CALCULUS 2

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Infinite Series:

Sequences:

- Write the terms of a sequence.
- Determine whether a sequence converges or diverges.
- Write a formula for the nth term of a sequence.

1,	2,	3,	4,	. . . ,	n,	. . .	
↓	↓	↓	↓	↓	↓	↓	Sequence
a_1,	a_2,	a_3,	a_4,	. . . ,	a_n,	. . .	

- $a_1, a_2, a_3,$........ are the terms of the sequence.
- The number a_n is the nth term of the sequence,

Example 1

Writing the Terms of a Sequence

a. The terms of the sequence $\{a_n\} = \{3 + (-1)^n\}$ are

$$3 + (-1)^1,\ 3 + (-1)^2,\ 3 + (-1)^3,\ 3 + (-1)^4, \ldots$$
$$2, \quad 4, \quad 2, \quad 4, \quad \ldots.$$

b. The terms of the sequence $\{b_n\} = \left\{\dfrac{n}{1-2n}\right\}$ are

$$\frac{1}{1-2\cdot 1},\quad \frac{2}{1-2\cdot 2},\quad \frac{3}{1-2\cdot 3},\quad \frac{4}{1-2\cdot 4},\quad \ldots$$

$$-1,\quad -\frac{2}{3},\quad -\frac{3}{5},\quad -\frac{4}{7},\quad \ldots.$$

c. The terms of the sequence $\{c_n\} = \left\{\dfrac{n^2}{2^n-1}\right\}$ are

$$\frac{1^2}{2^1-1},\quad \frac{2^2}{2^2-1},\quad \frac{3^2}{2^3-1},\quad \frac{4^2}{2^4-1},\quad \ldots$$

$$\frac{1}{1},\quad \frac{4}{3},\quad \frac{9}{7},\quad \frac{16}{15},\quad \ldots.$$

d. The terms of the **recursively defined** sequence $\{d_n\}$, where $d_1 = 25$ and $d_{n+1} = d_n - 5$, are

$$25,\quad 25-5=20,\quad 20-5=15,\quad 15-5=10,\ \ldots.$$

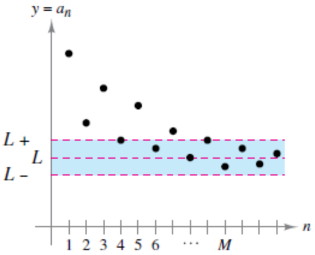
Definition of the Limit of a Sequence

Let L be a real number. The limit of a sequence $\{a_n\}_{n\in\mathbb{N}}$ is L, written as

$$\lim_{n\to\infty} a_n = L$$

$$\Updownarrow$$

$$\forall \varepsilon > 0, \exists M > 0; \forall n > M \Longrightarrow |a_n - L| < \varepsilon$$



For $n > M$, the terms of the sequence all lie within ε units of L.

Limit of a Sequence:

EXAMPLE2: Finding the Limit of a Sequence

Find the limit of the sequence whosenth term is $a_n = \left(1 + \frac{1}{n}\right)^n$

In Example 5, Lecture 1, you learned that

$\lim\limits_{x \to \infty} = \left(1 + \frac{1}{x}\right)^x = e$ so $\lim\limits_{n \to \infty} a_n = \lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

There are different ways in which a sequence can fail to have a limit. One way is that the terms of the sequence increase without bound or decrease without bound.

- Terms increase without bound: $\lim\limits_{n \to \infty} a_n = +\infty$
- Terms decrease without bound: $\lim\limits_{n \to \infty} a_n = -\infty$

Properties of Limits of Sequences

Let $\lim_{n \to \infty} a_n = L$ and $\lim_{n \to \infty} b_n = K$.

1. Scalar multiple: $\lim_{n \to \infty} (ca_n) = cL$, c is any real number.
2. Sum or difference: $\lim_{n \to \infty} (a_n \pm b_n) = L \pm K$
3. Product: $\lim_{n \to \infty} (a_n b_n) = LK$
4. Quotient: $\lim_{n \to \infty} \dfrac{a_n}{b_n} = \dfrac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

EXAMPLE 3 Determining Convergence or Divergence

1. Because the sequence $(a_n)_{n\in\mathbb{N}} = \{3 + (-1)^n\}$

   has terms $2, 4, 2, 4$....that alternate between 2 and 4,

   the limit does not exist. So, the sequence diverges.

2. For $(b_n)_{n\in\mathbb{N}} = \left(\frac{n}{1-2n}\right)$

$$\lim_{n\to\infty} \frac{n}{1-2n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n} - 2} = -\frac{1}{2}$$

   which implies that the sequence converges to $-\frac{1}{2}$.

Example 4: Using L'hôpital's rule to Determine Convergence

Show that the sequence whose nth term is $a_n = \frac{n^2}{2^n - 1}$ converges.

Solution Consider the function of a real variable

$$f(x) = \frac{x^2}{2^x - 1}.$$

Applying L'Hôpital's Rule twice produces

$$\lim_{x \to \infty} \frac{x^2}{2^x - 1} = \lim_{x \to \infty} \frac{2x}{(\ln 2)2^x} = \lim_{x \to \infty} \frac{2}{(\ln 2)^2 2^x} = 0.$$

Because $f(n) = a_n$ for every positive integer n,

$$\lim_{n \to \infty} \frac{n^2}{2^n - 1} = 0.$$

So, the sequence converges to 0.

Finding the nth Term of a Sequence

Example 5: Finding the nth Term of a Sequence

$$\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots\ldots\ldots$$

and then determine whether the sequence you have chosen converges or diverges.

First, note that the numerators are successive powers of 2, and the denominators form the sequence of positive odd integers. By comparing a_n with n, you have the following pattern.

$$\frac{2^1}{1}, \frac{2^2}{3}, \frac{2^3}{5}, \frac{2^4}{7}, \frac{2^5}{9}, \ldots, \frac{2^n}{2n-1}, \ldots$$

Consider the function of a real variable $f(x) = 2^x/(2x - 1)$. Applying L'Hôpital's Rule produces

$$\lim_{x \to \infty} \frac{2^x}{2x-1} = \lim_{x \to \infty} \frac{2^x(\ln 2)}{2} = \infty.$$

$$\lim_{n \to \infty} \frac{2^n}{2n-1} = \infty.$$

So, the sequence diverges.

Example 6: Finding the nth Term of a Sequence

$$-\frac{2}{1}, \frac{8}{2}, -\frac{26}{6}, \frac{80}{24}, -\frac{242}{120}, \ldots\ldots$$

and then determine whether the sequence you have chosen converges or diverges.

Note that the numerators are 1 less than 3^n.

$$3^1 - 1 = 2 \quad 3^2 - 1 = 8 \quad 3^3 - 1 = 26 \quad 3^4 - 1 = 80 \quad 3^5 - 1 = 242$$

So, you can reason that the numerators are given by the rule

$$3^n - 1.$$

Factoring the denominators produces

$$1 = 1$$
$$2 = 1 \cdot 2$$
$$6 = 1 \cdot 2 \cdot 3$$
$$24 = 1 \cdot 2 \cdot 3 \cdot 4$$

and

$$120 = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5.$$

Finding the nth Term of a Sequence

This suggests that the denominators are represented by $n!$. Finally, because the signs alternate, you can write the nth term as

$$a_n = (-1)^n \left(\frac{3^n - 1}{n!}\right)$$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{3^n - 1}{n!} = 0$$

$$\lim_{n\to\infty} a_n = 0$$

So, the sequence converges to zero.

Monotonic Sequences

Definition of Monotonic Sequence

A sequence $\{a_n\}$ is **monotonic** when its terms are nondecreasing

$$a_1 \le a_2 \le a_3 \le \cdots \le a_n \le \cdots$$

or when its terms are nonincreasing

$$a_1 \ge a_2 \ge a_3 \ge \cdots \ge a_n \ge \cdots.$$

Example 7: Determine whether each sequence having the given nth term is monotonic.

1. $a_n = 3 + (-1)^n$
2. $b_n = \frac{2n}{1+n}$
3. $c_n = \frac{n^2}{2^n - 1}$

1. This sequence alternates between 2 and 4. So, it is not monotonic.
2. This sequence is monotonic because

$$b_n = \frac{2n}{1+n} \overset{?}{<} \frac{2(n+1)}{1+(n+1)} = b_{n+1}$$

$$2n(2+n) \overset{?}{<} (1+n)(2n+2)$$

$$4n + 2n^2 \overset{?}{<} 2 + 4n + 2n^2$$

$$0 < 2$$

3. This sequence is not monotonic because the second term is greater than both the first term and the third term.

Definition of Bounded Sequence

1. A sequence $\{a_n\}$ is **bounded above** when there is a real number M such that $a_n \leq M$ for all n. The number M is called an **upper bound** of the sequence.
2. A sequence $\{a_n\}$ is **bounded below** when there is a real number N such that $N \leq a_n$ for all n. The number N is called a **lower bound** of the sequence.
3. A sequence $\{a_n\}$ is **bounded** when it is bounded above and bounded below.

If a sequence $(a_n)_{n\in\mathbb{N}}$ is bounded and monotonic, then it converges.

Bounded and Monotonic Sequences

a. The sequence $\{a_n\} = \{1/n\}$ is both bounded and monotonic. So, by , it must converge.

b. The divergent sequence $\{b_n\} = \{n^2/(n+1)\}$ is monotonic but not bounded. (It is bounded below.)

c. The divergent sequence $\{c_n\} = \{(-1)^n\}$ is bounded but not monotonic.

Thank you for your attention