CALCULUS 2

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Improper Integrals:

Evaluate an improper integral that has an infinite limit of integration

Definition:

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx; \quad c \in \mathbb{R}$$

- In the first two cases,
  - the improper integral converges when the limit exists
  - otherwise, the improper integral diverges.
- In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges

An Improper Integral That Diverges

EXAMPLE 1: Evaluate $\int_1^\infty \frac{dx}{x}$

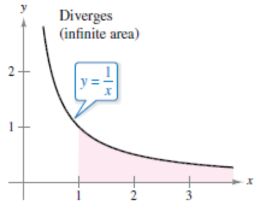
Solution:

$$\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} dx \quad \text{Take limit as } b \to \infty$$

$$= \lim_{b\to\infty} [\ln x]_1^b \quad \text{Apply Log Rule.}$$

$$= \lim_{b\to\infty} (\ln b - 0) = \infty$$

The limit does not exist. $\Longrightarrow$ the improper integral diverges.



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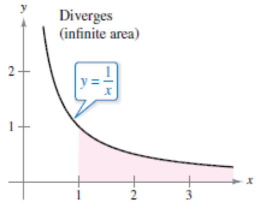
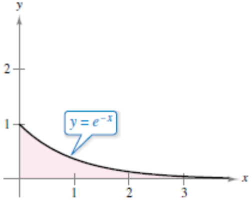


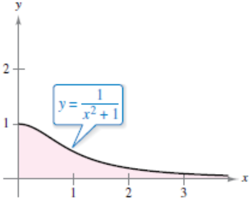
EXAMPLE 2: Evaluate

- $\int_0^\infty e^{-x} dx$
- $\int_0^\infty \frac{1}{x^2+1} dx$

a.
$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b = \lim_{b\to\infty}(-e^{-b}+1) = 1$$

b.
$$\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty}\Big[\arctan x\Big]_0^b = \lim_{b\to\infty}\arctan b = \frac{\pi}{2}$$





Using L'Hôpital's Rule with an Improper Integral:

EXAMPLE 3: Evaluate $\int_1^\infty (1-x)\, e^{-x} dx$

Solution:

- Use integration by parts, $dv = e^{-x}$,$u = (1-x)$

$$\int_1^\infty (1-x)\, e^{-x} dx = -e^{-x}(1-x) - \int e^{-x} dx$$
$$= -e^{-x} + xe^{-x} + e^{-x} + c$$
$$= xe^{-x} + c$$

- Now, apply the definition of an improper integral.

$$\int_1^\infty (1-x)\, e^{-x} dx = \lim_{b\to\infty} \int_1^b \frac{dx}{x} dx$$
$$= \lim_{b\to\infty} \left[xe^{-x}\right]_1^b$$

Using L'Hôpital's Rule with an Improper Integral:

$$\int_1^\infty (1-x)\,e^{-x}dx = \lim_{b\to\infty}\left[be^{-b} - \frac{1}{e}\right]$$

- For the first limit, use L'Hôpital's Rule.

$$\lim_{b\to\infty}\frac{b}{e^b} = \lim_{b\to\infty}\frac{1}{e^b} = 0$$

- So, you can conclude that

$$\int_1^\infty (1-x)\,e^{-x}dx = \lim_{b\to\infty}\left[be^{-b} - \frac{1}{e}\right]$$

$$= 0 - \frac{1}{e} = -\frac{1}{e}$$

Infinite Upper and Lower Limits of Integration

EXAMPLE 4 Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}} dx$

Solution:

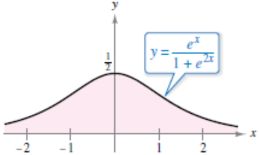
$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx = \int_{-\infty}^{0} \frac{e^x}{1+e^{2x}}\,dx + \int_{0}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$

$$= \lim_{a \to -\infty} \Big[\arctan e^x\Big]_a^0 + \lim_{b \to \infty} \Big[\arctan e^x\Big]_0^b$$

$$= \lim_{a \to -\infty} \left(\frac{\pi}{4} - \arctan e^a\right) + \lim_{b \to \infty} \left(\arctan e^b - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - 0 + \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$



Improper Integrals with Infinite Discontinuities

Definition:

1. If f is continuous on the interval $[a, b)$, and has an infinite discontinuity at b, then
$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

2. If f is continuous on the interval $(a, b]$, and has an infinite discontinuity at a, then
$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

3. If f is continuous on the interval $[a, b]$, except for some c in $[a, b]$ at which f has an infinite discontinuity, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx; \quad c \in \mathbb{R}$$

- In the first two cases,
  - the improper integral converges when the limit exists
  - otherwise, the improper integral diverges.
- In the third case, the improper integral on the left diverges when either of the improper integrals on the right diverges.

Improper Integrals with Infinite Discontinuities

EXAMPLE 4:

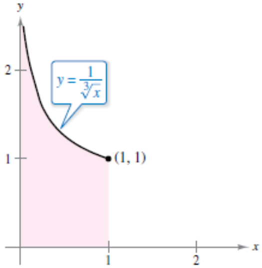
Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Solution The integrand has an infinite discontinuity at $x = 0$, as shown in the figure at the right. You can evaluate this integral as shown below.

$$\int_0^1 x^{-1/3}\,dx = \lim_{b \to 0^+} \left[\frac{x^{2/3}}{2/3}\right]_b^1$$

$$= \lim_{b \to 0^+} \frac{3}{2}(1 - b^{2/3})$$

$$= \frac{3}{2}$$



Infinite discontinuity at $x = 0$

EXAMPLE 5:

Evaluate $\displaystyle\int_0^2 \frac{dx}{x^3}$.

Solution Because the integrand has an infinite discontinuity at $x = 0$, you can write

$$\int_0^2 \frac{dx}{x^3} = \lim_{b \to 0^+} \left[-\frac{1}{2x^2}\right]_b^2$$

$$= \lim_{b \to 0^+} \left(-\frac{1}{8} + \frac{1}{2b^2}\right)$$

$$= \infty.$$

So, you can conclude that the improper integral diverges.

Improper Integrals with Infinite Discontinuities

EXAMPLE 6:

Evaluate $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}(x+1)}$.

Solution To evaluate this integral, split it at a convenient point (say, $x = 1$) and write

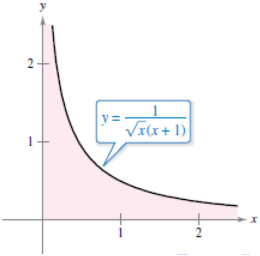
$$\int_0^\infty \frac{dx}{\sqrt{x}(x+1)} = \int_0^1 \frac{dx}{\sqrt{x}(x+1)} + \int_1^\infty \frac{dx}{\sqrt{x}(x+1)}$$

$$= \lim_{b \quad 0^+} \Big[2 \arctan \sqrt{x}\Big]_b^1 + \lim_{c \quad \infty} \Big[2 \arctan \sqrt{x}\Big]_1^c$$

$$= \lim_{b \quad 0^+} \left(2 \arctan 1 - 2 \arctan \sqrt{b}\right) + \lim_{c \quad \infty} \left(2 \arctan \sqrt{c} - 2 \arctan 1\right)$$

$$= 2\left(\frac{\pi}{4}\right) - 0 + 2\left(\frac{\pi}{2}\right) - 2\left(\frac{\pi}{4}\right)$$

$$= \pi.$$



Thank you for your attention