Lecture 6: Inner Product Spaces

CECC122: Linear Algebra and Matrix Theory

Manara University

2023-2024

جامعة المنارة
MANARA UNIVERSITY

5.2 Inner Product Spaces

- **Inner Product:**

Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in a vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and satisfies the following axioms.

(1) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$

(2) $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$

(3) $c\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle c\boldsymbol{u}, \boldsymbol{v} \rangle$

(4) $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ and $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$

- Notes:

$\mathbf{u}.\mathbf{v}$ = dot product (Euclidean inner product for R^n)

$<\mathbf{u}, \mathbf{v}>$ = general inner product for vector space V

- Notes:

A vector space V with an inner product is called an inner product space.

Vector space: $(V, +, .)$

Inner product space: $(V, +, ., <, >)$

- Ex 1: (Euclidean inner product for R^n)

Show that the dot product in R^n satisfies the four axioms of an inner product.

Sol:

$$\boldsymbol{u} = (u_1, u_2, \ldots, u_n), \ \boldsymbol{v} = (v_1, v_2, \ldots, v_n)$$

$$< \boldsymbol{u}, \boldsymbol{v} > = \boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

By Theorem 5.3, this dot product satisfies the required four axioms. Thus it is an inner product on R^n.

- **Ex 2: (A different inner product for R^n)**

Show that the function defines an inner product on R^2, where $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$

$$<\boldsymbol{u}, \boldsymbol{v}> = u_1 v_1 + 2 u_2 v_2$$

Sol:

(1) $<\boldsymbol{u}, \boldsymbol{v}> = u_1 v_1 + 2 u_2 v_2 = v_1 u_1 + 2 v_2 u_2 = <\boldsymbol{v}, \boldsymbol{u}>$

(2) $\boldsymbol{w} = (w_1, w_2)$

$$< \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} > = u_1 (v_1 + w_1) + 2u_2 (v_2 + w_2)$$
$$= u_1 v_1 + u_1 w_1 + 2u_2 v_2 + 2u_2 w_2$$
$$= (u_1 v_1 + 2u_2 v_2) + (u_1 w_1 + 2u_2 w_2)$$
$$= < \boldsymbol{u}, \boldsymbol{v} > + < \boldsymbol{u}, \boldsymbol{w} >$$

(3) $c < \boldsymbol{u}, \boldsymbol{v} > = c(u_1 v_1 + 2u_2 v_2) = (cu_1)v_1 + 2(cu_2)v_2 = < c\boldsymbol{u}, \boldsymbol{v} >$

(4) $< \boldsymbol{v}, \boldsymbol{v} > = v_1^2 + 2v_2^2 \geq 0$

$$< \boldsymbol{v}, \boldsymbol{v} > = 0 \Rightarrow v_1^2 + 2v_2^2 = 0 \Rightarrow v_1 = v_2 = 0 \quad (\boldsymbol{v} = \boldsymbol{0})$$

- Note: (An inner product on R^n)

$$< \boldsymbol{u}, \boldsymbol{v} > = c_1 u_1 v_1 + c_2 u_2 v_2 + \cdots + c_n u_n v_n, \quad c_i > 0 \qquad \text{(weights)}$$

- Ex 3: (A function that is not an inner product)

Show that the following function is not an inner product on R^3

$$< \boldsymbol{u}, \boldsymbol{v} > = u_1 v_1 - 2u_2 v_2 + u_3 v_3$$

Sol:

Let $\mathbf{v} = (1, 2, 1)$, then $<\mathbf{v}, \mathbf{v}> = (1)(1) - 2(2)(2) + (1)(1) = -6 < 0$

Axiom 4 is not satisfied. Thus this function is not an inner product on R^3

- Theorem 5.7: (Properties of inner products)

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in an inner product space V, and let c be any real number.

(1) $<\mathbf{0}, \mathbf{v}> = <\mathbf{v}, \mathbf{0}> = 0$

(2) $<\mathbf{u} + \mathbf{v}, \mathbf{w}> = <\mathbf{u}, \mathbf{w}> + <\mathbf{v}, \mathbf{w}>$

(3) $<\mathbf{u}, c\mathbf{v}> = c<\mathbf{u}, \mathbf{v}>$

- Norm (length) of $\boldsymbol{u}$: $\|\boldsymbol{u}\| = \sqrt{<\boldsymbol{u}, \boldsymbol{u}>}$

- Note: $\|\boldsymbol{u}\|^2 = <\boldsymbol{u}, \boldsymbol{u}>$

- Distance between $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle}$$

- Angle between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\| \|\boldsymbol{v}\|}, \ 0 \le \theta \le \pi$$

- Orthogonal: $(\boldsymbol{u} \perp \boldsymbol{v})$

$\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $<\boldsymbol{u}, \boldsymbol{v}> = 0$

- Notes:

  (1) If $\|\boldsymbol{v}\| = 1$, then $\boldsymbol{v}$ is called a unit vector

  (2) $\|\boldsymbol{v}\| \neq 1$ $\xrightarrow{\text{Normalizing}}$ $\dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$ (the unit vector in the direction of $\boldsymbol{v}$)

  $\boldsymbol{v} \neq \mathbf{0}$

  not a unit vector

- Properties of norm:

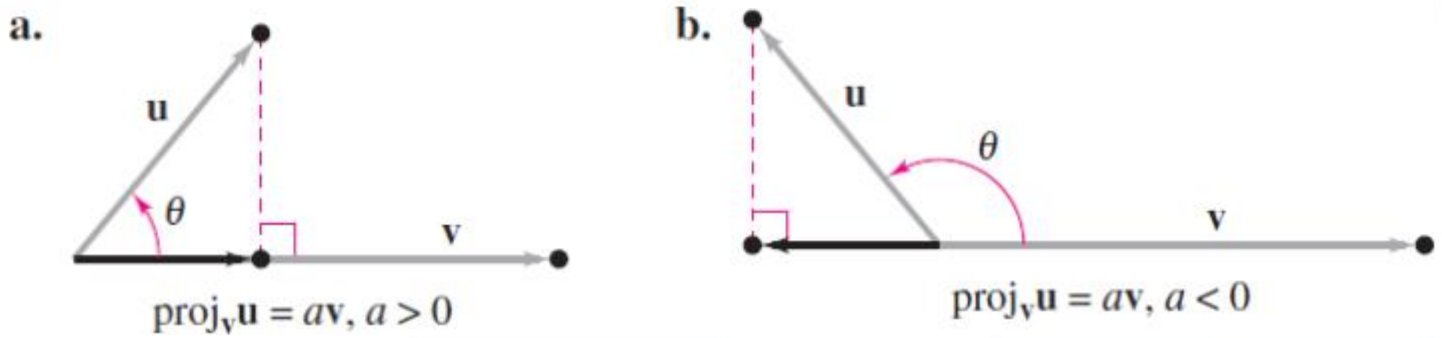
  (1) $\|\boldsymbol{u}\| \geq 0$

  (2) $\|\boldsymbol{u}\| = 0$ if and only if $\boldsymbol{u} = \mathbf{0}$

  (3) $\|c\boldsymbol{u}\| = |c|\|\boldsymbol{u}\|$

- **Orthogonal projections in inner product spaces:**

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in an inner product space V, such that $\boldsymbol{v} \neq 0$. Then the orthogonal projection of u onto $\boldsymbol{v}$ is given by $\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \dfrac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}\boldsymbol{v}$



- **Note:**

If $\boldsymbol{v}$ is a unit vector, then $< \boldsymbol{v}, \boldsymbol{v} > = \|\boldsymbol{v}\|^2 = 1.$

The formula for the orthogonal projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ takes the following simpler form:

$$\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle \boldsymbol{v}$$

- **Ex 4: (Finding an orthogonal projection in R^3)**

Use the Euclidean inner product in R^3 to find the orthogonal projection of $\boldsymbol{u} = (6, 2, 4)$ onto $\boldsymbol{v} = (1, 2, 0)$.

Sol:

$$<\boldsymbol{u}, \boldsymbol{v}> = (6)(1) + (2)(2) + (4)(0) = 10$$

$$<\boldsymbol{v}, \boldsymbol{v}> = 1^2 + 2^2 + 0^2 = 5$$

$$\text{proj}_{\boldsymbol{v}}\boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}\boldsymbol{v} = \frac{10}{5}(1, 2, 0) = (2, 4, 0)$$

- **Note:**

$$\boldsymbol{u} - \text{proj}_{\boldsymbol{v}}\boldsymbol{u} = (6, 2, 4) - (2, 4, 0) = (4, -2, 4) \text{ is orthogonal to } \boldsymbol{v} = (1, 2, 0)$$

5.1 Length and Dot Product in Rn

5.2 Inner Product Spaces

5.3 Orthonormal Bases: Gram-Schmidt Process

5.4 Mathematical Models and Least Square Analysis

5.3 Orthonormal Bases: Gram-Schmidt Process

- Orthogonal:

A set S of vectors in an inner product space V is called an orthogonal set if every pair of vectors in the set is orthogonal.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$< \mathbf{v}_i, \mathbf{v}_j > = 0, \quad i \neq j$$

- Orthonormal:

An orthogonal set in which each vector is a unit vector is called orthonormal

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\} \subseteq V$$

$$< \mathbf{v}_i, \mathbf{v}_j > = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Note:

If S is a basis, then it is called an orthogonal basis or an orthonormal basis.

- Ex 1: (A nonstandard orthonormal basis for R^3)

Show that the following set is an orthonormal basis.

$$S = \left\{ \underbrace{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)}_{v_1}, \underbrace{\left(-\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right)}_{v_2}, \underbrace{\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)}_{v_3} \right\}$$

Sol:

$$v_1 \cdot v_2 = -\frac{1}{6} + \frac{1}{6} + 0 = 0$$

$$v_1 \cdot v_3 = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$$

$$v_2 \cdot v_3 = -\frac{\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$$

Show that the three vectors are mutually orthogonal.

$$\|v_1\| = \sqrt{v_1 \cdot v_1} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

$$\|v_2\| = \sqrt{v_2 \cdot v_2} = \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{9}} = 1$$

$$\|v_3\| = \sqrt{v_3 \cdot v_3} = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$$

Show that each vector is of length 1

Thus S is an orthonormal set

- Theorem 5.9: (Orthogonal sets are linearly independent)

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space V, then S is linearly independent.

- Corollary to Theorem 5.9:

  If V is an inner product space of dimension n, then any orthogonal set of n nonzero vectors is a basis for V.

- Ex 2: (Using orthogonality to test for a basis)

Show that the following set is a basis for R^4

$$S = \{\underset{v_1}{(2, 3, 2, -2)}, \underset{v_2}{(1, 0, 0, 1)}, \underset{v_3}{(-1, 0, 2, 1)}, \underset{v_4}{(-1, 2, -1, 1)}\}$$

Sol:

v_1, v_2, v_3, v_4: nonzero vectors

$$v_1 \cdot v_2 = 2+0+0-2=0 \qquad v_2 \cdot v_3 = -1+0+0+1=0$$

$$v_1 \cdot v_3 = -2+0+4-2=0 \qquad v_2 \cdot v_4 = -1+0+0+1=0$$

$$v_1 \cdot v_4 = -2+6-2-2=0 \qquad v_3 \cdot v_4 = 1+0-2+1=0$$

$\Rightarrow S$ is orthogonal $\Rightarrow S$ is a basis for R^4

- **Theorem 5.10: (Coordinates relative to an orthonormal basis)**

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, then the coordinate representation of a vector $\mathbf{w}$ with respect to B is

$$\mathbf{w} = <\mathbf{w}, \mathbf{v}_1> \mathbf{v}_1 + <\mathbf{w}, \mathbf{v}_2> \mathbf{v}_2 + \cdots + <\mathbf{w}, \mathbf{v}_n> \mathbf{v}_n$$

- **Note:**

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for V and $\mathbf{w} \in V$, then the corresponding coordinate matrix of $\mathbf{w}$ relative to B is

$$[\mathbf{w}]_B = \begin{bmatrix} <\mathbf{w}, \mathbf{v}_1> \\ <\mathbf{w}, \mathbf{v}_2> \\ \vdots \\ <\mathbf{w}, \mathbf{v}_n> \end{bmatrix}$$

- Ex 3: (Representing vectors relative to an orthonormal basis)

Find the coordinates of vector $\mathbf{w} = (5, -5, 2)$ relative to the following orthonormal basis for R^3 .

$$B = \{(\frac{3}{5}, \frac{4}{5}, 0), (-\frac{4}{5}, \frac{3}{5}, 0), (0, 0, 1)\}$$

Sol:

$$< \mathbf{w}, \mathbf{v}_1 > = \mathbf{w} . \mathbf{v}_1 = (5, -5, 2) . (\frac{3}{5}, \frac{4}{5}, 0) = -1$$

$$< \mathbf{w}, \mathbf{v}_2 > = \mathbf{w} . \mathbf{v}_2 = (5, -5, 2) . (-\frac{4}{5}, \frac{3}{5}, 0) = -7$$

$$< \mathbf{w}, \mathbf{v}_3 > = \mathbf{w} . \mathbf{v}_3 = (5, -5, 2) . (0, 0, 1) = 2$$

$$\Rightarrow [\mathbf{w}]_B = \begin{bmatrix} -1 \\ -7 \\ 2 \end{bmatrix}$$

- Theorem 5.11: (Gram-Schmidt orthonormalization process)

(1) Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for an inner product space V

(2) Let $B' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, where

$$\mathbf{w}_1 = \mathbf{v}_1$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{<\mathbf{v}_2, \mathbf{w}_1>}{<\mathbf{w}_1, \mathbf{w}_1>}\mathbf{w}_1$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{<\mathbf{v}_3, \mathbf{w}_1>}{<\mathbf{w}_1, \mathbf{w}_1>}\mathbf{w}_1 - \frac{<\mathbf{v}_3, \mathbf{w}_2>}{<\mathbf{w}_2, \mathbf{w}_2>}\mathbf{w}_2$$

$$\vdots$$

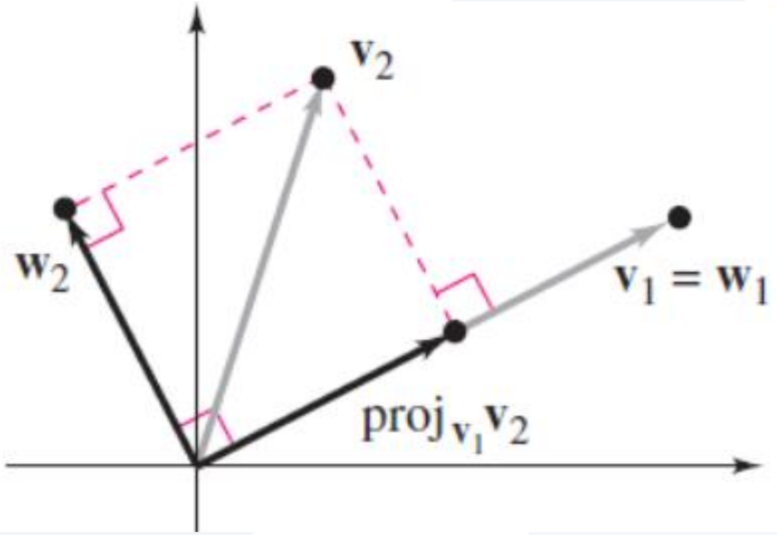
$$\mathbf{w}_n = \mathbf{v}_n - \sum_{i=1}^{n-1}\frac{<\mathbf{v}_n, \mathbf{w}_i>}{<\mathbf{w}_i, \mathbf{w}_i>}\mathbf{w}_i$$

Then B' is an orthogonal basis for V

(3) Let $\boldsymbol{u}_i = \frac{\boldsymbol{w}_i}{\|\boldsymbol{w}_i\|}$

Then $B'' = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ is an orthonormal basis for V

Also, $\text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ for $k = 1, 2, \ldots, n$



- Ex 4: (Applying the Gram-Schmidt orthonormalization process)

Apply the Gram-Schmidt orthonormalization process to the basis B for R^2

$$B = \{\underset{\mathbf{v}_1}{(1, 1)}, \underset{\mathbf{v}_2}{(0, 1)}\}$$

Sol:

$$\mathbf{w}_1 = \mathbf{v}_1 = (1, 1)$$

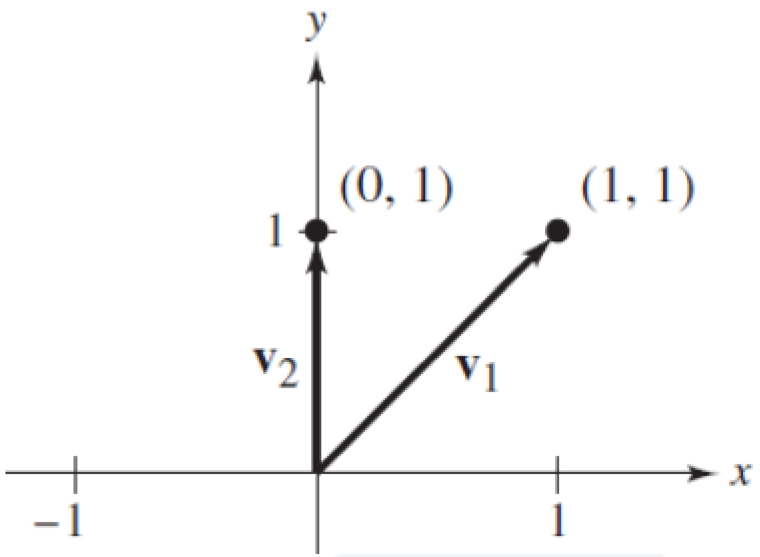
$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{<\mathbf{v}_2, \mathbf{w}_1>}{<\mathbf{w}_1, \mathbf{w}_1>}\mathbf{w}_1 = (0, 1) - \frac{1}{2}(1, 1) = (-\frac{1}{2}, \frac{1}{2})$$

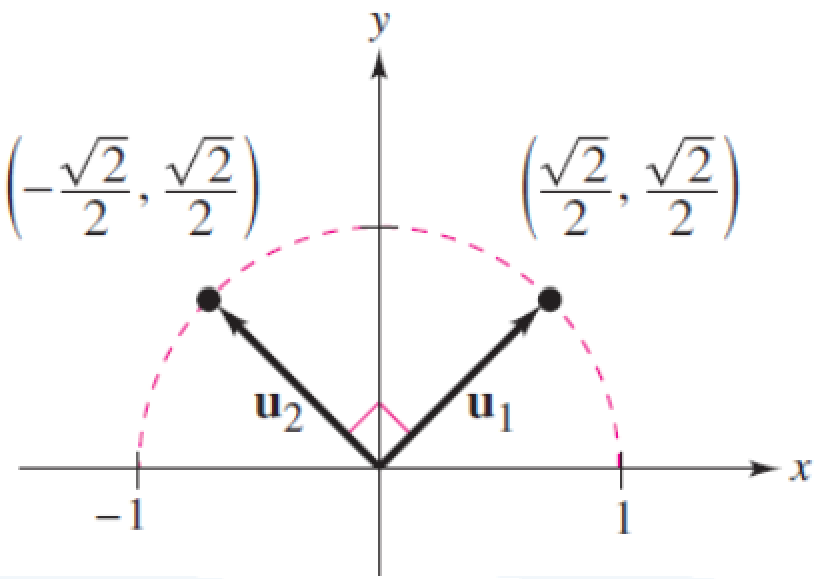
The set $B' = \{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis for R^2

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}}(1, 1) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$$

$$u_2 = \frac{w_2}{\|w_2\|} = \frac{1}{1/\sqrt{2}}\left(-\frac{1}{2}, \frac{1}{2}\right) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

The set $B'' = \{u_1, u_2\}$ is an orthonormal basis for R^2





- **Ex 5: (Applying the Gram-Schmidt orthonormalization process)**

Apply the Gram-Schmidt orthonormalization process to the basis B for R^3

$$B = \{\underset{v_1}{(1, 1, 0)}, \underset{v_2}{(1, 2, 0)}, \underset{v_3}{(0, 1, 2)}\}$$

Sol:

$$w_1 = v_1 = (1, 1, 0)$$

$$w_2 = v_2 - \frac{< v_2, w_1 >}{< w_1, w_1 >} w_1 = (1, 2, 0) - \frac{3}{2}(1, 1, 0) = (-\frac{1}{2}, \frac{1}{2}, 0)$$

$$w_3 = v_3 - \frac{< v_3, w_1 >}{< w_1, w_1 >} w_1 - \frac{< v_3, w_2 >}{< w_2, w_2 >} w_2$$

$$= (1, 2, 0) - \frac{1}{2}(1, 1, 0) - \frac{1/2}{1/2}(-\frac{1}{2}, \frac{1}{2}, 0) = (0, 0, 2)$$

The set $B' = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis for R^3

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{2}}(1, 1, 0) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{1/\sqrt{2}}(-\frac{1}{2}, \frac{1}{2}, 0) = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$$

$$\mathbf{u}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{1}{2}(0, 0, 2) = (0, 0, 1)$$

The set $B'' = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis for R^3