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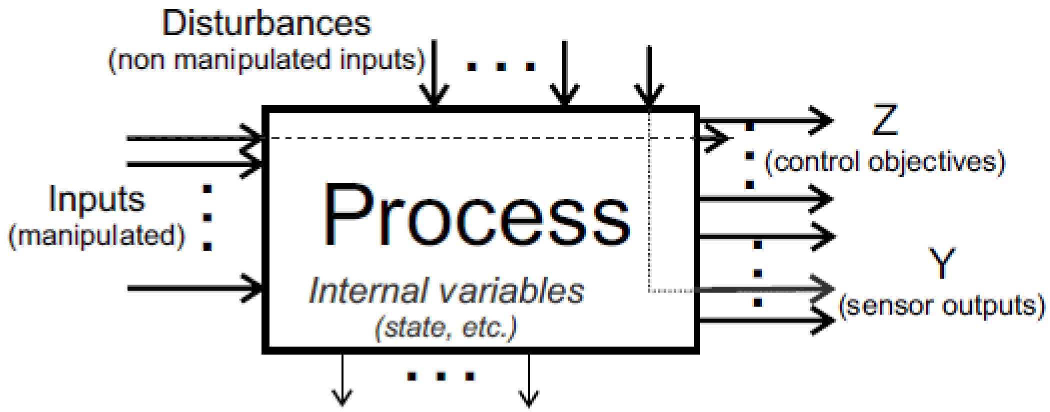
جامعة المنارة

MANARA UNIVERSITY

نظم متعددة المتغيرات

Multivariable Systems



الروبوت والأنظمة الذكية

مدرس المقرر

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Control Systems Design in State Space

DESIGN OF SERVO SYSTEMS

In this section we shall discuss the pole-placement approach to the design of **type 1** servo systems. Here we shall limit our systems each to have a scalar control signal ***u*** and a scalar output ***y***.

In what follows we shall first discuss a problem of designing a **type 1** servo system when the plant involves an integrator. Then we shall discuss the design of a **type 1** servo system when the plant has no integrator.

Design of Type 1 Servo System when the Plant Has An Integrator

Assume that the plant is defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$\mathbf{x}$ = state vector for the plant (n-vector)

u = control signal (scalar)

y = output signal (scalar)

$\mathbf{A}$ = $n \times n$ constant matrix

$\mathbf{B}$ = $n \times 1$ constant matrix

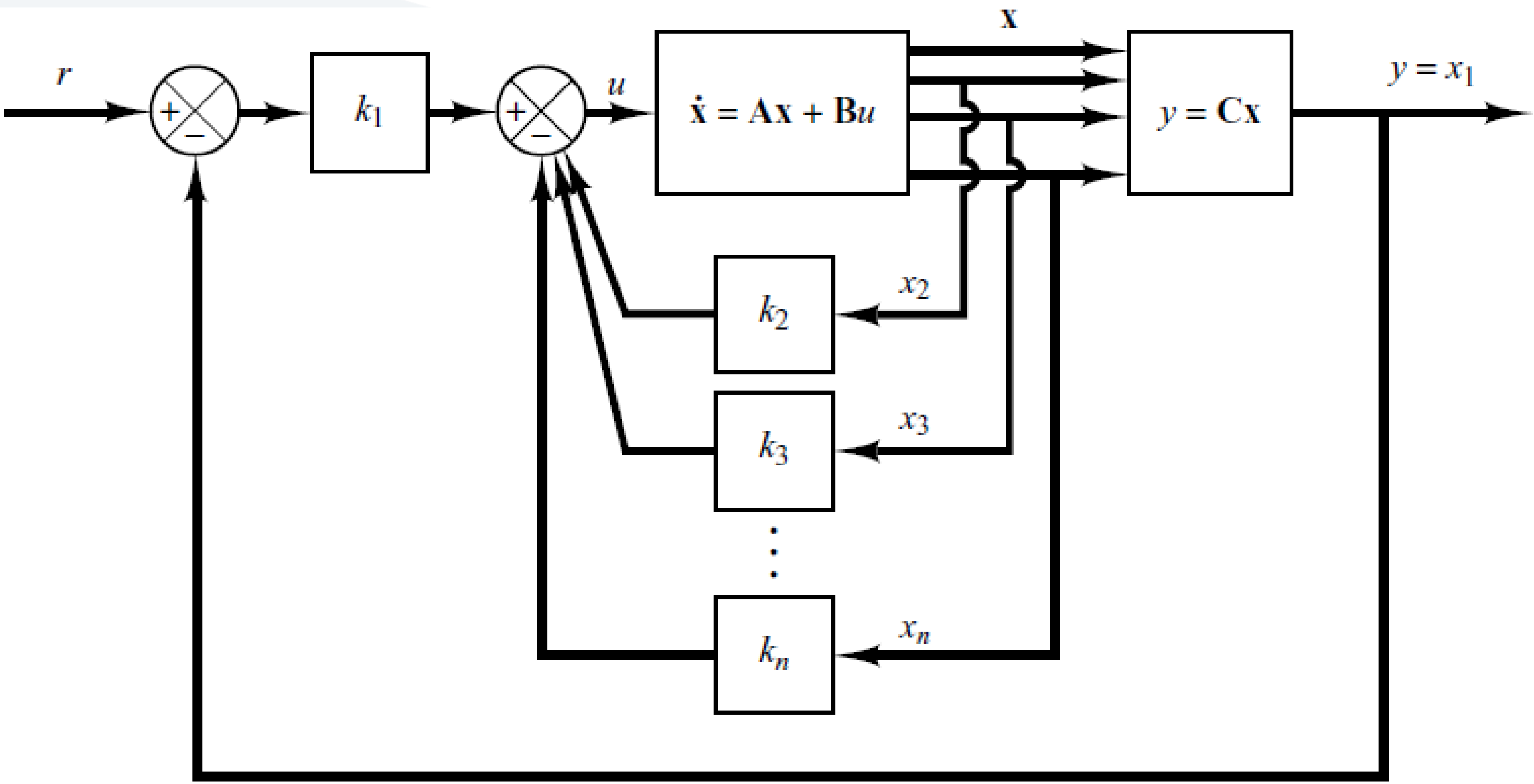
$\mathbf{C}$ = $1 \times n$ constant matrix

DESIGN OF SERVO SYSTEMS

As stated earlier, we assume that both the control signal ***u*** and the output signal ***y*** are scalars. By a proper choice of a set of state variables, it is possible to choose the output to be equal to one of the state variables. (which the output ***y*** becomes equal to x_1.)

Figure shows a general configuration of the **type 1** servo system when the plant has an integrator. Here we assumed that $\boldsymbol{y} = \boldsymbol{x_1}$. In the present analysis we assume that

DESIGN OF SERVO SYSTEMS



DESIGN OF SERVO SYSTEMS

the reference input ***r*** is a step function. In this system we use the following state-feedback control scheme:

$$u = -\begin{bmatrix} 0 & k_2 & k_3 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + k_1(r - x_1)$$

$$= -\mathbf{K}\mathbf{x} + k_1 r$$

Where

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$

DESIGN OF SERVO SYSTEMS

Assume that the reference input (step function) is applied at $t = 0$. Then, for $t > 0$, the system dynamics can be described

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1 r$$

We shall design the type **1** servo system such that the closed-loop poles are located at desired positions. The designed system will be an asymptotically stable system, $y(\infty)$will approach the constant value ***r***, and $u(\infty)$ will approach zero. (***r*** is a step input.)

Notice that at steady state we have

$$\dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(\infty) + \mathbf{B}k_1 r(\infty)$$

DESIGN OF SERVO SYSTEMS

Noting that $\boldsymbol{r}(\boldsymbol{t})$ is a step input, we have $\boldsymbol{r}(\infty) = \boldsymbol{r}(\boldsymbol{t}) = \boldsymbol{r}$ (constant) for $\boldsymbol{t} > \mathbf{0}$. By subtracting , we obtain

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{BK})[\mathbf{x}(t) - \mathbf{x}(\infty)]$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{e}(t)$$

Then

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{BK})\mathbf{e}$$

This equation describes the error dynamics.

The design of the **type 1** servo system here is converted to the design of an asymptotically stable regulator system such that $\boldsymbol{e}(\boldsymbol{t})$ approaches zero, given any initial condition $\boldsymbol{e}(\mathbf{0})$.

DESIGN OF SERVO SYSTEMS

If the system defined by $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

is completely state controllable, then, by specifying the desired eigenvalues μ_1 , μ_2 ,… , μ_n for the matrix $A - BK$, matrix K can be determined by the pole-placement technique.

The steady-state values of $X(t)$ and $u(t)$ can be found as follows: At steady state ($t = \infty$), we have, from Equation

($\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1 r$)

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(\infty) + \mathbf{B}k_1 r$$

Since the desired eigenvalues of $\boldsymbol{A}-\boldsymbol{BK}$ are all in the left-half $\boldsymbol{s}$ plane, the inverse of matrix $\boldsymbol{A}-\boldsymbol{BK}$ exists. Consequently, $\mathbf{X}(\infty)$ can be determined as

$$\mathbf{x}(\infty) = -(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}k_1 r$$

Also, $\boldsymbol{u}(\infty)$ can be obtained as

$$u(\infty) = -\mathbf{Kx}(\infty) + k_1 r = 0$$

EXAMPLE

Design a **type 1** servo system when the plant transfer function has an integrator. Assume that the plant transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{1}{s(s+1)(s+2)}$$

The desired closed-loop poles are $s = -2 \pm j2\sqrt{3}$ and $s = -10$. Assume that the system configuration is the same as that shown in Figure and the reference input ***r*** is a step function.

Obtain the unit-step response of the designed system.

Define state variables x_1 , x_2 and x_3 as follows:

EXAMPLE

$$x_1 = y$$
$$x_2 = \dot{x}_1$$
$$x_3 = \dot{x}_2$$

Then the state-space representation of the system becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

Where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

EXAMPLE

the control signal ***u*** is given by

$$u = -(k_2 x_2 + k_3 x_3) + k_1(r - x_1) = -\mathbf{K}\mathbf{x} + k_1 r$$

Where

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

The state feedback gain matrix ***K*** is thus $K = [160 \quad 54 \quad 11]$

Unit-Step Response of the Designed System: The unit-step response of the designed system can be obtained as follows:

Since

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [160 \quad 54 \quad 11] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix}$$

EXAMPLE

the state equation for the designed system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 160 \end{bmatrix} r$$

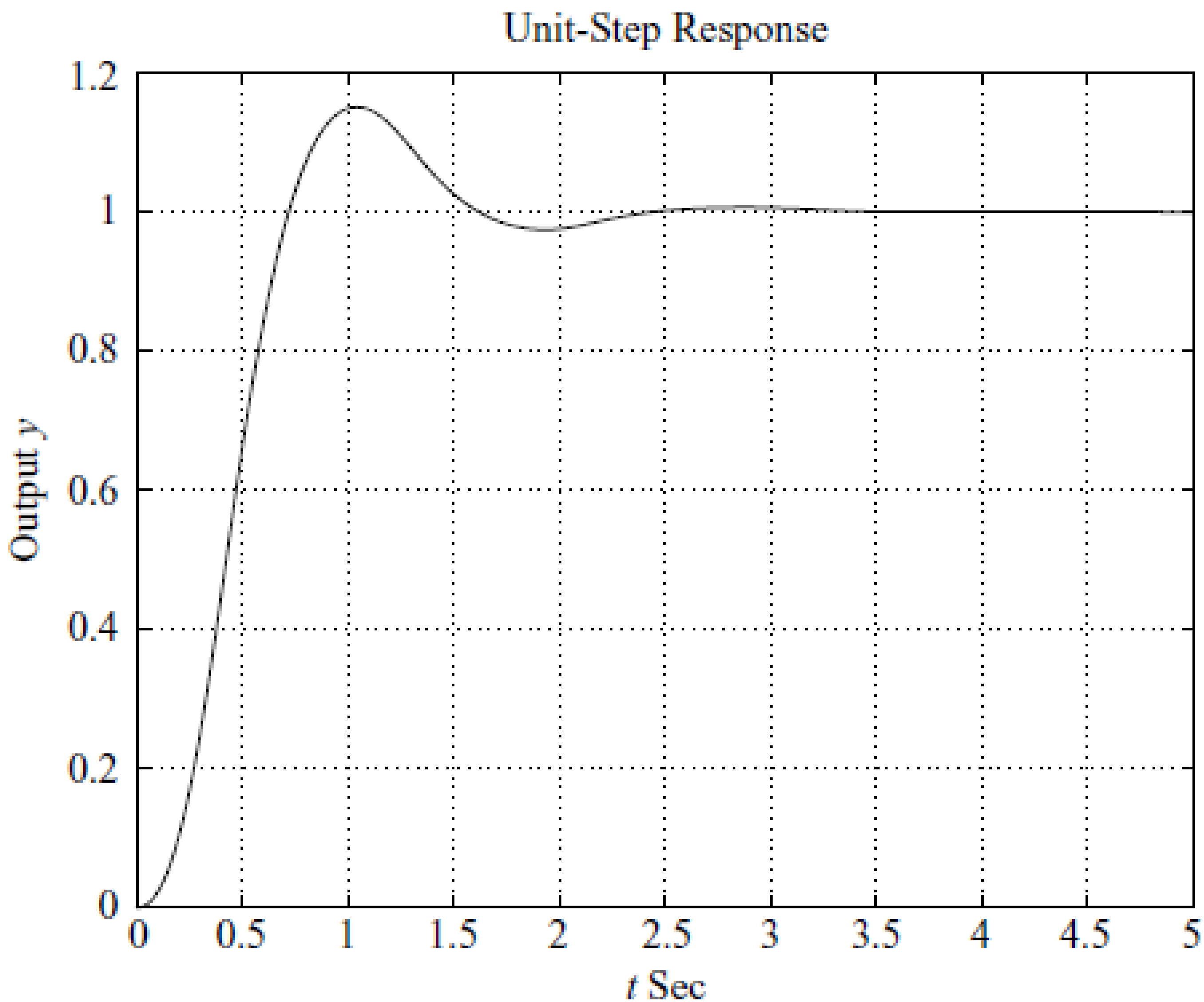
and the output equation is

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solving Equations for ***y*(*t*)** when ***r*** is a unit-step function gives the unit-step response curve ***y*(*t*)** versus ***t***.

The resulting unit-step response curve is shown in this Figure

EXAMPLE



Unit-Step Response
1.2
1
0.8
0.6
0.4
0.2
0
Output y
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
t Sec

EXAMPLE

Note that since

$$u(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r(\infty) = -\mathbf{K}\mathbf{x}(\infty) + k_1 r$$

we have

$$u(\infty) = -[160 \quad 54 \quad 11]\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{bmatrix} + 160r$$

$$= -[160 \quad 54 \quad 11]\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} + 160r = 0$$

At steady state the control signal u becomes zero.

Design of Type 1 Servo System when the Plant Has No Integrator

If the plant has no integrator (**type 0** plant), the basic principle of the design of a **type 1** servo system is to insert an integrator in the feedforward path between the error comparator and the plant, as shown in Figure. (The block diagram of Figure is a basic form of the **type 1** servo system where the plant has no integrator.) From the diagram, we obtain

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\mathbf{x} + k_I \xi$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x}$$

where $\mathbf{X}$= state vector of the plant (***n***-vector)

Design of Type 1 Servo System when the Plant Has No Integrator

where $\mathbf{X}$= state vector of the plant (***n***-vector)

u = control signal (scalar)

y = output signal (scalar)

ξ = output of the integrator (state variable of the system, scalar)

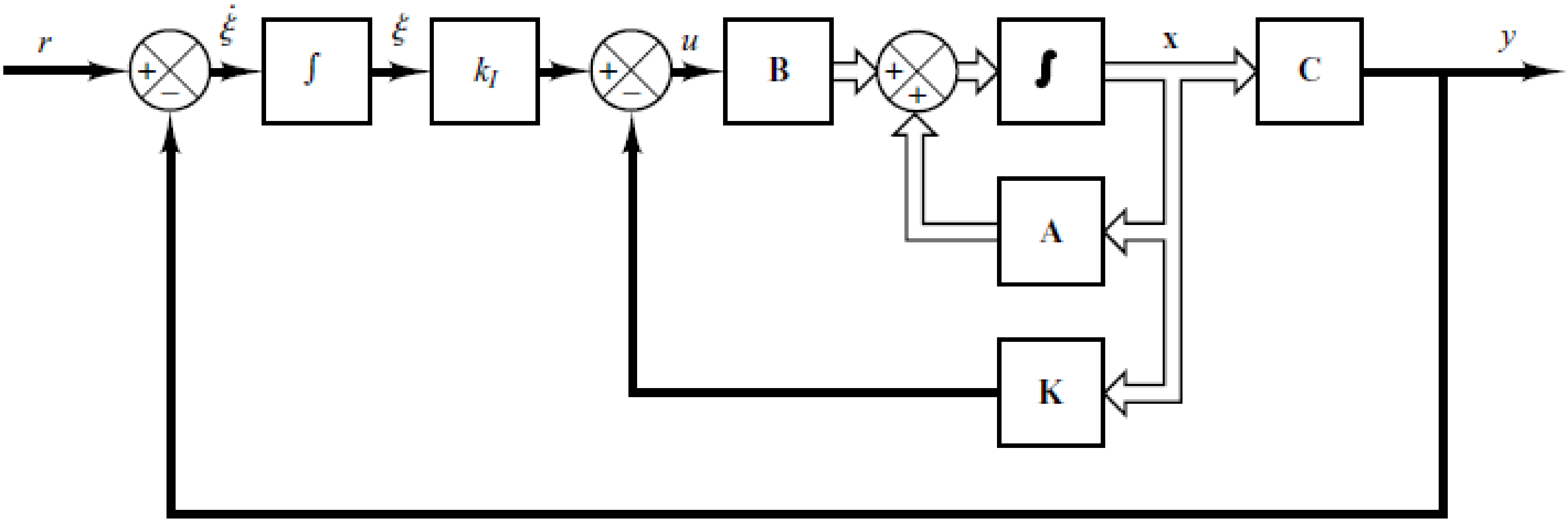
r = reference input signal (step function, scalar)

$\mathbf{A}$ = $n \times n$ constant matrix

$\mathbf{B}$ = $n \times 1$ constant matrix

$\mathbf{C}$ = $1 \times n$ constant matrix

Design of Type 1 Servo System when the Plant Has No Integrator



Design of Type 1 Servo System when the Plant Has No Integrator

We assume that the plant is completely state controllable. The transfer function of the plant can be given by

$$G_p(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

To avoid the possibility of the inserted integrator being canceled by the zero at the origin of the plant, we assume that $G_p(s)$ has no zero at the origin.

Assume that the reference input (step function) is applied at $t = 0$. Then, for $t > 0$, the system dynamics can be described by an equation that is a combination of Equations:

Design of Type 1 Servo System when the Plant Has No Integrator

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \xi(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(t)$$

We shall design an asymptotically stable system such that $\boldsymbol{X}(\infty)$, $\xi(\infty)$, and $\boldsymbol{u}(\infty)$ approach constant values, respectively. Then, at steady state, $\dot{\xi}(\infty) = \mathbf{0}$ and we get $\boldsymbol{y}(\infty) = \boldsymbol{r}$.

Notice that at steady state we have

$$\begin{bmatrix} \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(\infty) \\ \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u(\infty) + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r(\infty)$$

Design of Type 1 Servo System when the Plant Has No Integrator

Noting that $\boldsymbol{r(t)}$ is a step input, we have $\boldsymbol{r(t)} = \boldsymbol{r}(\infty) = \boldsymbol{r}$(constant) for $\boldsymbol{t} > \boldsymbol{0}$. By subtracting, we obtain

$$\begin{bmatrix} \dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) \\ \dot{\xi}(t) - \dot{\xi}(\infty) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) - \mathbf{x}(\infty) \\ \xi(t) - \xi(\infty) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} [u(t) - u(\infty)]$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{x}_e(t)$$
$$\xi(t) - \xi(\infty) = \xi_e(t)$$
$$u(t) - u(\infty) = u_e(t)$$

Then

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u_e(t)$$

Design of Type 1 Servo System when the Plant Has No Integrator

Where $$u_e(t) = -\mathbf{K}\mathbf{x}_e(t) + k_I \xi_e(t)$$

Define a new **(*n*+1)th-order** error vector $\boldsymbol{e}(\boldsymbol{t})$ by

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} = (n + 1)\text{-vector}$$

Then $$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e$$

Where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix}, \qquad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}$$

Design of Type 1 Servo System when the Plant Has No Integrator

and

$$u_e = -\hat{\mathbf{K}}\mathbf{e}$$

Where $\hat{\mathbf{K}} = [\mathbf{K} \;\vdots\; -k_I]$

The state error equation can be obtained by

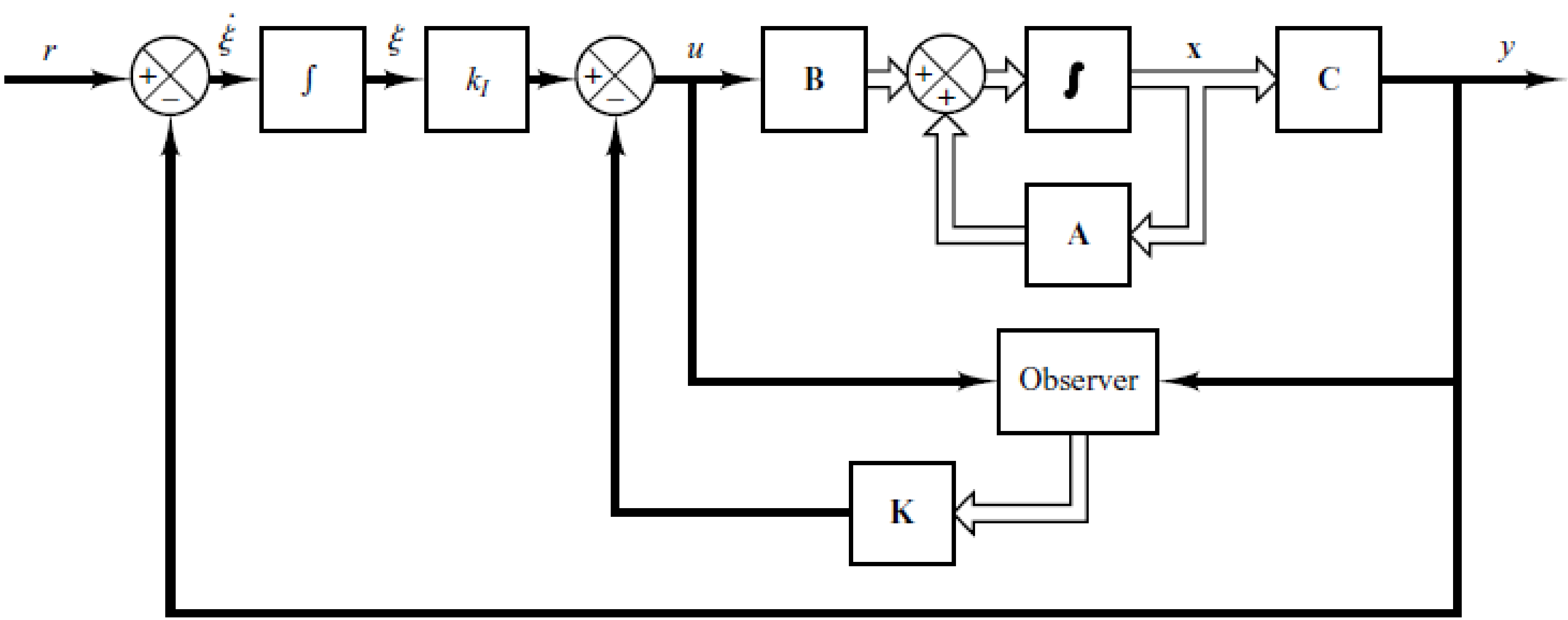
$$\dot{\mathbf{e}} = (\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}})\mathbf{e}$$

If the desired eigenvalues of matrix (that is, the desired closed-loop poles) are specified as $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$,… , $\boldsymbol{\mu}_{n+1}$, then the state-feedback gain matrix $\mathbf{K}$ and the integral gain constant $\boldsymbol{k_I}$ can be determined by the pole-placement technique, provided that the system is completely state controllable. Note that if the matrix

Design of Type 1 Servo System when the Plant Has No Integrator

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{C} & 0 \end{bmatrix}$$

has **rank n+1**, then the system defined by is completely state controllable.



Design of Type 1 Servo System when the Plant Has No Integrator

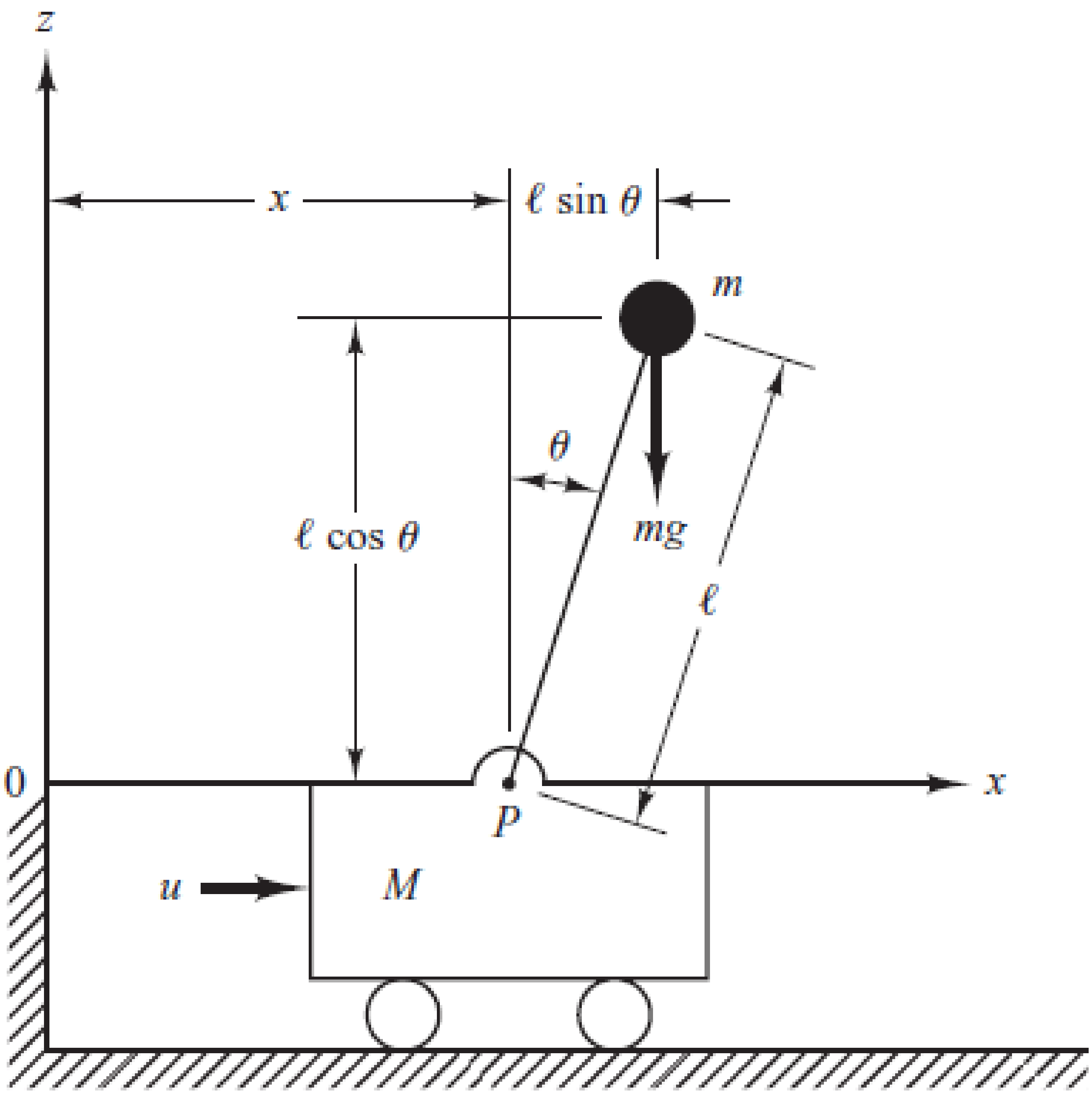
As is usually the case, not all state variables can be directly measurable. If this is the case, we need to use a state observer. Figure shows a block diagram of a **type 1** servo system with a state observer. [In the figure, each block with an integral symbol represents an integrator (**1/s**).]

EXAMPLE

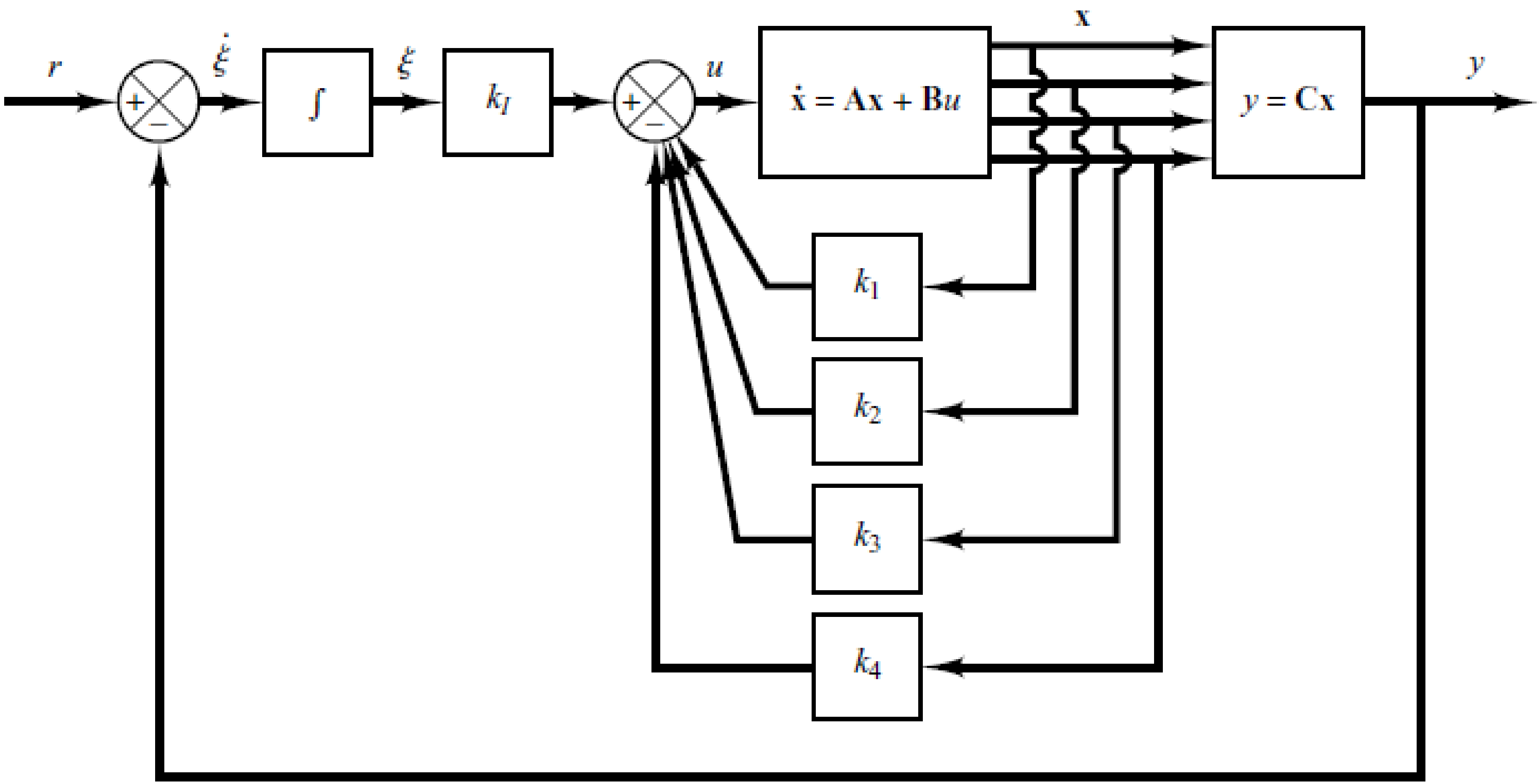
Consider the inverted-pendulum control system shown in Figure. In this example, we are concerned only with the motion of the pendulum and motion of the cart in the plane of the page.

It is desired to keep the inverted pendulum upright as much as possible and yet control the position of the cart—for instance, move the cart in a step fashion. To control the position of the cart, we need to build a **type 1** servo system. The inverted-pendulum system mounted on a cart does not have an integrator. Therefore, we feed the position signal ***y*** (which indicates the position of the cart) back to the input and insert an integrator in the feedforward path, as shown

EXAMPLE



EXAMPLE



EXAMPLE

in Figure. We assume that the pendulum angle $\boldsymbol{\theta}$ and the angular velocity are small, so that $\mathbf{sin}\,\boldsymbol{\theta} := \boldsymbol{\theta}$, $\mathbf{cos}\,\boldsymbol{\theta} := \mathbf{1}$ and $\boldsymbol{\theta}\acute{\boldsymbol{\theta}}^{2} := \mathbf{0}$ We also assume that the numerical values for ***M***, ***m***, and ***l*** are given as

$$M = 2\ \mathrm{kg}, \qquad m = 0.1\ \mathrm{kg}, \qquad l = 0.5\ \mathrm{m}$$

Earlier we derived the equations for the inverted-pendulum system. The equations for the inverted-pendulum control system shown in Figure are

$$Ml\ddot{\theta} = (M + m)g\theta - u$$

$$M\ddot{x} = u - mg\theta$$

EXAMPLE

When the given numerical values are substituted, become

$$\ddot{\theta} = 20.601\theta - u$$

$$\ddot{x} = 0.5u - 0.4905\theta$$

Let us define the state variables **x_1, x_2, x_3** , and **x_4** as

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$x_3 = x$$

$$x_4 = \dot{x}$$

EXAMPLE

Then, considering the cart position ***x*** as the output of the system, we obtain the equations for the system as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\mathbf{x} + k_I \xi$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x}$$

Where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{C} = [0 \quad 0 \quad 1 \quad 0]$$

EXAMPLE

For the **type 1** servo system, we have the state error equation as

$$\dot{\mathbf{e}} = \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e$$

Where

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 20.601 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -0.4905 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

and the control signal is

$$u_e = -\hat{\mathbf{K}}\mathbf{e}$$

Where

$$\hat{\mathbf{K}} = [\mathbf{K} \; \vdots \; -k_I] = [k_1 \quad k_2 \quad k_3 \quad k_4 \; \vdots \; -k_I]$$

EXAMPLE

To obtain a reasonable speed and damping in the response of the designed system (for example, the settling time of approximately **4 ~ 5** *sec* and the maximum overshoot of **15% ~ 16%** in the step response of the cart), let us choose the desired closed-loop poles at $s = \mu_i$(i=1, 2, 3, 4, 5), where

$$\mu_1 = -1 + j\sqrt{3}, \quad \mu_2 = -1 - j\sqrt{3}, \quad \mu_3 = -5, \quad \mu_4 = -5, \quad \mu_5 = -5$$

Thus, we get

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \begin{bmatrix} -157.6336 & -35.3733 & -56.0652 & -36.7466 \end{bmatrix}$$

And $k_I = -50.9684$

EXAMPLE

Unit Step-Response Characteristics of the Designed System

Once we determine the feedback gain matrix **K** and the integral gain constant k_I, the step response in the cart position can be obtained by solving the following equation:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{B}k_I \\ -\mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

The output $y(t)$ of the system is $x_3(t)$, or

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + [0]r$$

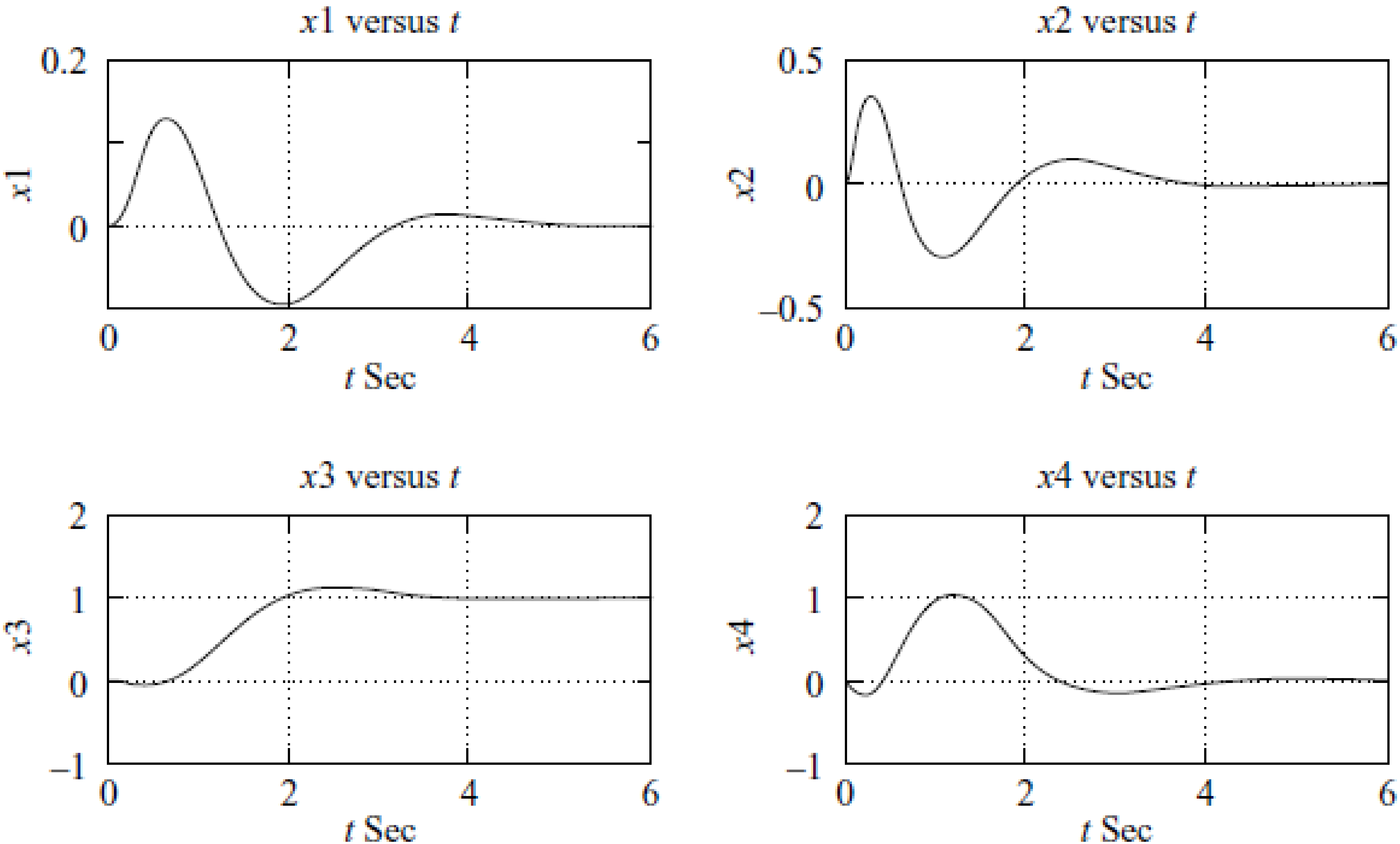
EXAMPLE

Figure shows curves $\boldsymbol{x_1}$ versus $\boldsymbol{t}$, $\boldsymbol{x_2}$ versus $\boldsymbol{t}$, $\boldsymbol{x_3}$ (= output $\boldsymbol{y}$) versus $\boldsymbol{t}$, $\boldsymbol{x_4}$ versus $\boldsymbol{t}$, and x_5 (= **ξ**) versus $\boldsymbol{t}$. Notice that $\boldsymbol{y(t)}$ = $\boldsymbol{x_3(t)}$ has approximately **15%** overshoot and the settling time is approximately **4.5** sec. **ξ**(t) [= $\boldsymbol{x_5(t)}$] approaches **1.1**.This result can be derived as follows: Since

$$\dot{\mathbf{x}}(\infty) = \mathbf{0} = \mathbf{A}\mathbf{x}(\infty) + \mathbf{B}u(\infty)$$

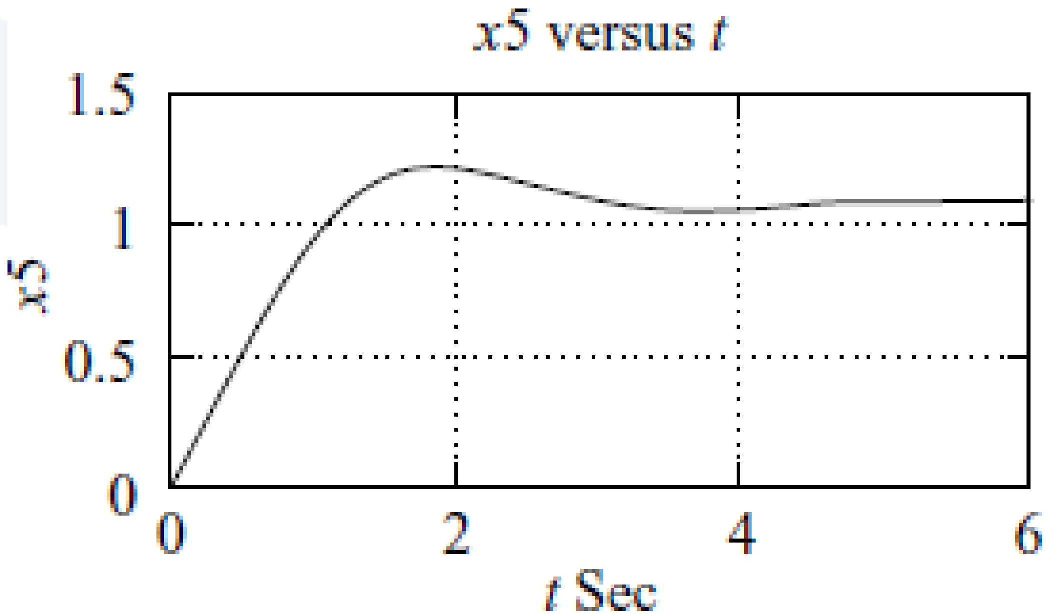
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20.601 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.4905 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0.5 \end{bmatrix} u(\infty)$$

EXAMPLE



x1 versus t
0.2
0
x1
0
2
4
6
t Sec
x2 versus t
0.5
0
–0.5
x2
0
2
4
6
t Sec
x3 versus t
2
1
0
–1
x3
0
2
4
6
t Sec
x4 versus t
2
1
0
–1
x4
0
2
4
6
t Sec

EXAMPLE



we get

$$u(\infty) = 0$$

Since $u(\infty) = 0$, we have,

$$u(\infty) = 0 = -\mathbf{K}\mathbf{x}(\infty) + k_I \xi(\infty)$$

EXAMPLE

and so

$$\xi(\infty) = \frac{1}{k_I}[\mathbf{Kx}(\infty)] = \frac{1}{k_I}k_3x_3(\infty) = \frac{-56.0652}{-50.9684}r = 1.1r$$

Hence, for $\boldsymbol{r = 1}$, we have $\xi(\infty) = 1.1$

It is noted that, as in any design problem, if the speed and damping are not quite satisfactory, then we must modify the desired characteristic equation and determine a new matrix $\hat{\mathbf{K}}$ Computer simulations must be repeated until a satisfactory result is obtained.