditions, the truth of this principle is apparent. By use of this principle, the observability of a given system can be checked by testing the state controllability of its dual.

## Detectability

For a partially observable system, if the unobservable modes are stable and the observable modes are unstable, the system is said to be detectable. Note that the concept of detectability is dual to the concept of stabilizability.

# Control Systems Design in State Space

# INTRODUCTION

This chapter discusses state-space design methods based on the pole-placement method, observers, the quadratic optimal regulator systems, and introductory aspects of robust control systems. The pole-placement method is somewhat similar to the root-locus method in that we place closed-loop poles at desired locations. The basic difference is that in the root-locus design we place only the dominant closed-loop poles at the desired locations, while in the pole-placement design we place all closed-loop poles at desired locations.

# INTRODUCTION

We begin by presenting the basic materials on pole placement in regulator systems.

We then discuss the design of state observers, followed by the design of regulator systems and control systems using the pole-placement-with-state-observer approach. Then, we discuss the quadratic optimal regulator systems.

# POLE PLACEMENT

In this section we shall present a design method commonly called the *pole-placement* or *pole-assignment technique.* We assume that all state variables are measurable and are available for feedback. It will be shown that if the system considered is completely state controllable, then poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.

The present design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements.

# POLE PLACEMENT

Let us assume that we decide that the desired closed-loop poles are to be at $\boldsymbol{s = \mu_1, s = \mu_2, \ldots, s = \mu_n}$. By choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is completely state controllable.

we limit our discussions to single-input, single-output systems. That is, we assume the control signal ***u*(*t*)** and output signal ***y*(*t*)** to be scalars. In the derivation in this section we assume that the reference input ***r*(*t*)** is zero.

# POLE PLACEMENT

In what follows we shall prove that a necessary and sufficient condition that the closed-loop poles can be placed at any arbitrary locations in the ***s*** plane is that the system be completely state controllable. Then we shall discuss methods for determining the required state feedback gain matrix.

It is noted that when the control signal is a vector quantity, the mathematical aspects of the pole-placement scheme become complicated. (When the control signal is a vector quantity, the state feedback gain matrix is not unique. It is possible to choose freely more than ***n*** parameters; that is, in addition to being able to place ***n*** closed-loop poles properly, we have the freedom to satisfy some or all of the other requirements, if any, of the closed-loop system.)

# Design by Pole Placement

In the conventional approach to the design of a single input, single-output control system, we design a controller (compensator) such that the dominant closed-loop poles have a desired damping ratio $\zeta$ and a desired undamped natural frequency $\omega_n$. In this approach, the order of the system may be raised by 1 or 2 unless pole–zero cancellation takes place. Note that in this approach we assume the effects on the responses of nondominant closed-loop poles to be negligible.

# Design by Pole Placement

Different from specifying only dominant closed-loop poles (the conventional design approach), the present pole-placement approach specifies all closed-loop poles. (There is a cost associated with placing all closed-loop poles, however, because placing all closed loop poles requires successful measurements of all state variables or else requires the inclusion of a state observer in the system.) There is also a requirement on the part of the system for the closed-loop poles to be placed at arbitrarily chosen locations. The requirement is that the system be completely state controllable. We shall prove this fact in this section.

# Design by Pole Placement

Consider a control system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

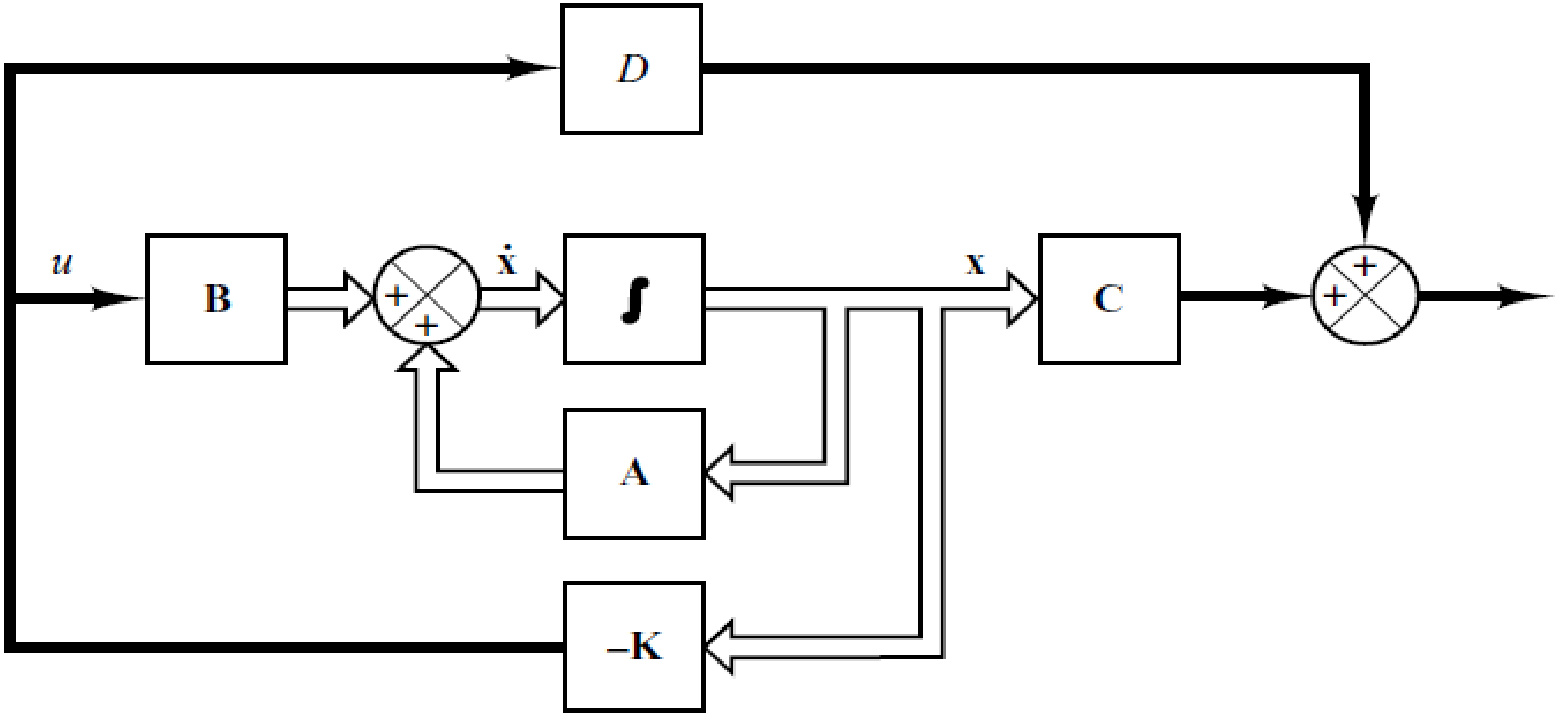
where $\mathbf{x}$ = state vector ($n$-vector)
$y$ = output signal (scalar)
$u$ = control signal (scalar)
$\mathbf{A}$ = $n \times n$ constant matrix
$\mathbf{B}$ = $n \times 1$ constant matrix
$\mathbf{C}$ = $1 \times n$ constant matrix
$D$ = constant (scalar)

We shall choose the control signal to be

$$u = -\mathbf{K}\mathbf{x}$$

Design by Pole Placement

This means that the control signal ***u*** is determined by an instantaneous state. Such a scheme is called state feedback. The 1xn matrix ***K*** is called the state feedback gain matrix. We assume that all state variables are available for feedback. In the following analysis we assume that ***u*** is unconstrained.



Design by Pole Placement

This closed-loop system has no input. Its objective is to maintain the zero output. Because of the disturbances that may be present, the output will deviate from zero. The nonzero output will be returned to the zero reference input because of the state feedback scheme of the system. Such a system where the reference input is always zero is called a regulator system. (Note that if the reference input to the system is always a nonzero constant, the system is also called a regulator system.) Then

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t)$$

The solution of this equation is given by

$$\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{B}\mathbf{K})t}\mathbf{x}(0)$$

Design by Pole Placement

where **x**(0) is the initial state caused by external disturbances. The stability and transient response characteristics are determined by the eigenvalues of matrix **A-BK**. If matrix **K** is chosen properly, the matrix **A**-**BK** can be made an asymptotically stable matrix, and for all **x**(0) ≠ **0**, it is possible to make **x**(***t***) approach **0** as ***t*** approaches infinity. The eigenvalues of matrix **A**-**BK** are called the regulator poles. If these regulator poles are placed in the left-half s plane, then **x**(***t***) approaches **0** as ***t*** approaches infinity. The problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.

# Necessary and Sufficient Condition for Arbitrary Pole Placement

In what follows, we shall prove that arbitrary pole placement for a given system is possible if and only if the system is completely state controllable.

We shall now prove that a necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable. We shall first derive the necessary condition. We begin by proving that if the system is not completely state controllable, then there are eigenvalues of matrix **A-BK** that cannot be controlled by state feedback.

Suppose that the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

is not completely state controllable.

Then the rank of the controllability matrix is less than n, or

$$\text{rank}\left[\mathbf{B} \mid \mathbf{AB} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right] = q < n$$

## Necessary and Sufficient Condition for Arbitrary Pole Placement

This means that there are ***q*** linearly independent column vectors in the controllability matrix. Let us define such ***q*** linearly independent column vectors as $\boldsymbol{f}_1$ , $\boldsymbol{f}_2$ , .... , $\boldsymbol{f}_q$. Also, let us choose ***n-q*** additional ***n***-vectors $\boldsymbol{V}_{q+1}$, $\boldsymbol{V}_{q+2}$, .... , $\boldsymbol{V}_q$ such that

$$\mathbf{P} = \left[\mathbf{f}_1 \;\vdots\; \mathbf{f}_2 \;\vdots\; \cdots \;\vdots\; \mathbf{f}_q \;\vdots\; \mathbf{v}_{q+1} \;\vdots\; \mathbf{v}_{q+2} \;\vdots\; \cdots \;\vdots\; \mathbf{v}_n\right]$$

is of rank ***n***. Then it can be shown that

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \left[\begin{array}{c|c} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right], \qquad \hat{\mathbf{B}} = \mathbf{P}^{-1}\mathbf{B} = \left[\begin{array}{c} \mathbf{B}_{11} \\ \hline \mathbf{0} \end{array}\right]$$

# Necessary and Sufficient Condition for Arbitrary Pole Placement

Now define $\hat{\mathbf{K}} = \mathbf{K}\mathbf{P} = \left[\mathbf{k}_1 \;\vdots\; \mathbf{k}_2\right]$

Then we have

$$
\begin{aligned}
|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| &= \left|\mathbf{P}^{-1}(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{P}\right| \\
&= \left|s\mathbf{I} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P} + \mathbf{P}^{-1}\mathbf{B}\mathbf{K}\mathbf{P}\right| \\
&= \left|s\mathbf{I} - \hat{\mathbf{A}} + \hat{\mathbf{B}}\hat{\mathbf{K}}\right| \\
&= \left|s\mathbf{I} - \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{0} \end{bmatrix}\left[\mathbf{k}_1 \;\vdots\; \mathbf{k}_2\right]\right| \\
&= \begin{vmatrix} s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1 & -\mathbf{A}_{12} + \mathbf{B}_{11}\mathbf{k}_2 \\ \mathbf{0} & s\mathbf{I}_{n-q} - \mathbf{A}_{22} \end{vmatrix} \\
&= \left|s\mathbf{I}_q - \mathbf{A}_{11} + \mathbf{B}_{11}\mathbf{k}_1\right| \cdot \left|s\mathbf{I}_{n-q} - \mathbf{A}_{22}\right| = 0
\end{aligned}
$$

# Necessary and Sufficient Condition for Arbitrary Pole Placement

Where $\mathbf{I}_q$ is a $\boldsymbol{q}$-dimensional identity matrix and $\mathbf{I}_{n-q}$ is an $\boldsymbol{n} - \boldsymbol{q}$ -dimensional identity matrix.

Notice that the eigenvalues of $\mathbf{A}_{22}$ do not depend on $\mathbf{K}$. Thus, if the system is not completely state controllable, then there are eigenvalues of matrix $\mathbf{A}$ that cannot be arbitrarily placed. Therefore, to place the eigenvalues of matrix $\mathbf{A} - \mathbf{BK}$ arbitrarily, the system must be completely state controllable (necessary condition).

Next we shall prove a sufficient condition: that is, if the system is completely state controllable, then all eigenvalues of matrix $\mathbf{A}$ can be arbitrarily placed.

In proving a sufficient condition, it is convenient to transform the state equation into the controllable canonical form.

## Necessary and Sufficient Condition for Arbitrary Pole Placement

Define a transformation matrix $\mathbf{T}$ by $\quad \mathbf{T} = \mathbf{MW}$

where $\mathbf{M}$ is the controllability matrix

$$\mathbf{M} = \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]$$

And

$$\mathbf{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where the $a_i$'s are coefficients of the characteristic polynomial

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

## Necessary and Sufficient Condition for Arbitrary Pole Placement

Define a new state vector $\widehat{\mathbf{X}}$ by $\mathbf{x} = \mathbf{T}\hat{\mathbf{x}}$

If the rank of the controllability matrix $\mathbf{M}$ is ***n*** (meaning that the system is completely state controllable), then the inverse of matrix $\mathbf{T}$ exists, and

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\hat{x} + \mathbf{T}^{-1}\mathbf{B}u$$

where

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \qquad \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix}$$

# Necessary and Sufficient Condition for Arbitrary Pole Placement

Thus, given a state equation, it can be transformed into the controllable canonical form if the system is completely state controllable and if we transform the state vector $\mathbf{X}$ into state vector by use of the transformation matrix $\mathbf{T}$.

Let us choose a set of the desired eigenvalues as $\boldsymbol{\mu_1}$, $\boldsymbol{\mu_2}$, … , $\boldsymbol{\mu_n}$. Then the desired characteristic equation becomes

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

Let us write

$$\mathbf{KT} = \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix}$$

When

$$u = -\mathbf{KT}\hat{\mathbf{x}}$$

## Necessary and Sufficient Condition for Arbitrary Pole Placement

is used to control the system given by $\dot{\hat{x}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\hat{x} + \mathbf{T}^{-1}\mathbf{B}u$

the system equation becomes

$$\dot{\hat{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\hat{\mathbf{x}} - \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}\hat{\mathbf{x}}$$

The characteristic equation is $\left|s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T} + \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}\right| = 0$

This characteristic equation is the same as the characteristic equation for the system, defined by Equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x} + Du$$

When $u = -\mathbf{K}\mathbf{x}$ is used as the control signal.

This can be seen as follows: Since

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

# Necessary and Sufficient Condition for Arbitrary Pole Placement

the characteristic equation for this system is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = |\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{A} + \mathbf{BK})\mathbf{T}| = |s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT} + \mathbf{T}^{-1}\mathbf{BKT}| = 0$$

Now let us simplify the characteristic equation of the system in the controllable canonical form.

we have

$$|s\mathbf{I} - \mathbf{T}^{-1}\mathbf{AT} + \mathbf{T}^{-1}\mathbf{BKT}|$$

$$= \left| s\mathbf{I} - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} [\delta_n \ \delta_{n-1} \ \cdots \ \delta_1] \right|$$

# Necessary and Sufficient Condition for Arbitrary Pole Placement

$$\left|s\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T} + \mathbf{T}^{-1}\mathbf{B}\mathbf{K}\mathbf{T}\right|$$

$$= \left| s\mathbf{I} - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n + \delta_n & a_{n-1} + \delta_{n-1} & \cdots & s + a_1 + \delta_1 \end{vmatrix}$$

$$= s^n + (a_1 + \delta_1)s^{n-1} + \cdots + (a_{n-1} + \delta_{n-1})s + (a_n + \delta_n) = 0$$

# Necessary and Sufficient Condition for Arbitrary Pole Placement

This is the characteristic equation for the system with state feedback. Therefore, it must be equal desired characteristic equation. By equating the coefficients of like powers of ***s***, we get

$$
\begin{aligned}
a_1 + \delta_1 &= \alpha_1 \\
a_2 + \delta_2 &= \alpha_2 \\
&\cdot \\
&\cdot \\
&\cdot \\
a_n + \delta_n &= \alpha_n
\end{aligned}
$$

Solving the preceding equations for the $\delta$'s and substituting them into Equation

$$\mathbf{KT} = \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix}$$

we obtain

$$\mathbf{K} = \begin{bmatrix} \delta_n & \delta_{n-1} & \cdots & \delta_1 \end{bmatrix} \mathbf{T}^{-1}$$

## Necessary and Sufficient Condition for Arbitrary Pole Placement

$$\mathbf{K} = [\delta_n \ \delta_{n-1} \ \cdots \ \delta_1]\mathbf{T}^{-1}$$
$$= [\alpha_n - a_n \ \vdots \ \alpha_{n-1} - a_{n-1} \ \vdots \ \cdots \ \vdots \ \alpha_2 - a_2 \ \vdots \ \alpha_1 - a_1]\mathbf{T}^{-1}$$

Thus, if the system is completely state controllable, all eigenvalues can be arbitrarily placed by choosing matrix **K** according to last Equation. We have thus proved that a necessary and sufficient condition for arbitrary pole placement is that the system be completely state controllable. It is noted that if the system is not completely state controllable, but is stabilizable, then it is possible to make the entire system stable by placing the closed-loop poles at desired locations for ***q*** controllable modes. The remaining ***n-q*** uncontrollable modes are stable. So the entire system can be made stable.

# Determination of Matrix $\mathbf{K}$ Using Transformation Matrix $\mathbf{T}$

Suppose that the system is defined by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

and the control signal is given by $u = -\mathbf{K}\mathbf{x}$

The feedback gain matrix **K** that forces the eigenvalues of **A**-**BK** to be $\mu_1, \mu_2, \dots, \mu_n$ (desired values) can be determined by the following steps :

***Step 1***: Check the controllability condition for the system. If the system is completely state controllable, then use the following steps:

***Step 2***: From the characteristic polynomial for matrix **A**, that is,

$$|s\mathbf{I} - \mathbf{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

determine the values of $a_1, a_2, \dots, a_n$.

## Determination of Matrix $\mathbf{K}$ Using Transformation Matrix $\mathbf{T}$

***Step 3***: Determine the transformation matrix $\mathbf{T}$ that transforms the system state equation into the controllable canonical form. (If the given system equation is already in the controllable canonical form, then $\mathbf{T} = \mathbf{I}$.) It is not necessary to write the state equation in the controllable canonical form. All we need here is to find the matrix $\mathbf{T}$. The transformation matrix $\mathbf{T}$ is given by Equation

where $\mathbf{M}$ an $\mathbf{T} = \mathbf{MW}$ given above.

***Step 4***: Using the desired eigenvalues (desired closed-loop poles), write the desired characteristic polynomial:

$$(s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n$$

and determine the values of $\alpha_1, \alpha_2, \dots, \alpha_n$.

# Determination of Matrix $\mathbf{K}$ Using Transformation Matrix $\mathbf{T}$

***Step 5***: The required state feedback gain matrix **K** can be determined from Equation :

$$\mathbf{K} = \left[\alpha_n - a_n \;\vdots\; \alpha_{n-1} - a_{n-1} \;\vdots\; \cdots \;\vdots\; \alpha_2 - a_2 \;\vdots\; \alpha_1 - a_1\right]\mathbf{T}^{-1}$$

# Determination of Matrix $\mathbf{K}$ Using Direct Substitution Method

If the system is of low order ($\boldsymbol{n \leq 3}$), direct substitution of matrix $\mathbf{K}$ into the desired characteristic polynomial may be simpler. For example, if $\boldsymbol{n = 3}$, then write the state feedback gain matrix $\mathbf{K}$ as

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

Substitute this $\mathbf{K}$ matrix into the desired characteristic polynomial |**s I** - **A** + **BK**| and equate it to $(\boldsymbol{s - \mu_1})$, $(\boldsymbol{s - \mu_2})$, $(\boldsymbol{s - \mu_3})$, or

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = (s - \mu_1)(s - \mu_2)(s - \mu_3)$$

Since both sides of this characteristic equation are polynomials in s, by equating the coefficients of the like powers of s on both sides, it is possible to determine the values of $k_1$, $k_2$ , and $k_3$ . This approach is convenient if **n=2** or **3**. (For **n=4, 5, 6, … ,** this approach may become very tedious.)

## Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

Note that if the system is not completely controllable, matrix $\mathbf{K}$ cannot be determined. (No solution exists.)

There is a well-known formula, known as Ackermann's formula, for the determination of the state feedback gain matrix $\mathbf{K}$. We shall present this formula in what follows.

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where we use the state feedback control ***u=–Kx***. We assume that the system is completely state controllable. We also assume that the desired closed-loop poles are at

$$s = \mu_1, s = \mu_2, \ldots, s = \mu_n.$$

## Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

Use of the state feedback control $u = -\mathbf{K}\mathbf{x}$

modifies the system equation to $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$

Let us define $\widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{B}\mathbf{K}$

The desired characteristic equation is

$$|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = |s\mathbf{I} - \widetilde{\mathbf{A}}| = (s - \mu_1)(s - \mu_2)\cdots(s - \mu_n)$$
$$= s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

Since the Cayley–Hamilton theorem states that $\widetilde{\mathbf{A}}$ satisfies its own characteristic equation, we have

$$\phi(\widetilde{\mathbf{A}}) = \widetilde{\mathbf{A}}^n + \alpha_1 \widetilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1}\widetilde{\mathbf{A}} + \alpha_n \mathbf{I} = \mathbf{0}$$

We shall utilize Equation to derive Ackermann's formula.

# Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

To simplify the derivation, we consider the case where ***n=3***. (For any other positive integer ***n***, the following derivation can be easily extended.)

Consider the following identities:

$$\mathbf{I} = \mathbf{I}$$

$$\widetilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$$

$$\widetilde{\mathbf{A}}^2 = (\mathbf{A} - \mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\widetilde{\mathbf{A}}$$

$$\widetilde{\mathbf{A}}^3 = (\mathbf{A} - \mathbf{BK})^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2$$

Multiplying the preceding equations in order by $\alpha_3$, $\alpha_2$ , $\alpha_1$ , and $\alpha_0$ (where $\alpha_0$=1), respectively, and adding the results, we obtain

## Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

$$\alpha_3 \mathbf{I} + \alpha_2 \widetilde{\mathbf{A}} + \alpha_1 \widetilde{\mathbf{A}}^2 + \widetilde{\mathbf{A}}^3$$
$$= \alpha_3 \mathbf{I} + \alpha_2(\mathbf{A} - \mathbf{BK}) + \alpha_1(\mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\widetilde{\mathbf{A}}) + \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK}$$
$$- \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2$$
$$= \alpha_3 \mathbf{I} + \alpha_2 \mathbf{A} + \alpha_1 \mathbf{A}^2 + \mathbf{A}^3 - \alpha_2 \mathbf{BK} - \alpha_1 \mathbf{ABK} - \alpha_1 \mathbf{BK}\widetilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK}$$
$$- \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2$$

Referring to Equation

$$\phi(\widetilde{\mathbf{A}}) = \widetilde{\mathbf{A}}^n + \alpha_1 \widetilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1} \widetilde{\mathbf{A}} + \alpha_n \mathbf{I} = \mathbf{0}$$

we have

$$\alpha_3 \mathbf{I} + \alpha_2 \widetilde{\mathbf{A}} + \alpha_1 \widetilde{\mathbf{A}}^2 + \widetilde{\mathbf{A}}^3 = \phi(\widetilde{\mathbf{A}}) = \mathbf{0}$$

Also, we have

$$\alpha_3 \mathbf{I} + \alpha_2 \mathbf{A} + \alpha_1 \mathbf{A}^2 + \mathbf{A}^3 = \phi(\mathbf{A}) \neq \mathbf{0}$$

# Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

Substituting the last two equations, we have

$$\phi(\widetilde{\mathbf{A}}) = \phi(\mathbf{A}) - \alpha_2 \mathbf{BK} - \alpha_1 \mathbf{BK}\widetilde{\mathbf{A}} - \mathbf{BK}\widetilde{\mathbf{A}}^2 - \alpha_1 \mathbf{ABK} - \mathbf{ABK}\widetilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK}$$

Since $\phi(\widetilde{\mathbf{A}}) = \mathbf{0}$, we obtain

$$\phi(\mathbf{A}) = \mathbf{B}(\alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2) + \mathbf{AB}(\alpha_1 \mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}}) + \mathbf{A}^2\mathbf{BK}$$

$$= [\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}] \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

Since the system is completely state controllable, the inverse of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]$$

# Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

exists. Premultiplying both sides of Equation ( $\phi(\mathbf{A})$ ) by the inverse of the controllability matrix, we obtain

$$[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A}) = \begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

Premultiplying both sides of this last equation by $[0 \quad 0 \quad 1]$, we obtain

$$[0 \quad 0 \quad 1][\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A}) = [0 \quad 0 \quad 1]\begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\widetilde{\mathbf{A}} + \mathbf{K}\widetilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\widetilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} = \mathbf{K}$$

which can be rewritten as

$$\mathbf{K} = [0 \quad 0 \quad 1][\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$$

This last equation gives the required state feedback gain matrix $\mathbf{K}$.

# Determination of Matrix $\mathbf{K}$ Using Ackermann's Formula

For an arbitrary positive integer ***n***, we have

$$\mathbf{K} = [0 \quad 0 \quad \cdots \quad 0 \quad 1]\left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \cdots \;\vdots\; \mathbf{A}^{n-1}\mathbf{B}\right]^{-1}\phi(\mathbf{A})$$

Equation ($\mathbf{K}$) is known as Ackermann's formula for the determination of the state feedback gain matrix $\mathbf{K}$.

## Choosing the Locations of Desired Closed-Loop Poles.

The first step in the pole-placement design approach is to choose the locations of the desired closed-loop poles. The most frequently used approach is to choose such poles based on experience in the root-locus design, placing a dominant pair of closed-loop poles and choosing other poles so that they are far to the left of the dominant closed-loop poles.

Note that if we place the dominant closed-loop poles far from the $j\omega$ axis, so that the system response becomes very fast, the signals in the system become very large, with the result that the system may become nonlinear. This should be avoided.

# EXAMPLE

Consider the regulator system shown in Figure . The plant is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

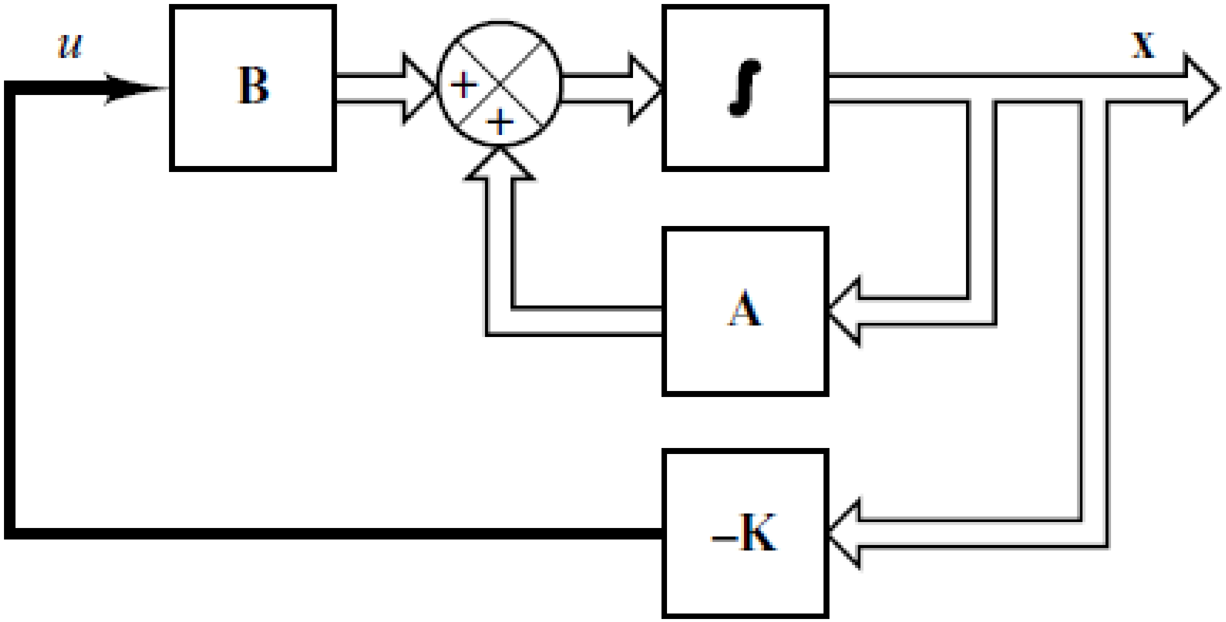
$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The system uses the state feedback control **u=–Kx.** Let us choose the desired closed-loop poles at

$$s = -2 + j4, \qquad s = -2 - j4, \qquad s = -10$$

(We make such a choice because we know from experience that such a set of closed-loop poles will result in a reasonable or acceptable transient response.) Determine the state feedback gain matrix **K.**

# EXAMPLE




First, we need to check the controllability matrix of the system. Since the controllability matrix **M** is given by

$$\mathbf{M} = \left[\mathbf{B} \;\vdots\; \mathbf{AB} \;\vdots\; \mathbf{A}^2\mathbf{B}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

# EXAMPLE

we find that |**M**|=–1, and therefore, rank **M**=3. Thus, the system is completely state controllable and arbitrary pole placement is possible.

Next, we shall solve this problem. We shall demonstrate each of the three methods presented.

***Method 1***: The characteristic equation for the system is

$$
\begin{aligned}
|s\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 5 & s+6 \end{vmatrix} \\
&= s^3 + 6s^2 + 5s + 1 \\
&= s^3 + a_1 s^2 + a_2 s + a_3 = 0
\end{aligned}
$$

# EXAMPLE

Hence, $a_1 = 6, \quad a_2 = 5, \quad a_3 = 1$

The desired characteristic equation is

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$
$$= s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = 0$$

Hence, $\alpha_1 = 14, \quad \alpha_2 = 60, \quad \alpha_3 = 200$

we have $\mathbf{K} = \left[\alpha_3 - a_3 \mid \alpha_2 - a_2 \mid \alpha_1 - a_1\right]\mathbf{T}^{-1}$

where **T=I** for this problem because the given state equation is in the controllable canonical form.

Then we have
$$\mathbf{K} = [200 - 1 \mid 60 - 5 \mid 14 - 6]$$
$$= [199 \quad 55 \quad 8]$$

# EXAMPLE

***Method 2***: By defining the desired state feedback gain matrix **K** as $\mathbf{K} = [k_1 \quad k_2 \quad k_3]$

and equating $|s\mathbf{I} - \mathbf{A} + \mathbf{BK}|$

with the desired characteristic equation, we btain

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{vmatrix}$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

# EXAMPLE

Thus, $6 + k_3 = 14, \qquad 5 + k_2 = 60, \qquad 1 + k_1 = 200$

from which we obtain $k_1 = 199, \qquad k_2 = 55, \qquad k_3 = 8$

Or

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

***Method 3***: Th $\mathbf{K} = [0 \quad 0 \quad 1][\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}]^{-1}\phi(\mathbf{A})$ nn's formula.

Since $\phi(\mathbf{A}) = \mathbf{A}^3 + 14\mathbf{A}^2 + 60\mathbf{A} + 200\mathbf{I}$

# EXAMPLE

$$\phi(\mathbf{A}) = \mathbf{A}^3 + 14\mathbf{A}^2 + 60\mathbf{A} + 200\mathbf{I}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2$$

$$+ 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

and

$$[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$\mathbf{K} = [0 \quad 0 \quad 1]\begin{bmatrix} 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix} \begin{bmatrix} -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

we obtain

$$= [0 \quad 0 \quad 1]\begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [199 \quad 55 \quad 8]$$

As a matter of course, the feedback gain matrix **K** obtained by the three methods are the same. With this state feedback, the closed-loop poles are placed at s=–2 ; j4 and s=–10, as desired.

It is noted that if the order ***n*** of the system were **4** or higher, methods 1 and 3 are recommended, since all matrix computations can be carried out by a computer. If method 2 is used, hand computations become necessary because a computer may not handle the characteristic equation with unknown parameters $k_1, k_2, \ldots, k_n$.

# DESIGN OF SERVO SYSTEMS