

Exercises 6: Inner Product Spaces

CECC122: Linear Algebra and Matrix Theory

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Find (a) $\|\mathbf{u}\|$, (b) $\|\mathbf{v}\|$, and (c) $\|\mathbf{u} + \mathbf{v}\|$

① $\mathbf{u} = (3, 1, 3)$, $\mathbf{v} = (0, -1, 1)$

(a) $\|\mathbf{u}\| = \sqrt{3^2 + 1^2 + 3^2} = \sqrt{19}$ (b) $\|\mathbf{v}\| = \sqrt{0^2 + (-1)^2 + 1^2} = \sqrt{2}$

(c) $\|\mathbf{u} + \mathbf{v}\| = \|(3 - 0, 1 - (-1), 3 - 1)\| = \|(3, 2, 2)\| = \sqrt{3^2 + 2^2 + 2^2} = \sqrt{17}$

Find a unit vector (a) in the direction of \mathbf{u} and (b) in the direction opposite that of \mathbf{u}

① $\mathbf{u} = (3, 2, -5)$, $\mathbf{v} = (0, -1, 1)$

(a) A unit vector \mathbf{v} in the direction of \mathbf{u} is given by

$$\mathbf{v} = \frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{3^2 + 2^2 + (-5)^2}} (3, 2, -5) = \frac{1}{\sqrt{38}} (3, 2, -5) = \left(\frac{3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{-5}{\sqrt{38}} \right)$$

(b) A unit vector in the direction opposite that of \mathbf{u} is given by

$$-\mathbf{v} = -\left(\frac{3}{\sqrt{38}}, \frac{2}{\sqrt{38}}, \frac{-5}{\sqrt{38}}\right) = \left(\frac{-3}{\sqrt{38}}, \frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}}\right)$$

Find the vector \mathbf{v} with the given length and the same direction as \mathbf{u}

① $\|\mathbf{v}\| = 5, \mathbf{u} = (\sqrt{5}, 5, 0)$

First find a unit vector in the direction of \mathbf{u}

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = \frac{1}{\sqrt{5+25}}(\sqrt{5}, 5, 0) = \frac{1}{\sqrt{30}}(\sqrt{5}, 5, 0) = \left(\frac{1}{\sqrt{6}}, \frac{\sqrt{5}}{\sqrt{6}}, 0\right)$$

Then \mathbf{v} is five times this vector $\mathbf{v} = 5\left(\frac{1}{\sqrt{6}}, \frac{\sqrt{5}}{\sqrt{6}}, 0\right) = \left(\frac{5}{\sqrt{6}}, \frac{5\sqrt{5}}{\sqrt{6}}, 0\right)$

For what values of c is $\|c(1, 2, 3)\| = 1$?

$$\|c(1, 2, 3)\| = |c|\|(1, 2, 3)\| = 1 \Rightarrow |c| = \frac{1}{\|(1, 2, 3)\|} = \frac{1}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{1}{\sqrt{14}}$$

$$\Rightarrow c = \pm \frac{1}{\sqrt{14}}$$

Find (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{v} \cdot \mathbf{v}$, (c) $\|\mathbf{v}\|^2$, (d) $d(\mathbf{u}, \mathbf{v})$, (e) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, and (f) $\mathbf{u} \cdot (5\mathbf{v})$

① $\mathbf{u} = (3, 4)$, $\mathbf{v} = (2, -3)$

(a) $\mathbf{u} \cdot \mathbf{v} = 3(2) + 4(-3) = -6$

(b) $\mathbf{v} \cdot \mathbf{v} = 2(2) + (-3)(-3) = 13$

(c) $\|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = 3^2 + 4^2 = 25$

(d) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(3 - 2, 4 - (-3))\| = \|(1, 7)\| = \sqrt{1^2 + 7^2} = \sqrt{50} = 5\sqrt{2}$

$$(e) (\mathbf{u} \cdot \mathbf{v})\mathbf{v} = -6\mathbf{v} = (-12, 18)$$

$$(f) \mathbf{u}(5\mathbf{v}) = 5\mathbf{u} \cdot \mathbf{v} = 5(-6) = -30$$

$$\textcircled{2} \quad \mathbf{u} = (2, -2, 1), \mathbf{v} = (2, -1, -6)$$

$$(a) \mathbf{u} \cdot \mathbf{v} = 2(2) + (-2)(-1) + 1(-6) = 0$$

$$(b) \mathbf{v} \cdot \mathbf{v} = 2(2) + (-1)(-1) + (-6)(-6) = 41$$

$$(c) \|\mathbf{u}\|^2 = \mathbf{u} \cdot \mathbf{u} = 2^2 + (-2)^2 + 1^2 = 9$$

$$(d) d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(2 - 2, -2 - (-1), 1 - (-6))\| = \|(0, -1, 7)\|$$
$$= \sqrt{0^2 + (-1)^2 + 7^2} = \sqrt{50}$$

$$(e) (\mathbf{u} \cdot \mathbf{v})\mathbf{v} = 0\mathbf{v} = \mathbf{0} = (0, 0, 0)$$

$$(f) \mathbf{u}(5\mathbf{v}) = 5\mathbf{u} \cdot \mathbf{v} = 0$$

Find $(3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v})$ when $\mathbf{u} \cdot \mathbf{u} = 8$, $\mathbf{u} \cdot \mathbf{v} = 7$, and $\mathbf{v} \cdot \mathbf{v} = 6$

$$\begin{aligned} (3\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - 3\mathbf{v}) &= 3\mathbf{u} \cdot \mathbf{u} - 9\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + 3\mathbf{v} \cdot \mathbf{v} = 3\mathbf{u} \cdot \mathbf{u} - 10\mathbf{u} \cdot \mathbf{v} + 3\mathbf{v} \cdot \mathbf{v} \\ &= 3(8) - 10(7) + 3(6) = -28 \end{aligned}$$

Find the angle θ between the vectors

① $\mathbf{u} = (1, -1)$, $\mathbf{v} = (0, 1)$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1(0) + (-1)(1)}{\sqrt{1^2 + (-1)^2} \sqrt{0^2 + (-1)^2}} = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1} \left(\frac{-1}{\sqrt{2}} \right) = \frac{3\pi}{4} \text{ rad } (135^\circ)$$

② $\mathbf{u} = \left(\cos \frac{\pi}{6}, \sin \frac{\pi}{6} \right)$, $\mathbf{v} = \left(\cos \frac{3\pi}{4}, \sin \frac{3\pi}{4} \right)$



$$\begin{aligned}\cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{\cos \frac{\pi}{6} \cos \frac{3\pi}{4} + \sin \frac{\pi}{6} \sin \frac{3\pi}{4}}{\sqrt{\cos^2 \frac{\pi}{6} + \sin^2 \frac{\pi}{6}} \sqrt{\cos^2 \frac{3\pi}{4} + \sin^2 \frac{3\pi}{4}}} = \frac{\cos \left(\frac{\pi}{6} - \frac{3\pi}{4} \right)}{1 \times 1} \\ &= \cos \left(-\frac{7\pi}{12} \right) = \cos \left(\frac{7\pi}{12} \right) \Rightarrow \theta = \frac{7\pi}{12} \text{ radians } (105^\circ)\end{aligned}$$

③ $\mathbf{u} = (1, 1, 1), \mathbf{v} = (2, 1, -1)$

$$\begin{aligned}\cos \theta &= \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{1(2) + 1(1) + 1(-1)}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{2^2 + 1^2 + (-1)^2}} = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{\sqrt{2}}{3} \\ \Rightarrow \theta &= \cos^{-1}(\sqrt{2}/3) \approx 1.08 \text{ radians } (62^\circ)\end{aligned}$$

Determine whether \mathbf{u} and \mathbf{v} are orthogonal, parallel, or neither

① $\mathbf{u} = (1, -1, 1), \mathbf{v} = (2, 1, -1)$

$\mathbf{u} \cdot \mathbf{v} = 1(2) + (-1)(1) + 1(-1) = 0$, the vectors \mathbf{u} and \mathbf{v} are orthogonal

② $\mathbf{u} = (1, -1, 1), \mathbf{v} = (2, 1, 1)$

$\mathbf{u} \cdot \mathbf{v} = 1(2) + (-1)(1) + 1(1) = 2 \neq 0, \Rightarrow \mathbf{u}$ and \mathbf{v} are not orthogonal. Moreover, because one is not a scalar multiple of the other, they are not parallel

③ $\mathbf{u} = (1, -2, 4), \mathbf{v} = (-3, 6, -12)$

$\mathbf{u} = -3\mathbf{v}$, the vectors \mathbf{u} and \mathbf{v} are parallel

④ $\mathbf{u} = (\cos \theta, \sin \theta, -1), \mathbf{v} = (\sin \theta, -\cos \theta, 0)$

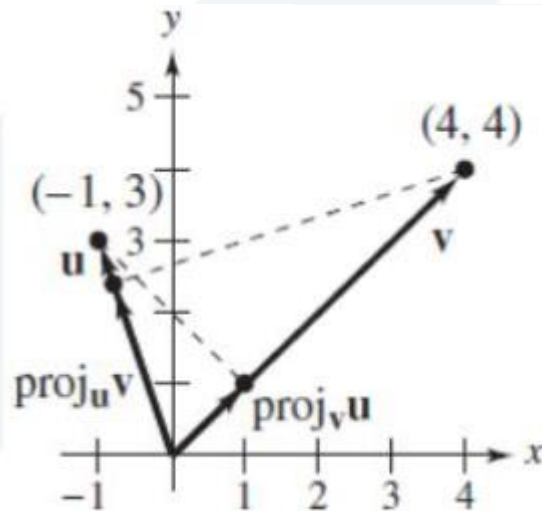
$\mathbf{u} \cdot \mathbf{v} = \cos \theta (\sin \theta) + (\sin \theta)(-\cos \theta) + (-1)(0) = 0$, the vectors \mathbf{u} and \mathbf{v} are orthogonal

Find (a) $\text{proj}_v \mathbf{u}$ and (b) $\text{proj}_u \mathbf{v}$

① $\mathbf{u} = (-1, 3)$, $\mathbf{v} = (4, 4)$

$$(a) \text{proj}_v \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{-1(4) + 3(4)}{4^2 + 4^2} (4, 4) = \frac{1}{4} (4, 4) = (1, 1)$$

$$(b) \text{proj}_u \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{4(-1) + 4(3)}{(-1)^2 + 3^2} (-1, 3) = \frac{4}{5} (-1, 3) = \left(-\frac{4}{5}, \frac{12}{5}\right)$$



② $\mathbf{u} = (1, 3, -2)$, $\mathbf{v} = (0, -1, 1)$

$$(a) \text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{1(0) + 3(-1) - 2(1)}{0^2 + (-1)^2 + 1^2} (0, -1, 1) = \frac{-5}{2} (0, -1, 1) = \left(0, \frac{5}{2}, \frac{-5}{2}\right)$$

$$(b) \text{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{0(1) + (-1)(3) + 1(-2)}{1^2 + 3^2 + (-2)^2} (1, 3, -2) \\ = \frac{-5}{14} (1, 3, -2) = \left(\frac{-5}{14}, \frac{-15}{14}, \frac{5}{7}\right)$$

(a) Determine whether the set of vectors in \mathbf{R}^n is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for \mathbf{R}^n

① $\mathbf{u} = (2, -4), \mathbf{v} = (2, 1)$

(a) $(2, -4) \cdot (2, 1) = 2(2) + (-4)(1) = 0$, the set is orthogonal

(b) $\|(2, -4)\| = \sqrt{2^2 + (-4)^2} = \sqrt{20} \neq 1$, the set is not orthonormal

(c) 2 independent vectors, the set is a basis for \mathbf{R}^2

② $\{\mathbf{u} = (3/5, 4/5), \mathbf{v} = (-4/5, 3/5)\}$

(a) $(\frac{3}{5}, \frac{4}{5}) \cdot (-\frac{4}{5}, \frac{3}{5}) = \frac{3}{5}(-\frac{4}{5}) + \frac{4}{5}(\frac{3}{5}) = 0$, the set is orthogonal

(b) $\|(\frac{3}{5}, \frac{4}{5})\| = \sqrt{(\frac{3}{5})^2 + (\frac{4}{5})^2} = 1, \quad \|(-\frac{4}{5}, \frac{3}{5})\| = \sqrt{(-\frac{4}{5})^2 + (\frac{3}{5})^2} = 1$, the set is orthonormal



(c) 2 independent vectors, the set is a basis for \mathbb{R}^2

$$\left\| \left(\frac{3}{5}, \frac{4}{5} \right) \right\| = \sqrt{\left(\frac{3}{5} \right)^2 + \left(\frac{4}{5} \right)^2} = 1, \quad \left\| \left(-\frac{4}{5}, \frac{3}{5} \right) \right\| = \sqrt{\left(-\frac{4}{5} \right)^2 + \left(\frac{3}{5} \right)^2} = 1$$

$$\textcircled{3} \left\{ \mathbf{u} = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right), \mathbf{v} = \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right), \mathbf{w} = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right) \right\}$$

(a) The set is orthogonal because

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right) \cdot \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right) = -\frac{\sqrt{12}}{12} + 0 + \frac{\sqrt{12}}{12} = 0$$

$$\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right) = \frac{\sqrt{6}}{6} + 0 - \frac{\sqrt{6}}{6} = 0$$

$$\left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \cdot \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) = -\frac{\sqrt{18}}{18} + \frac{\sqrt{18}}{9} - \frac{\sqrt{18}}{18} = 0$$

(b) The set is orthonormal because

$$\left\| \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right) \right\| = \sqrt{\frac{2}{4} + 0 + \frac{2}{4}} = 1, \quad \left\| \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) \right\| = \sqrt{\frac{3}{9} + \frac{3}{9} + \frac{3}{9}} = 1$$

$$\left\| \left(-\frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right) \right\| = \sqrt{\frac{6}{36} + \frac{6}{9} + \frac{6}{36}} = 1$$

(c) 3 independent vectors, the set is a basis for \mathbb{R}^3

④ $\left\{ \mathbf{u} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \mathbf{v} = (0, 1, 0) \right\}$

(a) The set is orthogonal because $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \cdot (0, 1, 0) = 0$

(b) The set is orthonormal because

$$\left\| \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right) \right\| = \sqrt{\frac{1}{2} + 0 + \frac{1}{2}} = 1, \quad \|(0, 1, 0)\| = \sqrt{0 + 1 + 0} = 1$$

(c) Only 2 independent vectors, the set is not a basis for \mathbb{R}^3

(a) Show that the set of vectors in \mathbf{R}^n is orthogonal, and (b) normalize the set to produce an orthonormal set

① $\mathbf{u} = (-1, 4), \mathbf{v} = (8, 2)$

(a) $(-1, 4) \cdot (8, 2) = -1(8) + 4(2) = 0$, the set is orthogonal

(b) $\|(-1, 4)\| = \sqrt{(-1)^2 + 4^2} = \sqrt{17} \neq 1$, the set is not orthonormal. Normalizing

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{17}}(-1, 4) = \left(-\frac{\sqrt{17}}{17}, \frac{4\sqrt{17}}{17} \right)$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{2\sqrt{17}}(8, 2) = \left(\frac{4\sqrt{17}}{17}, \frac{\sqrt{17}}{17} \right)$$

$$\textcircled{2} \left\{ \mathbf{v}_1 = (\sqrt{3}, \sqrt{3}, \sqrt{3}), \mathbf{v}_2 = (-\sqrt{2}, 0, \sqrt{2}) \right\}$$

(a) $(\sqrt{3}, \sqrt{3}, \sqrt{3}) \cdot (-\sqrt{2}, 0, \sqrt{2}) = -\sqrt{6} + 0 + \sqrt{6} = 0$, the set is orthogonal

(b) $\|(\sqrt{3}, \sqrt{3}, \sqrt{3})\| = \sqrt{\sqrt{3}^2 + \sqrt{3}^2 + \sqrt{3}^2} = \sqrt{9} = 3 \neq 1$

the set is not orthonormal. Normalizing

$$\mathbf{u}_1 = \frac{1}{3}(\sqrt{3}, \sqrt{3}, \sqrt{3}) = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right)$$

$$\mathbf{u}_2 = \frac{1}{2}(-\sqrt{2}, 0, \sqrt{2}) = \left(-\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2} \right)$$

Find the coordinate matrix of w relative to the orthonormal basis B in \mathbb{R}^n

$$\textcircled{1} \quad w = (-3, 4), B = \left\{ \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right), \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) \right\}$$

$$(-3, 4) \cdot \left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \right) = \frac{-3\sqrt{5}}{5} + \frac{8\sqrt{5}}{5} = \frac{5\sqrt{5}}{5} = \sqrt{5}$$

$$(-3, 4) \cdot \left(-\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right) = \frac{6\sqrt{5}}{5} + \frac{4\sqrt{5}}{5} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$$

$$\text{So, } [w]_B = \begin{bmatrix} \sqrt{5} \\ 2\sqrt{5} \end{bmatrix}$$

$$\textcircled{2} \quad \mathbf{w} = (2, -2, 1), B = \left\{ \left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right), (0, 1, 0), \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10} \right) \right\}$$

$$(2, -2, 1) \cdot \left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right) = \frac{2\sqrt{10}}{10} + 0 + \frac{3\sqrt{10}}{10} = \frac{5\sqrt{10}}{10} = \frac{\sqrt{10}}{2}$$

$$(2, -2, 1) \cdot (0, 1, 0) = 0 - 2 + 0 = -2$$

$$(2, -2, 1) \cdot \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10} \right) = -\frac{6\sqrt{10}}{10} + 0 + \frac{\sqrt{10}}{10} = -\frac{5\sqrt{10}}{10} = -\frac{\sqrt{10}}{2}$$

$$\text{So, } [\mathbf{w}]_B = \begin{bmatrix} \frac{\sqrt{10}}{2} \\ -2 \\ -\frac{\sqrt{10}}{2} \end{bmatrix}$$

Apply the Gram-Schmidt orthonormalization process to transform the given basis for \mathbb{R}^n into an orthonormal basis

① $\mathbf{v}_1 = (3, 4), \mathbf{v}_2 = (1, 0)$

$$\mathbf{w}_1 = \mathbf{v}_1 = (3, 4)$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 = (1, 0) - \frac{1(3) + 0(4)}{3^2 + 4^2} (3, 4) = \left(\frac{16}{25}, -\frac{12}{25} \right)$$

Normalize the vectors

$$\mathbf{u}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|} = \frac{1}{\sqrt{3^2 + 4^2}} (3, 4) = \left(\frac{3}{5}, \frac{4}{5} \right)$$

$$\mathbf{u}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{1}{\sqrt{\left(\frac{16}{25}\right)^2 + \left(-\frac{12}{25}\right)^2}} \left(\frac{16}{25}, -\frac{12}{25} \right) = \left(\frac{4}{5}, -\frac{3}{5} \right)$$

$$\textcircled{2} \quad B = \{ \mathbf{v}_1 = (1, -2, 2), \mathbf{v}_2 = (2, 2, 1), \mathbf{v}_3 = (2, -1, -2) \}$$

Because $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$, the given vectors are orthogonal. Normalize the vectors

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2}} (1, -2, 2) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \right)$$

$$\mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{2^2 + 2^2 + 1^2}} (2, 2, 1) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right)$$

$$\mathbf{u}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{1^2 + (-2)^2 + 2^2}} (2, -1, -2) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right)$$

$$\textcircled{3} \quad B = \{ \mathbf{v}_1 = (4, -3, 0), \mathbf{v}_2 = (1, 2, 0), \mathbf{v}_3 = (0, 0, 4) \}$$

Orthogonalize each vector in B vectors

$$\mathbf{w}_1 = \mathbf{v}_1 = (4, -3, 0)$$

$$w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = (1, 2, 0) - \frac{2}{25} (4, -3, 0) = \left(\frac{33}{25}, \frac{44}{25}, 0 \right)$$

$$w_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = (0, 0, 4) - 0(4, -3, 0) - 0 \left(\frac{33}{25}, \frac{44}{25}, 0 \right) = (0, 0, 4)$$

Normalize the vectors

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{4^2 + (-3)^2 + 0^2}} (4, -3, 0) = \left(\frac{4}{5}, -\frac{3}{5}, 0 \right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{\left(\frac{33}{25}\right)^2 + \left(\frac{44}{25}\right)^2 + 0^2}} \left(\frac{33}{25}, \frac{44}{25}, 0 \right) = \left(\frac{3}{5}, \frac{4}{5}, 0 \right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{0^2 + 0^2 + 4^2}} (0, 0, 4) = (0, 0, 1)$$

Find (a) $\|\mathbf{u}\|$, (b) $\|\mathbf{v}\|$, and (c) $\|\mathbf{u} + \mathbf{v}\|$

1. $\mathbf{u} = (2, 1, -1)$, $\mathbf{v} = (0, -1, 1)$

2. $\mathbf{u} = (2, -1)$, $\mathbf{v} = (1, 1)$

Find a unit vector (a) in the direction of \mathbf{u} and (b) in the direction opposite that of \mathbf{u}

1. $\mathbf{u} = (-1, 3, 4)$

2. $\mathbf{u} = (2, -2)$

Find the vector \mathbf{v} with the given length and the same direction as \mathbf{u}

1. $\|\mathbf{v}\| = 4$, $\mathbf{u} = (1, 2, -1)$

2. $\|\mathbf{v}\| = 2$, $\mathbf{u} = (3, 4)$

Find (a) $\mathbf{u} \cdot \mathbf{v}$, (b) $\mathbf{v} \cdot \mathbf{v}$, (c) $\|\mathbf{v}\|^2$, (d) $d(\mathbf{u}, \mathbf{v})$, (e) $(\mathbf{u} \cdot \mathbf{v})\mathbf{v}$, and (f) $\mathbf{u} \cdot (5\mathbf{v})$

1. $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (2, -2)$

2. $\mathbf{u} = (0, -2, 1)$, $\mathbf{v} = (2, 1, -2)$

For what values of c is $\|c(-2, 2, 1)\| = 6$?

Find $(\mathbf{u} + \mathbf{v}) \cdot (2\mathbf{u} - \mathbf{v})$ when $\mathbf{u} \cdot \mathbf{u} = 4$, $\mathbf{u} \cdot \mathbf{v} = -5$, and $\mathbf{v} \cdot \mathbf{v} = 10$

Find the angle θ between the vectors

1. $\mathbf{u} = (3, 1)$, $\mathbf{v} = (-2, 4)$

2. $\mathbf{u} = (2, 3, 1)$, $\mathbf{v} = (-3, 2, 0)$

3. $\mathbf{u} = \left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3}\right)$, $\mathbf{v} = \left(\cos \frac{\pi}{4}, \sin \frac{\pi}{4}\right)$

Determine whether \mathbf{u} and \mathbf{v} are orthogonal, parallel, or neither

1. $\mathbf{u} = (0, 3, -4)$, $\mathbf{v} = (1, -8, 6)$

2. $\mathbf{u} = (1, -1, 2)$, $\mathbf{v} = (-3, 3, -6)$

3. $\mathbf{u} = (1, -1)$, $\mathbf{v} = (0, -1)$

Find (a) $\text{proj}_{\mathbf{v}} \mathbf{u}$ and (b) $\text{proj}_{\mathbf{u}} \mathbf{v}$

1. $\mathbf{u} = (1, 2)$, $\mathbf{v} = (2, 1)$

2. $\mathbf{u} = (5, -3, 1)$, $\mathbf{v} = (1, -1, 0)$

(a) Determine whether the set of vectors in \mathbf{R}^n is orthogonal, (b) if the set is orthogonal, then determine whether it is also orthonormal, and (c) determine whether the set is a basis for \mathbf{R}^n

1. $\{\mathbf{u} = (-3, 5), \mathbf{v} = (4, 0)\}$

2. $\{\mathbf{u} = (2, 1), \mathbf{v} = (1/3, -2/3)\}$

3. $\left\{ \mathbf{u} = \left(\frac{\sqrt{2}}{3}, 0, -\frac{\sqrt{2}}{6} \right), \mathbf{v} = \left(0, \frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5} \right), \mathbf{w} = \left(\frac{\sqrt{5}}{5}, 0, \frac{1}{2} \right) \right\}$

4. $\{(2, 5, -3), (4, 2, 6)\}$

(a) Show that the set of vectors in \mathbf{R}^n is orthogonal, and (b) normalize the set to produce an orthonormal set

1. $\{ \mathbf{v}_1 = (-1, 3), \mathbf{v}_2 = (12, 4) \}$

2. $\{ \mathbf{v}_1 = (1, 3, 1), \mathbf{v}_2 = (3, 0, -3) \}$

Find the coordinate matrix of \mathbf{w} relative to the orthonormal basis \mathbf{B} in \mathbf{R}^n

1. $\mathbf{w} = (1, 2), \mathbf{B} = \left\{ \left(-\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13} \right), \left(\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13} \right) \right\}$

2. $\mathbf{w} = (1, -3, 2), \mathbf{B} = \left\{ \left(\frac{\sqrt{10}}{10}, 0, \frac{3\sqrt{10}}{10} \right), (0, 1, 0), \left(-\frac{3\sqrt{10}}{10}, 0, \frac{\sqrt{10}}{10} \right) \right\}$

Apply the Gram-Schmidt orthonormalization process to transform the given basis for \mathbb{R}^n into an orthonormal basis

1. $B = \{v_1 = (4, -3), v_2 = (3, 2)\}$

2. $B = \{v_1 = (2, 1, -2), v_2 = (1, 2, 2), v_3 = (2, -2, 1)\}$

3. $B = \{v_1 = (1, 0, 0), v_2 = (1, 1, 1), v_3 = (1, 1, -1)\}$